An Optimal Design of $H_\infty$ Static Output Feedback Controller using LMI for Collocated Gyroscopic System

Yuki Kurotaki, Tomoyuki Nagashio and Takashi Kida

Abstract—A class of symmetric static output feedback controllers are known to robustly stabilize symmetric collocated second-order linear time invariant systems having positive definite or positive semi-definite coefficient matrices. This paper extends the result to the asymmetric systems which include skew-symmetric gyroscopic terms. We first obtain the condition for static output feedback controllers to guarantee the closed-loop robust stability against model errors. Then an optimal $H_\infty$ design method based on LMI. By analyzing the upper bound of $H_\infty$ norm, it is clarified that the optimal feedback control works to cancel the skew-symmetric terms. Some numerical studies are performed to show the validity and to discuss the feasibility.

II. DESCRIPTION OF PLANT AND CONTROLLER

We consider the gyroscopic mechanical system described by the linear matrix second-order differential equation [10],

$$M \ddot{p} + D \dot{p} + G \dot{p} + Kp = Lu + Fw \tag{1}$$

where $p \in \mathbb{R}^n$ is the physical displacement vector, $u \in \mathbb{R}^m$ is the control input vector, and $w \in \mathbb{R}^l$ is the disturbance input vector. The coefficient matrices $M, D, G$ and $K$ are respectively associated with inertial, damping, gyroscopic, and stiffness forces and satisfy

$$M^T = M > 0, \quad D^T = D \geq 0, \quad K^T = K \geq 0, \quad G^T = -G. \tag{2}$$

The matrices $L$ and $F$ define the locations and directions of control inputs $u$ and disturbance inputs $w$. Since we consider collocated systems, the measurement outputs are supposed to be described as follows.

$$y_d = L^T p, \quad y_v = L^T \dot{p} \tag{3}$$

where $y_d \in \mathbb{R}^m$ and $y_v \in \mathbb{R}^m$ are respectively the displacement and velocity outputs measured at the same locations as control inputs. The collocated system (1), (3) is known to be stabilizable and detectable if the rank conditions

$$\text{rank}[D L] = n, \quad \text{rank}[K L] = n \tag{4}$$

hold [4]. It is noted here that the transfer function matrices $G_{y_d u}$ from $u$ to $y_d$ and $G_{y_v u}$ from $u$ to $y_v$ are given as follows.

$$G_{y_d u}(s) = L^T (Ms^2 + Ds + Gs + K)^{-1} L, \quad G_{y_v u}(s) = sG_{y_d u}(s) \tag{5}$$

Since the transfer function matrices in (5) are not symmetric, we refer to (1), (3) as the asymmetric system. It is symmetric only when $G = 0$, that is, (1) is a non-gyroscopic system whose control problem has been already discussed in [8].

Now, for gyroscopic systems, we consider the static output feedback control having arbitrary square feedback gain matrices $K_c$ and $D_c$ as

$$u = -K_c y_d - D_c y_v. \tag{6}$$

Then we can state the following lemma on the closed-loop internal stability.

I. INTRODUCTION

Symmetric systems have some interesting properties from the viewpoints of stability and control. A class of mechanical systems described by second-order linear matrix differential equations having symmetric coefficient matrices corresponding to mass, spring and damper is a typical example. For such a system with collocated sensors and actuators, a symmetric static output feedback controller having positive definite gain matrices is known to be effective because it guarantees the robust stability irrespective of the system parameters [1], [2]. In addition, several design methods of optimal positive definite feedback gain matrices have been proposed [3], [4], [5], [6], [7]. Among these, it has been shown that an optimal design in the $H_\infty$ control framework is feasible using the linear matrix inequality (LMI) derived from the bounded real lemma (BRL) [5], [6], [7]. The result in [7] has been generalized so as to treat a wider class of exogenous inputs and control outputs [8]. We also have demonstrated its effectiveness through an in-orbit large flexible spacecraft attitude control experiment [9].

By extending our previous results, this paper studies a robust control problem of collocated gyroscopic systems which include skew-symmetric terms. The gyroscopic system plays an important role in rotating mechanics such as spacecraft attitude motions. To this end, we first investigate the stability condition of the closed-loop system by static output feedback controller having arbitrary square feedback gain matrices. Then we propose an optimal $H_\infty$ design method based on LMI. By analyzing the upper bound of $H_\infty$ norm, it is clarified that the optimal feedback control works to cancel the skew-symmetric terms. Some numerical studies are performed to show the validity and to discuss the feasibility.

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Y. Kurotaki is with the Department of Mechanical Engineering and Intelligent Systems, University of Electro-Communications, Chofu, 1-5-1, Tokyo, Japan kurotaki @ctr.mce.uec.ac.jp

T. Nagashio is with the Department of Aerospace Engineering, Osaka Prefecture University, Sakai, 1-1, Osaka, Japan nagashio @aero.osaka pref u.ac.jp

T. Kida is with the Department of Mechanical Engineering and Intelligent Systems, University of Electro-Communications, Chofu, 1-5-1, Tokyo, Japan kida @mce.uec.ac.jp
Lemma 1 The closed-loop system of (1) and (3) with (6) is internally stable if the static output feedback gain matrices satisfy
\[ K_c = K_c^T > 0, \quad D_c + D_c^T > 0 \] (7)
when the disturbance input \( w = 0 \).

Proof When we select feedback gain matrices as (7), the closed-loop system becomes
\[ M \dot{p} + D_c \dot{p} + G_c \dot{p} + K_c p = Fw \] (8)
\[ K_{cl} = K + L K_c L_c^T, \quad D_{cl} = D + L D_c L_c^T, \quad G_{cl} = G + L D_c L_c^T \] (9)
where \( D_c = D_s + D_k \) with \( D_s \) being symmetric and \( D_k \) skew-symmetric defined by
\[ D_s = \frac{1}{2} (D_c + D_c^T), \quad D_k = \frac{1}{2} (D_c - D_c^T) \] (10)
Then the coefficient matrices \( D_{cl}, K_{cl} \) in (8) are obviously symmetric and \( G_{cl} \) is skew-symmetric. Furthermore it can be shown that \( D_{cl} > 0 \) and \( K_{cl} > 0 \) if (1), (3) satisfy (4) when \( D_c > 0 \) and \( K_c > 0 \) [4]. Therefore (7) is a sufficient condition for (8) being stable from the Kelvin-Tait-Chetaev theorem [11], [12].

It is noted that the stability condition (7) only requires the sign definiteness of the coefficient matrices of plant (1) and controller (6) and is independent from system parameters. In this sense, the controller (6) is a highly robust control law. By using the salient stability property, we consider to optimize feedback gain matrices \( K_c \) and \( D_c \) in the \( H_{\infty} \) control framework.

III. ANALYSIS AND SYNTHESIS OF CONTROL SYSTEM

Now let us consider the transfer function matrix of the closed-loop system (8) from the disturbance input \( w \) to the control output \( z = E^T p \in \mathbb{R}^c \):
\[ G(s) = E^T (M s^2 + D_c s + G_c s + K_c)^{-1} F, \] (11)
and design feedback gain matrices \( K_c \) and \( D_c \) minimizing
\[ \gamma > 0 \text{ s.t.} \]
\[ \|G(s)\|_{\infty} < \gamma \] (12)
so that the disturbance attenuation ability is maximized. For the purpose, we describe the closed-loop system (8) and the control output \( z \) by the state equation
\[ \dot{x} = Ax + Bw \] (13)
\[ z = Cx. \] (14)
When the state variable is \( x = [p^T \; \dot{p}^T]^T \), we obtain
\[ A = \begin{bmatrix} 0 & I \\ -M^{-1} K_c & -M^{-1} (D_c + G_c) \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ M^{-1} F \end{bmatrix} \] (15)
\[ C = \begin{bmatrix} E^T \\ 0 \end{bmatrix}. \] (16)
From the bounded rela lemma (BRL), the stable transfer function matrix \( G(s) \) satisfies (12) if and only if there exists an \( X > 0 \) such that
\[ \begin{bmatrix} X A + A^T X & X B & C^T \\ B^T X & -\gamma I & 0 \\ C & 0 & -\gamma I \end{bmatrix} < 0. \] (17)

For the standard dynamic \( H_{\infty} \) output feedback control problems, it is known that the inequality condition of BRL (17) is feasible if the system is stabilizable and detectable [13], and some optimization methods using LMIs are well known [14], [15]. However, in the case of static output feedback controllers, it is also known that (17) is bilinear and this cannot be transformed to LMIs which allow the efficient solution utilizing the convex property [15]. To overcome the difficulty, we state the following lemma related to a candidate solution of the Lyapunov inequality.

Lemma 2 A solution of the Lyapunov inequality \( X A + A^T X < 0 \) is given by
\[ X_c = \begin{bmatrix} K_c + \alpha D_c & \alpha M \\ \alpha M & M \end{bmatrix} \] (18)
for a sufficiently small \( \alpha > 0 \).

Proof From \( X_c (18) \) and \( A \) in (15), we obtain \( X_c A + A^T X_c = -Q_c \) where
\[ Q_c = \begin{bmatrix} 2 \alpha K_c & \alpha G_c \\ \alpha G_c^T & 2 D_c - 2 \alpha M \end{bmatrix}. \] (19)
The matrices \( X_c \) and \( Q_c \) are apparently positive definite from the Schur complement, since
\[ M > 0, \quad K_c + \alpha D_c - \alpha^2 M > 0 \] (20)
and
\[ 2 \alpha K_c > 0, \quad 2 D_c - 2 \alpha M - \frac{\alpha}{2} G_c^T K_c^{-1} G_c > 0 \] (21)
hold for a sufficiently small positive scalar \( \alpha \).

Using Lemma 2, we can derive the following theorem.

Theorem 1 There always exists the controller (6) satisfying (7) such that the closed-loop system (13), (14) has a \( H_{\infty} \) norm less than \( \gamma > 0 \). And the lower bound of \( \gamma \) is given by
\[ \gamma_{\text{min}} \leq \lambda_{\text{max}} (Q_c^{-1}) \lambda_{\text{max}} (S S^T) \] (22)
where \( \lambda_{\text{max}} (\cdot) \) means the maximum eigenvalue, and
\[ S = \begin{bmatrix} \alpha F & E \\ F & 0 \end{bmatrix}. \] (23)

is a constant matrix.

Proof From the Schur complement, the inequality (17) is equivalent to \( \gamma > 0 \) and
\[ X A + A^T X + \frac{1}{\gamma} [X B \quad C^T] [B^T X \quad C] < 0. \] (24)
This is also equivalent to
\[ -Q_c + \frac{1}{\gamma} S S^T < 0 \] (25)
by substituting \( X_c \) (18) to X in (24). Therefore, the inequality (25) holds for all \( \gamma > \gamma_{\text{min}} \), where the lower bound \( \gamma_{\text{min}} \) is given as
\[ \gamma_{\text{min}} = \lambda_{\text{max}} (Q_c^{-1/2} S S^T Q_c^{-1/2}) \leq \lambda_{\text{max}} (Q_c^{-1}) \lambda_{\text{max}} (S S^T). \] (26)
by eigenvalue analysis.

The inequality (25) which is explicitly written as
\[
\begin{bmatrix}
-2\alpha K_{ce} & -\alpha G_{ce} & \alpha F_E \\
-\alpha G_{ce}^T & 2\alpha M - 2D_{ce} & F \\
\alpha F_E^T & F^T & -\gamma I & 0
\end{bmatrix}
< 0
\] (27)
is a LMI with respect to $K_{ce}$, $D_{ce}$ and $G_{ce}$ which are linear functions of the variables $K_c$, $D_c$ to be obtained as defined in (9) and (10). Therefore, we can design an optimal static output feedback controller (6), (7) by applying convex optimization algorithms to (27) with (7) if $\alpha$ is given. Furthermore, we can state the following property.

**Theorem 2** The upper bound $\gamma_{\text{min}}$ of $\gamma_{\text{min}}$ is minimal when the skew-symmetric coefficient matrix of the closed-loop system (8) satisfies $G_{ce} = 0$.

**Proof** From the matrix inversion formula, we first decompose $Q_{ce}^{-1}$ in (19) as
\[
Q_{ce}^{-1} = R^T (H_0 + H_G) R
\] (28)
where
\[
R = \begin{bmatrix}
I & 0 \\
-\frac{1}{\alpha} G_{ce} N_2^{-1} & I
\end{bmatrix},
H_0 = \begin{bmatrix}
\frac{1}{\alpha^2} N_2^{-1} & 0 \\
0 & N_2^{-1}
\end{bmatrix}
\] (29)
\[
H_G = \begin{bmatrix}
0 & 0 \\
N_1^{-1} G_{ce}^T (N_2 - G_{ce} N_1^{-1} G_{ce}^T)^{-1} G_{ce} N_1^{-1}
\end{bmatrix}
\] (30)
and the matrices $N_1$ and $N_2$ are
\[
N_1 = 2(D_{ce} - \alpha M) > 0, \quad N_2 = \frac{2}{\alpha} K_{ce} > 0.
\] (31)
Then we obtain the following relationship.
\[
\lambda_{\text{max}}(Q^{-1}) = \lambda_{\text{max}}(R^T (H_0 + H_G) R)
\]
\[
\leq \lambda_{\text{max}}(R^T R) \lambda_{\text{max}}(H_0 + H_G)
\]
\[
\leq \lambda_{\text{max}}(R^T R) (\lambda_{\text{max}}(H_0) + \lambda_{\text{max}}(H_G))
\] (32)
Therefore, from (26), the upper bound $\gamma_{\text{min}}$ of $\gamma_{\text{min}}$ becomes
\[
\gamma_{\text{min}} = \lambda_{\text{max}}(R^T R) (\lambda_{\text{max}}(H_0) + \lambda_{\text{max}}(H_G)) \lambda_{\text{max}}(SS^T)
\] (33)
It is noted here that
\[
\lambda_{\text{max}}(R^T R) \geq 1, \quad \lambda_{\text{max}}(H_G) \geq 0.
\] (34)
and $\lambda_{\text{max}}(R^T R) = 1$ and $\lambda_{\text{max}}(H_G) = 0$ when $G_{ce} = 0$. Therefore
\[
\gamma_{\text{min}} = \lambda_{\text{max}}(H_0) \lambda_{\text{max}}(SS^T)
\] (35)
is the minimal upper bound.

**Remark 1** For the gyroscopic system (1) with (2), Theorem 2 suggests that the optimal $H_{ce}$ static output feedback controller (6) works to cancel the gyroscopic term $G$ of the plant by making the skew-symmetric part $D_k$ of the feedback gain $D_c$ to satisfy $G_{ce} = G + LD_c L^T = 0$.

**Remark 2** For the non-gyroscopic system (1) with $M > 0$, $D > 0$, $K > 0$ and $G = 0$, Theorem 2 also states that the static output feedback controller (6) is $H_{ce}$ optimal when feedback gains are symmetric and positive definite. This is equivalent to our previous result [8].

**IV. Numerical Studies**

A. Gyroscopic Mass-Spring-Damper Model

Let us consider a gyroscopic three point-mass system connected through springs and dampers, where the displacement vector is $p \in \mathbb{R}^3$ and the control input is $u \in \mathbb{R}^3$. The coefficient matrices are supposed to be
\[
D = \begin{bmatrix}
1 & -1 & 0 \\
-1 & 2 & -1 \\
0 & -1 & 1
\end{bmatrix},
G = \begin{bmatrix}
0 & -1 & 1 \\
1 & 0 & -1 \\
-1 & 1 & 0
\end{bmatrix}
\]
\[
M = L = F = I_3 \quad \text{and} \quad E = I_3.
\]
Then we obtain the following relationship.
\[
Ka = \begin{bmatrix}
1.63 e^1 \\
2.00 e^0 \\
-2.60 e^{-3}
\end{bmatrix},
\]
\[
Db = \begin{bmatrix}
3.72 e^2 \\
-1.71 e^0 \\
-1.76 e^{-2}
\end{bmatrix}
\]
This is a stabilizing controller since $K_c > 0$, $D_e \triangleq \frac{1}{2}(D_c + D_c^T) > 0$ and $D_k = \frac{1}{2}(D_c - D_c^T) = -D_c^T$ as shown in Lemma 1. In fact all closed-loop eigenvalues are negative real. Additionally, since $D_e$ satisfies $G \ell = 0$, the statements in Theorem 2 and Remark 1 are verified.

From a practical viewpoint, however, there remains a problem: how to select a suitable design parameter $\alpha$ and how it relates to the control performance. Therefore we examine the relation between the design parameter $\alpha$ and the achievable $H_{ce}$ norm $\gamma$ based on this numerical example. For the system, the upper bound $\gamma_{\text{min}}$ in (35) becomes
\[
\gamma_{\text{min}} \leq \lambda_{\text{max}}(H_0) \left(1 + \alpha^2\right) \lambda_{\text{max}}(FF^T) + \lambda_{\text{max}}(EE^T)
\]
\[
= (2 + \alpha^2) \lambda_{\text{max}}(H_0)
\]
where from (29) and (31),
\[
\lambda_{\text{max}}(H_0) = \max \left\{ \lambda_{\text{max}}\left(\frac{1}{\alpha^2} N_2^{-1}\right), \lambda_{\text{max}}(N_2^{-1}) \right\}
\]
Therefore we obtain the relationship between $\gamma_{\text{min}}$ and $\alpha$ as
\[
\gamma_{\text{min}} = \max \{ \gamma_1, \gamma_2 \}
\]
where
\[
\gamma_1 = \left(1 + \frac{\alpha}{2}\right) \lambda_{\text{max}}(K_{ce}^{-1})
\]
\[
\gamma_2 = \left(1 + \frac{\alpha}{2}\right) \lambda_{\text{max}}(D_{ce} - \alpha M)^{-1}
\]
Then the optimal $\alpha^*$ minimizing $\gamma_1$ is given as $\alpha^* = \sqrt{2}$ from $d\gamma_1(\alpha^*)/d\alpha = 0$, since $K_{dc}$ does not depend on $\alpha$. In order to investigate the accuracy of estimate $\alpha^*$, we solve the optimal design problem numerically by iteratively changing $\alpha \in [0.01, 10]$, whose result is shown in Fig. 1. It is obvious that $\gamma_1$ is minimum near $\alpha = \sqrt{2}$. It is also observed that $\gamma_1$ is approximately 0.08 when $\alpha = 1$, while the "mincx" result is 0.0554 as already mentioned. The gap between the real performance $\gamma$ and its estimate $\gamma_{min}$ is not so large for this example.

**B. Gyroscopic Spacecraft Model**

As the second example, we consider a three-axis stabilized spacecraft model

$$
\begin{align*}
J_\phi \ddot{\phi} - J_\psi \dot{\phi} \omega_\psi &= u_\phi + w_\phi \\
J_\theta \ddot{\theta} &= u_\theta + w_\theta \\
J_\psi \ddot{\psi} + J_\theta \omega_\psi \dot{\phi} &= u_\psi + w_\psi
\end{align*}
$$

where $\phi$, $\theta$ and $\psi$ are respectively roll, pitch and yaw angles, $J_i$, $u_i$ and $w_i$ ($i = \phi, \theta, \psi$) are moments of inertia, control input torques and disturbance input torques around each axis. The orbital angular velocity is represented with $\omega_0$. It is noted that the angular motions around roll ($\phi$) and yaw ($\psi$) are coupled with each other through the gyroscopic terms including $\omega_0$ while the pitch motion is decoupled from others. Therefore control and disturbance inputs added around roll axis affect the yaw motion and vice versa. Since we are interested in controlling gyroscopic systems, only the roll and yaw coupled motions are described herein by defining the physical coordinate $p$ as $p = [\phi \ psi]^T$. When we select $J_i = 3130$ (kgm$^2$) and $\omega_0 = 1.21 \times 10^{-3}$ (rad/s), we have following coefficient matrices:

$$
M = \begin{bmatrix}
3.13e^3 & 0 \\
0 & 3.13e^3
\end{bmatrix}, \quad G = \begin{bmatrix}
0 & -3.27e^2 \\
3.27e^2 & 0
\end{bmatrix}
$$

and $K = 0, D = 0$. The optimal static output feedback controller design using "mincx" results in $\gamma = 0.387$ and

$$
K_c = \begin{bmatrix}
1.29e^2 & 0 \\
0 & 1.29e^2
\end{bmatrix}, \quad D_c = \begin{bmatrix}
6.43e^2 & 3.27e^2 \\
-3.27e^2 & 6.43e^2
\end{bmatrix}
$$

when $\alpha = 0.01$. It is obvious that $K_c > 0, D_c = \frac{1}{2}(D_c + D_c^T) > 0$ and $D = \frac{1}{2}(D_c - D_c^T)$ cancels the gyroscopic term $G$. This controller is referred to as the asymmetric controller since $D_c$ is not symmetric. For comparison purpose, we consider to design optimal controller under the constraints

$$
K_c = K_c^T > 0, \quad D_c = D_c^T > 0
$$

The optimization problem of LMI (25) with the constraints is also solved using "mincx" again, and we obtain $\gamma = 0.388$ and

$$
K_c = \begin{bmatrix}
1.29e^2 & 0 \\
0 & 1.29e^2
\end{bmatrix}, \quad D_c = \begin{bmatrix}
7.13e^2 & 0 \\
0 & 7.13e^2
\end{bmatrix}
$$

The controller can be referred to as the symmetric controller. Although the values of $\gamma$ of asymmetric and symmetric controllers is similar, some important differences are found in their time responses to disturbance inputs as shown in Fig. 2. In the numerical simulations, we suppose to apply the rectangular disturbance inputs $w_i$ during 0.0625 (s) as follows.

$$
t = 0(s) : \quad w_\phi = -0.3, \quad w_\theta = -0.5, \quad w_\psi = 1.8(Nm)
$$

$$
t = 150(s) : \quad w_\phi = 0.0, \quad w_\theta = 1.2, \quad w_\psi = -1.8(Nm)
$$

In Fig. 2, by comparing two roll angle responses during $t \in [0, 50]$ (s), we find the settling time of the closed-loop system with the asymmetric controller being substantially shorter than that with symmetric controller. Additionally, during the interval $t \in [150, 200]$ (s), the roll angle controlled by the asymmetric controller does not change while that controlled by the symmetric controller varies largely affected by disturbance inputs added around the yaw axis. This is because the asymmetric static output feedback controller works to decouple the roll and yaw motions by cancelling the gyroscopic term.
V. CONCLUSION

In this paper, we have investigated the control problem of collocated linear gyroscopic systems with static output feedback controllers. It has been shown that displacement feedback with positive definite gains and velocity feedback with the positive definite and skew-symmetric gains guarantees the robust stability of the closed-loop system irrespective of the system parameters. Furthermore, we have proposed an $H_\infty$ optimal design method based on the LMI. After some analysis, we have clarified that the skew-symmetric feedback gain minimizes the $H_\infty$ norm by cancelling the gyroscopic term of the plant. Two numerical study results are illustrated to verify the validity and to compare the performance of asymmetric and symmetric controllers for gyroscopic systems.

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