Relationship between Time-Invariant and Time-Variant Filtering Algorithms for a Class of Problems of Navigation Data Processing*

Alexei V. Loparev, Oleg A. Stepanov, and Igor B. Chelpanov

Abstract— The method of local approximation (MLA) of power spectral densities, which is widely used for solving Wiener filtering problems in navigation data processing, is considered. The relationship between a class of problems and the corresponding time-invariant algorithms derived by the MLA with a class of Kalman filtering problems and the corresponding algorithms is revealed and discussed. An illustrative example is given; the practical significance of the results obtained is discussed.

I. INTRODUCTION

The problems of navigation data processing are commonly solved with the use of Kalman filters. Their generally recognized advantage is that estimation problems can be solved for time-variant systems. In this case, the root mean square (RMS) estimation error is minimized for any instant of time, and not only in the steady-state mode [1–4]. However, frequency methods developed for optimal signal estimation from noisy measurements are still relevant [5–7]. In these methods, the signal and measurement errors are assumed to be stationary processes with known stochastic properties. The approach consists in determining the transfer function (TF) of the optimal filter, which minimizes the RMS estimation error in the steady-state mode. Of great importance are approximate methods used to solve this kind of problems. In particular, the method of local approximation (MLA) of power spectral densities (PSD) of the signal and measurement errors was developed specially for navigation applications [8–10]. It is intended for approximate solution of Wiener filtering in the steady-state mode for stationary processes with rational PSDs. The basis for this method is an assumption that the TFs of the filters are determined by the properties of PSDs for the valid signal and measurement errors in the vicinity of their crosspoint. For the description in the logarithmic scale of PSDs, the MLA uses approximating straight lines corresponding to the so-called ‘conditional’ PSDs for the valid signal \( u \) and measurement noise \( v \), described as \( S_u(\omega) \approx A^q \omega^p \), \( S_v(\omega) \approx B^q \omega^p \), with \( p,q = \pm 0,1,2 \).

The advantage of this approximate method is that, in some cases, it allows a researcher to significantly simplify the problems of stationary process filtering and the algorithms used, so that the resulting simplified algorithms possess properties that are beneficial in practice. In particular, the MLA algorithms have the properties of ‘astatism’, which is of importance in practice. Astatism means that the filters allow for exact reproduction of the input signal in the form of polynomials, the degrees of which are the same as the order of astatism [8]. Sometimes, the statement of the filtering problem that is to be solved in the development of navigation devices may be rather obscure. At the same time, it can be formulated as a result of a preliminary analysis of the problem carried out within the framework of the frequency approach with the use of the MLA [10]. Furthermore, the filtering accuracy can be calculated with the use of simple convenient relationships obtained with the MLA. Actually, all these circumstances explain the fact why the MLA is widely used in processing of navigation data.

However, there are some questions to be settled in the case when the MLA is applied within the framework of the Wiener approach. In particular, the algorithms are derived with the use of ‘conditional’ PSDs—the notion that cannot be called rigorous. In this connection, one of the questions still awaits its answer: for what kind of random processes the filters designed by the MLA are optimal. Another question is why the solution of the Wiener filtering problem, which, as is well known, is formulated for stationary zero-mean processes, leads to a TF for the filter possessing the properties of astatism. As regards applied problems, the loss in accuracy of the resulting MLA algorithms as compared with time-variant Kalman filtering algorithms is also significant.

The answers to these questions can be obtained by studying the relationship between the problems and the corresponding time-invariant MLA algorithms with the problems and the corresponding time-variant Kalman filtering algorithms. It is revealing of such a relationship that this paper is devoted to. Also discussed is the practical significance of the results obtained.

The paper is organized as follows. Section 2 gives a summary of the MLA and describes the problems and the algorithms associated with them. Section 3 reveals a class of Kalman filtering problems for nonstationary processes, for which a steady-state solution coincides with the MLA solution. This result is generalized for an \( n \)-dimensional case in Section 4 where TFs and error covariances are derived for the steady-state mode of the Kalman filter (KF). An illustrative example is given. In conclusion, the practical significance of the results obtained is discussed.

II. METHOD OF LOCAL APPROXIMATION OF PSDS

Consider a classical problem of optimal signal estimation from noisy measurements. Let the scalar measurements \( y(t) = u(t) + v(t) \) include a valid signal \( u(t) \) (hereinafter called ‘signal’) and an additive measurement error (noise) \( v(t) \), which are assumed stationary zero-mean random processes

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with specified rational PSDs \( S_u(\omega) \) and \( S_v(\omega) \). It is known that the estimate for signal \( \hat{u}(t) \), optimal by criterion
\[
\sigma^2_e = E[(u(t) - \hat{u}(t))^2] \rightarrow \min,
\]
can be derived at the output of the Wiener filter with the frequency transfer function (hereinafter ‘TF’) [5, 6]
\[
W_q(j\omega) = \frac{1}{S^*_v(\omega)} \left[ \frac{S_x(\omega)}{S^*_y(\omega)} \right]_{+} = \frac{C_1(j\omega)}{C_2(j\omega)},
\]
where \( S^*_v(\omega), S_x(\omega) \) is the result of factorization \( S_v(\omega) = S_u(\omega) + S_y(\omega) \) (decomposition into a product of complex conjugate multipliers, such that \( S^*_v(\omega) \) has poles only in the upper half-plane of the complex plane \( \omega \)). The procedure \( \left[ \right]_+ \) is a separation of the relationship in square brackets, i.e., its representation as a sum of two fractions with the subsequent truncation of the fractions with the poles in the lower half-plane. \( C_1(j\omega), C_2(j\omega) \) are the polynomials of degrees \( m \) and \( n \), with \( m \leq n \). Formula (1) satisfies the case of uncorrelated \( u(t) \) and \( v(t) \).

Describing the MLA, first of all, we should emphasize that it is based on approximate plotting (piecewise approximation) of PSDs in a logarithmic scale in the form of broken lines. In the case of purely imaginary zeros and poles of PSDs, logarithmic curves are plotted as broken lines. Each pair of complex-conjugate roots of a TF gives an upward break of the curve, and each pair of complex-conjugate poles gives a downward break with a 40 dB/decade slope. If there are quadruples of complex roots (poles), they give a break with an 80 dB/decade (with account of root multiplicity). At the breakpoint, the poles give a local peak and the roots—a local dip.

The next simplifying procedure is based on the idea that when more than one process (disturbance) are taken into account, the PSD \( S_d(\omega) \) of the sum of uncorrelated processes can be derived approximately by plotting an upper envelope for the PSDs of components \( S_i(\omega) \), which follows from the relationship
\[
\log S_d(\omega) = \log \sum_i S_i(\omega) = \max_i \log S_i(\omega).
\]

This formula is graphically represented in the form of an upper envelope (dotted line) of the PSD summands, as shown in Fig. 1. Using (2) makes it possible to exclude from consideration the spectral components that do not approach the upper envelope throughout the frequency range, and, consequently, reduce the number of disturbances. And, at last, the key feature of the MLA is the assertion that the TF of the filter is determined by the properties of the PSDs in the vicinity of the crosspoint of spectral densities \( S_u(\omega) \) and \( S_v(\omega) \). This assertion allows using the following approximations for PSDs (Fig. 1):
\[
S_u(\omega) \approx a^2(\rho/\omega)^p, \ S_v(\omega) \approx a^2(\rho/\omega)^q,
\]
where \( a^2 \) and \( \rho \) are the ordinate and abscissa of the crosspoint, respectively, and \( p \) and \( q \) are the powers that take positive, negative, and zero values. In a logarithmic scale, (3) are represented by inclined lines.

For the representations (3) in [8], the authors introduced a notion ‘conditional’ PSDs, which, in a logarithmic scale, take the form of straight lines, the slope of which is determined by \( p \) and \( q \). It is clear that under the assumptions made, the roots of \( S(t) \) can be found by solution of the equation
\[
\omega^{2(p-q)} + \rho^{2(p-q)} = 0.
\]

Hence, it obviously follows that denominator \( C_2(j\omega) \) of the TF determined by using the MLA, will be identical with the Butterworth filter [6]. The roots of the denominator lie on a circle with radius \( \rho \). The coefficients of polynomial \( C_1(j\omega) \) in the TF numerator can be found in the MLA, based on the condition of minimizing the filtering error variance or the condition of finitude of the filtering error variance.

One of the main advantages of the MLA is the fact that the information about the PSDs in relatively narrow frequency ranges is sufficient for obtaining practical results. The results obtained by using the MLA in the solution of the filtering problem for different combinations of simplest approximations of the signal and measurement error PSDs are given in Table 1 below.

![Fig. 1. Summation of PSDs.](image)

<table>
<thead>
<tr>
<th>( S_u(\omega) )</th>
<th>( S_v(\omega) )</th>
<th>( W_u(j\omega) )</th>
<th>( \sigma^2_e )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( a^2(\rho/\omega)^2 )</td>
<td>( a^2 )</td>
<td>( \frac{\rho}{j\omega + \rho} )</td>
<td>( \sqrt{2a^2\rho} )</td>
</tr>
<tr>
<td>( a^2(\rho/\omega)^2 )</td>
<td>( a^2(\rho/\omega)^{-2} )</td>
<td>( \frac{\rho^2}{(j\omega)^2 + \sqrt{2}\rho/\omega + \rho^2} )</td>
<td>( 2a^2\rho )</td>
</tr>
<tr>
<td>( a^2(\rho/\omega)^2 )</td>
<td>( a^2(\rho/\omega)^{-4} )</td>
<td>( \frac{\rho^3}{(j\omega)^3 + 2\rho(j\omega)^2 + 2\rho^2(j\omega + \rho)^2} )</td>
<td>( \sqrt{2a^2\rho} )</td>
</tr>
<tr>
<td>( a^2(\rho/\omega)^4 )</td>
<td>( a^2 )</td>
<td>( \frac{\sqrt{2}\rho/\omega + \rho^2}{(j\omega)^2 + \sqrt{2}\rho/\omega + \rho^2} )</td>
<td>( 2a^2\rho )</td>
</tr>
<tr>
<td>( a^2(\rho/\omega)^4 )</td>
<td>( a^2(\rho/\omega)^{-2} )</td>
<td>( \frac{2\rho^2(j\omega + \rho)^3}{(j\omega)^3 + 2\rho(j\omega)^2 + 2\rho^2(j\omega + \rho)^2} )</td>
<td>( 3a^2\rho )</td>
</tr>
<tr>
<td>( a^2(\rho/\omega)^6 )</td>
<td>( a^2 )</td>
<td>( \frac{2\rho(j\omega)^2 + 2\rho^2(j\omega + \rho)^3}{(j\omega)^3 + 2\rho(j\omega)^2 + 2\rho^2(j\omega + \rho)^2} )</td>
<td>( 2a^2\rho )</td>
</tr>
</tbody>
</table>
We draw your attention to the fact that in the cases where the PSD of the signal is approximated by a straight line with a negative slope, the filters have the property of astatism, its order determined by \( p \), i.e., the PSD slope. Generally speaking, it is a strange feature because the Wiener filtering approach is intended to be applied to solve estimation problems of stationary random zero-mean processes. The reason why these filters have such properties is discussed in the next section.

### III. RELATIONSHIP WITH KALMAN FILTERING PROBLEMS AND ALGORITHMS

Let us reveal the relationship between a class of Wiener filtering problems solved by the MLA and the corresponding algorithms with a class of Kalman filtering problems and the corresponding algorithms. For simplicity, we will do it by an example of solution of filtering problems in which conditional densities of the signal and measurement errors have the form \( S_x(\omega) = a^2 (\omega/\rho)^p \), \( p = 3 \), \( S_v(\omega) = a^2 \), \( q = 0 \), i.e., the errors are the white noise with PSD \( a^2 \). From Table 1 it follows that the TF for the Wiener filter that fits with these spectral representations is determined as

\[
W_0(j\omega) = \frac{2\rho(j\omega)^3 + 2\rho^2 j\omega + \rho^3}{(j\omega)^3 + 2\rho(j\omega)^2 + 2\rho^2 j\omega + \rho^3}. \tag{5}
\]

In order to reveal the above-mentioned relationship, we derive the KF for estimation of the state vector described by the equations

\[
\dot{x}_1 = x_2, \quad \dot{x}_2 = x_3, \quad \dot{x}_3 = w
\]

using measurements

\[
y(t) = x_1(t) + v(t) . \tag{6}
\]

It is assumed that the initial vector \( x(0) = (x_1(0), x_2(0), x_3(0))^T \) is a zero-mean vector with a known covariance matrix. We also assume that the forcing noise and measurement noise are not correlated with each other and with the initial vector, and their PSDs are \( \sigma^2_x \) and \( r^2 \). In other words, we need to derive an estimate for the signal, which is a sum of two nonstationary processes

\[
x_1(t) = x_1(0) + x_2(0)t + x_3(0)t^2 + \int_0^t \int_0^\tau_1 \int_0^\tau_2 w(\tau_3)d\tau_1 d\tau_2 d\tau_3, \tag{8}
\]

where the first summand is a polynomial-type quasideterministic nonstationary process, and the second summand is a nonstationary process in the form of a triple integral of the white noise.

By intuition it is clear that for the problem under consideration, the KF in the steady-state mode is identical with the Wiener filter (5) for ‘conditional’ PSDs. Let us show the validity of this proposition for \( a = r \) and \( \rho = (\sigma^2_x/r^{1/3}) \). We should bear in mind that the TF for the steady-state mode of the KF is determined as

\[
W(\omega) = (j\omega E - F + K_x H)^{-1} K_x , \quad \text{where } H, F, \text{ and } G \text{ are the matrices of observation, dynamics, and forcing noises, respectively, which are easy to determine from (6), (7). In order to find the gain } K_x = \frac{1}{r} P_x H^T , \text{ we need to find a steadystate solution for the covariance equation }
\]

\[
FP_x + P_x F^T + \sigma_g^2 G G^T - \frac{1}{r^2} P_x H^T H P_x = 0 . \text{ It can be shown that for the case under consideration, the covariance matrix } P_x \text{ and the TF, corresponding to the steady-state mode in the filtering problem, have the following forms} [11]:
\]

\[
p_x = r^2 \begin{bmatrix}
\rho^3 & 2\rho^2j\omega & \rho^3 \\
2\rho^2j\omega & 2\rho^3 & 2\rho^3 \\
\rho^3 & 2\rho^3 & 2\rho^3
\end{bmatrix}
\]

\[
W(\omega) = W_g(j\omega) = \begin{bmatrix}
2(j\omega/\rho)^2 + 2j\omega/\rho + 1 \\
2(j\omega/\rho)^2 + 2j\omega/\rho + 1
\end{bmatrix} (j\omega)^3 . \tag{9}
\]

The last formula determines the TF for the third-order Butterworth filter. According to (7), we will take into account the first component of the TF (9), so that the KF for estimation (8) in the steady-state mode becomes identical with the Wiener filter (5). It is clear that these filters are optimal in the steady-state mode for the nonstationary processes (8), which are a sum of the third-order polynomial (quasideterministic component) and triple integrated white noise. It is the presence of a polynomial-type quasideterministic component in the signal being estimated that explains the fact that the resulting filter with the TF (5) has the property of third-order astatism.

It is easy to see that for the whole set of Wiener filtering problems, the results for which are shown in Table 1, one can easily find an appropriate analogy with the problems that follow from models (3). This can be illustrated by an example with conditional PSDs \( S_x(\omega) = a^2 (\omega/\rho)^p \) and \( S_v(\omega) = a^2 \omega^2 \), i.e. \( p = 2, \ q = -1 \). From Table 1 it follows that for this case, \( W_0(j\omega) \) is determined as

\[
W_0(j\omega) = \frac{2(j\omega)^2 + \rho^3}{(j\omega)^2 + 2(j\omega)^2 \rho + 2(j\omega)^2 \rho^2 + \rho^3} . \tag{10}
\]

In fact, this problem corresponds to estimation of the double integral of white noise in its measurements against the derivative of white noise. In operator form, this measurement can be written as

\[
z(t) = \frac{a \rho^2}{D^3} w(t) + \frac{a}{\rho} v(t) . \tag{11}
\]

Here, \( D \) is a differential operator. It is clear that in terms of the classical theory, white noise is not a differentiable process, and this formula can only be considered within the previously introduced notion of ‘conditional’ PSDs. It is also clear that in this case it is impossible to find a direct analogy with the filtering problem formulated within the framework of the Kalman approach. However, we can integrate (11) and provide for new measurements in the form of

\[
y(t) = \frac{a \rho^2}{D^3} w(t) + \frac{a}{\rho} v(t) + \frac{\tilde{a} \rho^3}{D^3} w(t) + \tilde{a} v(t) , \tag{12}
\]

2018
where \( \tilde{a} = \frac{a}{\rho} \). These measurements in the time domain are fully consistent with problem formulated by (6) and (7). It is worth noting that in the original problem, we need to estimate the second integral of white noise. The TF for the optimal KF for measurements (12) in the steady-state mode will be determined by the second component of vector (9), i.e.,
\[
W_o(j\omega) = \frac{(j\omega)^2 + \rho^2}{(j\omega)^3 + 2(j\omega)^2 \rho + 2(j\omega) \rho^2 + \rho^3}.
\]
It is clear that in order to obtain the TF for the initial measurements which are derived by their integration, this function must be multiplied by \(1/j\omega\). By doing this, we have a complete agreement with (10). As for the error variance estimate, it will coincide with the error variance estimate of the second integral of white noise with the use of measurements (12) (\( \sigma^2_j = 3\rho^2 \)) if we take into account that \( r = \tilde{a} = a/\rho \).

Similarly, we can show the correspondence for the case when \( S_o(\omega) = a^2(\rho/\omega)^2 \), \( S_x(\omega) = a^2(\omega/\rho)^4 \), i.e., at \( \rho = 1, q = -2 \).

The results obtained with the MLA for the three-dimensional example considered is a prerequisite for their generalization to an n-dimensional case.

IV. STEADY-STATE SOLUTION FOR AN n-DIMENSIONAL KALMAN FILTERING PROBLEM

Let us consider a Kalman filtering problem described by the following model
\[
\dot{x} = Fx + Gw, \quad y = Hx + v,
\]
where all the elements of matrix \( F \) are zeros with the exception of \( F_{i+1,i} = 1; \ G = (0,0,\ldots,0,1)^T; \ H = (1,0,\ldots,0)^T; \ w, \ v \) are the scalar independent on each other white noises with PSDs \( \sigma^2_z \) and \( \sigma^2_r \). It is obvious that this problem is a general case of the filtering problem of a nonstationary process, which is the sum of a polynomial and an n-multiple integrated white noise. As applied to this problem, we can prove the following confirmation.

Confirmation. The TF of the KF for steady-state solution of the filtering problem (13) is determined as
\[
W(j\omega) = \left( \begin{array}{c}
K_1(j\omega)^{n-1} + K_2(j\omega)^{n-2} + \ldots + K_n \\
(j\omega)^n + K_1(j\omega)^{n-1} + K_2(j\omega)^{n-2} + \ldots + K_n \\
(j\omega)^n + K_1(j\omega)^{n-1} + K_2(j\omega)^{n-2} + \ldots + K_n \\
\ldots
\end{array} \right) / (j\omega)^n + K_1(j\omega)^{n-1} + K_2(j\omega)^{n-2} + \ldots + K_n,
\]
where \( K_i \) are such that
\[
\left( (j\omega)^n + K_1(j\omega)^{n-1} + K_2(j\omega)^{n-2} + \ldots + K_n \right)^2 = \omega^2n + K^2.
\]

It is easy to see that the solutions of Wiener filtering problems solved with the use of ‘conditional’ PSDs and the MLA presented in Table 2 will coincide with the steady-state solutions of the corresponding Kalman filtering problems.

It can also be shown that coefficients \( C_i \) for the estimation error covariances in the last column of Table II are determined according to Table III.

TABLE II. MLA SOLUTIONS

<table>
<thead>
<tr>
<th>( S_o(\omega) )</th>
<th>( S_x(\omega) )</th>
<th>( W(j\omega) )</th>
<th>( \sigma^2 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( a^2 )</td>
<td>( a^2 )</td>
<td>( W_1(j\omega) )</td>
<td>( C_1a^2\rho^2 )</td>
</tr>
<tr>
<td>( a^2 )</td>
<td>( a^2 )</td>
<td>( W_2(j\omega) )</td>
<td>( C_2a^2\rho^2 )</td>
</tr>
<tr>
<td>( a^2 )</td>
<td>( a^2 )</td>
<td>( W_3(j\omega) )</td>
<td>( C_3a^2\rho^2 )</td>
</tr>
</tbody>
</table>

Note. \( W_k(j\omega) \), \( k = 1, \ldots, n \) are the elements of \( W(j\omega) \) in Table II.

TABLE III. COEFFICIENTS FOR ERROR VARIANCES

<table>
<thead>
<tr>
<th>( C_1 )</th>
<th>( C_2 )</th>
<th>( C_3 )</th>
<th>( C_4 )</th>
<th>\ldots</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \frac{2\pi}{n} )</td>
<td>( \frac{\pi}{2n} )</td>
<td>( \cos \frac{2\pi}{n} )</td>
<td>( \cos \frac{2\pi}{n} )</td>
<td>\ldots</td>
</tr>
<tr>
<td>( \sin \frac{\pi}{2n} )</td>
<td>( \sin \frac{3\pi}{2n} )</td>
<td>( 2n )</td>
<td>( 2n )</td>
<td>\ldots</td>
</tr>
<tr>
<td>( \sin \frac{\pi}{2n} )</td>
<td>( \sin \frac{3\pi}{2n} )</td>
<td>( 2n )</td>
<td>( 2n )</td>
<td>\ldots</td>
</tr>
</tbody>
</table>

V. SYNTHESIS OF FILTERING ALGORITHMS FOR ESTIMATION OF GRAVITY ANOMALY

Let us consider a problem of gravity anomaly (GA) estimation by using gravimeter measurements and measurements of the altitude and the vertical velocity of a vehicle obtained from the satellite navigation system (SNS) [11]. These measurements can be represented as
\[
\begin{align*}
\hat{h}^{SNS} &= h + \delta h^{SNS}, \\
V_z^{SNS} &= V_z + \delta V_z^{SNS}, \\
\hat{g} &= \bar{g} + \delta g ,
\end{align*}
\]
where \( \bar{g} \) is the GA; \( \delta g \) are the gravimeter errors; \( h, V_z, \bar{V}_z \) are the altitude, vertical velocity and acceleration; \( \delta h, \delta V_z \) are the errors in the measurements from the SNS. For the altitude errors \( \Delta h^G \) and the vertical velocity errors \( \Delta V_z^G \) determined from the gravimeter data, it is possible to write
\[
\begin{align*}
\Delta h^G &= \Delta h^{SNS}, \\
\Delta V_z^G &= \bar{V}_z + \delta g ,
\end{align*}
\]

2019
Forming the difference measurements between the first and second integrals from the gravimeter data (18) and measurements (15), (16), we can write

\[ y^h = \Delta h^G - \delta h^{\text{SNS}} , \]  
\[ y^V = \Delta V^G - \delta V^{\text{SNS}} . \]  

Taking into account Equations (18) – (20) and specifying the stochastic models for \( \tilde{g} \), \( \delta h \), \( \delta V \), and \( \delta g \), we can formulate the statistical estimation problem. We assume the measurement errors of the gravimeter, coordinates and velocities to be independent of each other white noises. For the GA model as a function \( \tilde{g}(t) \) of the length along a straight track, we have chosen a widely used Jordan model [12]. The PSD and the correlation function for this model are determined as

\[ K_\tilde{g}(l) = \sigma_{\tilde{g}}^2 \left( 1 + \alpha l - \frac{(\alpha l)^2}{2} \right) e^{-\alpha l} , \]  
\[ S_\tilde{g}(\omega) = 2\omega^2 \sigma_{\tilde{g}}^2 \left( \frac{5}{6} \omega^2 + \alpha^2 \right) , \]  

where \( \sigma_{\tilde{g}} \) is the RMS of the GA; \( \alpha \) is the value reverse to the correlation interval; \( l \) is the length along the straight track for the case when the vehicle is moving at a constant speed \( V \); \( \omega \) is the analog of circular frequency for the process which depends on the length of the straight track \( (l = V t) \). It is not difficult to show that this process can be described by using the following third-order shaping filter

\[ \begin{align*} \tilde{g}_1 &= -\beta g_1 + \tilde{g}_2, \\ \tilde{g}_2 &= -\beta \tilde{g}_2 + \tilde{g}_3, \\ \tilde{g}_3 &= -\beta \tilde{g}_3 + w, \end{align*} \]  

where \( \tilde{g} = \tilde{g}_1 \tilde{g}_2 \tilde{g}_3 \beta = V \alpha \); \( w \) is the generating noise with PSD \( \sigma_w^2 = 10 \beta^3 \sigma_{\tilde{g}}^2 \).

Under the assumptions made, the problem reduces to estimation of five-dimensional vector \( x \) with components \( x = (\Delta h, \Delta V, g_1, g_2, g_3) ^T \). Here, the first two components are described by (18), and the last three by (23). For the sake of simplicity, neglecting the gravimeter errors (which is admissible at the present level of their accuracy), it is easy to formulate a filtering problem similar to (13) with the following matrices

\[ H = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad R = \begin{bmatrix} R_{\delta h} & 0 \\ 0 & R_{\delta V} \end{bmatrix}, \quad Q = \sigma_w^2. \]  

Bearing in mind the results obtained and implementing the ideas of the MLA, we can simplify this problem to obtain simple approximate relationships which allow studying the accuracy dependence on the parameters that determine the variability of the GA, speed and measurement precision. Moreover, it is possible to suggest an effective algorithm which is insensitive to changes in the GA model.

Let us illustrate this, assuming that \( V=50 \text{m/s} \), \( R_{\delta V} = (0.01 \text{m/s})^2 \), \( R_{\delta h} = (0.005 \text{m})^2 \). In the calculations, the RMS values for GA \( \sigma_{\tilde{g}} = 30 \text{mGal} \) and its derivative \( \sigma_{\tilde{g}/\omega} \) were 3, 5, and 10 mGal/km, respectively. Here, we should take into account that

\[ \sigma_{\tilde{g}/\omega}^2 = -\frac{d^2}{dl^2} K_\tilde{g}(l) \biggr|_{l=0} = 2\alpha^2 \sigma_{\tilde{g}}^2. \]  

According to the MLA, the filter TF is determined by the crosspoint of the PSDs of the signal and noise, and their inclination at this point. The analysis of (22) shows that for the accepted conditions for the signal PSD in the vicinity of its intersection with \( \omega^2 R_{\delta V} \), \( \omega^4 R_{\delta h} \) (conditional PSDs for the velocity and altitude errors) at \( \omega >> \alpha \), we can use the following approximation

\[ S_\tilde{g}(\omega) \approx \frac{q}{\alpha^4}. \]  

This approximation is a description of GA as the second integral of white noise with PSD \( \sigma_w^2 \). This case is depicted in Fig. 2.

![Fig. 2. PSDs of the signal and the measurement errors in the problem of GA estimation.](image-url)
The analysis of the plots in Fig. 2 allows the following conclusions. At the present-day levels of accuracy of velocity measurements, their application in addition to altitude measurements does not lead to increase in estimation accuracy of GA. In this connection, only coordinate measurements can be taken into account without loss of accuracy in estimation of anomalies. Also, instead of model (24), (25), we can use a simpler model which coincides with (13) at \( n = 4 \), i.e., at

\[
F = \begin{bmatrix}
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0
\end{bmatrix}, \quad G = \begin{bmatrix}
0 \\
0 \\
0 \\
1
\end{bmatrix}, \quad H^T = \begin{bmatrix}
1 & 0 & 0 & 0
\end{bmatrix}. \tag{28}
\]

The results of the previous section can be used to obtain simple expressions that allow for analysis of the steady-state accuracy not only for GA, but also for the vertical velocity and altitude. In this case, the final formula will only depend on \( R \), which characterizes the altitude measurement accuracy, the GA variation determined by the correlation interval \( \alpha \) calculated by (27), and speed \( V \).

Moreover, the studies have shown that the algorithm adjusted to model (28) differs insignificantly in its accuracy from the algorithm adjusted to model (25), and it is robust to changes in the GA model themselves.

It should be noted that the duration of a transient process is of vital importance in the problem under consideration, therefore, it is necessary that the time-variant KF adjusted to model (28) be used for implementation of the algorithm. This is explained by the fact that using the time-invariant MLA filter makes the transient process rather long.

VI. CONCLUSION AND DISCUSSION

The paper has revealed the relationship between a class of Wiener filtering problems and the MLA algorithms used to solve these problems with the corresponding Kalman filtering problems and algorithms.

The results derived are useful for the following reasons.

Firstly, we have answered the question—for what kind of processes the filters designed with the use of the notion of 'conditional' PSDs are optimal in the steady-state mode. They are nonstationary processes, which are a sum of a polynomial-type quasideterministic component and multiple integrated white noise. It is the presence of a polynomial-type quasideterministic component in the signal being estimated that explains the fact that the resulting TFs possess the property of astatism.

Secondly, it has been made possible to remove a certain lack of rigor in designing MLA filters, which is caused by the use of the notion of 'conditional' PSDs.

The practical importance of the revealed relationship is that we can avoid the main drawback of the MLA algorithms: prolonged transient process. This is made possible owing to the KF which corresponds to the appropriate nonstationary processes mentioned above.

It is also worth noting that the revealed relationship can be interpreted as a certain generalization of the MLA for a class of Kalman filtering problems (13). The essence of this interpretation is to replace a certain original model of the filtering problem with a model for estimation of multiple integrated white noise.

The results obtained are of interest in themselves, because they have allowed analytical expressions for TFs and error variances for a particular filtering problem which is frequently dealt with in navigation data processing.

In the future, the results obtained can be generalized for smoothing problems [13]. Also, we should continue searching for another class of problems for which there is a relationship between the MLA algorithms and the corresponding Kalman filtering algorithms.

REFERENCES


