Wellposedness and stabilization of a class of infinite dimensional bilinear control systems

Jamal Daafouz, Marius Tucsnak and Julie Valein

Abstract—We consider a class of infinite dimensional systems involving a control function $u$ taking values in $[0,1]$. This class contains, in particular, the average models of some infinite dimensional switched systems. We prove that the system is wellposed and obtain some regularity properties. Moreover, when $u$ is given in an appropriate feedback form and the system satisfies appropriate observability assumptions, we show that the system is weakly stable. The main example concerns the analysis and stabilization of a model of Boost converter connected to a load via a transmission line. The main novelty consists in the fact that we give a rigorous wellposedness and stability analysis of coupled systems, in the presence of duty cycles.

I. INTRODUCTION AND MAIN RESULTS

Infinite dimensional bilinear control systems occur in a large number of applications. We mention here only the control of elastic structures (see [1]), quantum control (see [3]) or averaged models of Boost converters, which are electrical circuits controlled by switches (transistors, diodes) [6]. The main novelty brought in by this work is that, instead of studying “case by case” the systems corresponding to various settings, we introduce here a class of systems, containing several situations of interest, for which we are able to prove wellposedness and stabilization results.

To describe the class of considered systems, we consider the Hilbert space $X$, endowed with the inner product $\langle \cdot, \cdot \rangle$ and the corresponding norm $\| \cdot \|$ and a m-dissipative operator $A : D(A) \to X$. Recall that an operator $A : D(A) \to X$ is said $m$-dissipative if it is dissipative (i.e. $\langle AZ, Z \rangle \leq 0$ for any $Z \in D(A)$) and satisfies $\text{Ran}(\lambda I - A) = X$ for some $\lambda > 0$. Moreover, let $P \in L(X)$. The general form of the system considered in this work is

$$\dot{Z}(t) = AZ(t) + u(t)PZ(t) + F,$$

where $F \in X$.

Our first results says that the system (1) is well-posed, in the following sense:

Proposition 1. Assume that $Z_0 \in X$ and $u \in L^1_{\text{loc}}[0,\infty)$. Then (1) admits a unique solution $Z \in C([0,\infty), X)$.

Moreover, given $\tau > 0$, there exists $K_\tau > 0$ such that

$$\|Z\|_{C([0,\tau], X)} \leq K_\tau (\|Z_0\| + \|u\|_{W^{1,1}[0,\tau]} + \|F\|)$$

($Z_0 \in X$). (2)

Finally, if $Z_0 \in D(A)$ and $u \in W^{1,1}_{\text{loc}}(0,\infty)$, the solution $Z$ of (1) satisfies

$$Z \in C([0,\infty), D(A)) \cap C^1([0,\infty), X).$$

The main aim of this work consists in giving sufficient conditions allowing the stabilization of the system (1) around an equilibrium state $Z^*$. Recall that $Z^* \in D(A)$ is said an equilibrium state of (1) if

$$AZ^* + \rho^* PZ^* + F = 0,$$

for some $\rho^* \in (0,1)$. If $Z^*$ is as above a natural question consists in finding the control $u$ in (1) in order to have

$$\lim_{t \to \infty} Z(t) = Z^*,$$

in an appropriate sense. To obtain a good candidate for the feedback law, we assume that $P$ is a non-positive operator and, after a simple calculation we see that if $Z$ is a smooth enough solution of (1) then

$$\frac{1}{2} \frac{d}{dt} \|Z(t) - Z^*\|^2 \leq (u(t) - \rho^*) \langle PZ(t), Z(t) - Z^* \rangle \quad (t \geq 0).$$

The above formula suggests, taking in consideration the fact that the control should satisfy

$$u(t) \in [0,1] \quad (t \geq 0 \text{ a.e.}),$$

the following “saturated” feedback law

$$u(t) := \begin{cases} 
\rho^* - \langle PZ(t), Z(t) - Z^* \rangle & \text{if } \rho^* - \langle PZ(t), Z(t) - Z^* \rangle \in [0,1] \\
0 & \text{if } \rho^* - \langle PZ(t), Z(t) - Z^* \rangle < 0 \\
1 & \text{if } \rho^* - \langle PZ(t), Z(t) - Z^* \rangle > 1.
\end{cases}$$

Theorem 2. For every $Z_0 \in X$ the system (1), with $u$ given by the feedback law (5) admits a unique solution $Z \in C([0,\infty), X)$.

We next study the wellposedness of the closed loop problem obtained by taking $u$ as in (5).

Theorem 3. For every $Z_0 \in X$ the system (1), with $u$ given by the feedback law (5) admits a unique solution $Z \in C([0,\infty), X)$.

To state properly our stability result, we set, for $t > 0$, $S(t)Z_0 = Z(t)$ where $Z$ is the solution of (1) with $u$ given by the feedback law (5). We say that $S(t)$ is sequentially weakly continuous on $X$ if $Z_{0n} \to Z_\infty$ in $X$ weakly implies
$S(t)Z_0 \to S(t)Z_\infty$ in $X$ weakly. Moreover, for $Z \in X$, we set

$$g(Z) = |\langle A(Z - Z^*), Z - Z^* \rangle| + |u - \rho^*| |\langle PZ, Z - Z^* \rangle|.$$  \hfill (6)

Finally we show that under the action of the feedback law (5) the state of the system converges to the admissible reference state, in a weak sense.

**Theorem 3.** Assume that

$$\forall t > 0, S(t) \text{ is sequentially weakly continuous on } X,$$  \hfill (7)

and

$$S(t)Z_0 \to S(t)Z_\infty \text{ weakly in } X$$

$$\Rightarrow \lim_{n \to +\infty} \int_0^T g(S(\tau)Z_{0n})d\tau \geq \int_0^T g(S(\tau)Z_0)d\tau,$$ \hfill (8)

and

$$g(Z(t)) = 0 \text{ in } [0, T] \text{ where } Z \text{ solution of } (1), (5)$$

$$\Rightarrow Z(t) = Z^* \text{ in } [0, T].$$ \hfill (9)

Then, under the assumptions of Theorem 2, the solution $Z(t)$ of (1), with $u$ given by (5), converges weakly in $X$ to $Z^*$ when $t \to \infty$.

This kind of stability question has been investigated, for instance, in [2] for the stabilization to zero without saturation, in [8] for the stabilization of a linear control system with an a priori bounded control. We refer also to [4] for the strong stabilization of controlled vibrating systems.

The main application of these results concerns an averaged model of Boost converter coupled to a transmission line. The main novelty we bring for this application consists in the fact that we give a rigorous approach to the analysis and the control design for this system. The model we consider leads to an infinite dimensional control system since the power converter is connected to the load via a transmission line (which is modeled by a partial differential equation). This coupling has already been considered in [9], where a Lyapunov function approach has been used to design a hybrid stabilizing control scheme, by assuming that solutions exist, that they are smooth enough and that appropriate observability results hold. A rigorous mathematical analysis of the model considered in [9] would require proving the wellposedness and the stability in a hybrid infinite dimensional setting.

The mathematical analysis of the corresponding averaged model has been first given in Daafouz, Tucsnak and Valein [5]. The advantage of the approach in the present works lies in a simplification of some of the proofs and a better understanding of the underlying dynamics.

The outline of this paper is as follows. In Section II, we give the sketch of the proofs of this paper. In Section III we apply these results of the case of Boost converters connected to transmission lines.

II. Sketch of the proofs

The proof of Proposition 1 is classical (see for instance [7]), and then, for shortness, is left to the readers.

**Proof:** [Proof of Theorem 2] We first show the local in time existence and uniqueness of solutions. More precisely let $R > 0$. We prove that there exists $T_{\text{max}} > 0$, depending only on $R$, such that, for every $Z_0 \in X$ such that $\|Z_0\| \leq R$ and every $T \in [0, T_{\text{max}})$, there exists a unique solution $Z \in C([0, T], X)$ of (1) with $u(t)$ given by (5). For that we use a fixed point argument.

For every $Z \in C([0, T], X)$, we denote

$$\tilde{u}(Z(t)) := \begin{cases} 
\rho^* - \langle PZ(t), Z(t) - Z^* \rangle & \text{if } \rho^* - \langle PZ(t), Z(t) - Z^* \rangle \in [0, 1] \\
0 & \text{if } \rho^* - \langle PZ(t), Z(t) - Z^* \rangle < 0 \\
1 & \text{if } \rho^* - \langle PZ(t), Z(t) - Z^* \rangle > 1.
\end{cases} \hfill (10)

Note that the function $t \mapsto \tilde{u}(Z(t))$ is continuous. With the above notation, according to Proposition 1, the initial value problem

$$\dot{\tilde{Z}} = A\tilde{Z}(t) + \tilde{u}(Z(t))P\tilde{Z}(t) + F, \quad \tilde{Z}(0) = Z_0,$$ \hfill (11)

admits, for each $T > 0$, a unique solution $\tilde{Z} \in C([0, T], X)$. We can thus define, for each $T > 0$ the map $G_T : C([0, T], X) \to C([0, T], X)$ by

$$G_T(Z) = \tilde{Z} \quad (Z \in C([0, T], X)),$$

where $\tilde{Z}$ is the solution of (11) with $\tilde{u}$ given by (10). In order to obtain the local existence, we show that, for each $R > 0$, there exists $T_{\text{max}} > 0$ such that, for every $T \in (0, T_{\text{max}})$, the nonlinear operator $G_T$ is a strict contraction of $B_{2R} = \{ Z \in C([0, T], X) \mid \|Z\|_{C([0,T],X)} \leq 2R \}$.

First we show that $G_T$ is mapping from $B_{2R}$ into itself. Using the facts that $\langle A\tilde{Z}, \tilde{Z} \rangle \leq 0$, $\langle P\tilde{Z}, \tilde{Z} \rangle \leq 0$ and $\tilde{u}(Z) \geq 0$, we can easily verify that

$$\|\tilde{Z}(t)\|^2 \leq \|Z_0\|^2 + 2 \int_0^T \langle F, \tilde{Z}(t) \rangle dt$$

$$\leq \|Z_0\|^2 + T \|F\|^2 + \int_0^T \|\tilde{Z}(t)\|^2 dt.$$

This implies, using the Gronwall Lemma, that for every $t \in [0, T]$ we have

$$\|\tilde{Z}(t)\|^2 \leq (\|Z_0\|^2 + T \|F\|^2) e^T.$$  \hfill (12)

Since $\|Z_0\| \leq R$, we can clearly choose $T_{\text{max}} > 0$ small enough such that, for every $T \in (0, T_{\text{max}})$, we have

$$\left(\|Z_0\|^2 + T \|F\|^2\right) e^T \leq 4R^2,$$

which shows that indeed, for every $T \in (0, T_{\text{max}})$, $G_T$ invariates $B_{2R}$.

We next prove that, for $T$ small enough, the mapping $G_T$ is a strict contraction on $B_{2R}$. Let $Z_1, Z_2 \in B_{2R}$ and denote
\[ \tilde{Z}_1 = G_T(Z_1), \quad \tilde{Z}_2 = G_T(Z_2). \] We set \( Z = Z_1 - Z_2 \) and \( \tilde{Z} = G_T(Z_1) - G_T(Z_2). \) Then \( \tilde{Z} \) satisfies

\[
\dot{\tilde{Z}}(t) = A\tilde{Z}(t) + \overline{u}(Z(t))P\tilde{Z}(t) + (\overline{u}(Z_1(t)) - \overline{u}(Z_2(t)))P\tilde{Z}_2(t)
\]

with initial data equal to zero. Using the facts that \( \langle A\tilde{Z}, \tilde{Z} \rangle \leq 0, \) \( \langle P\tilde{Z}, \tilde{Z} \rangle \leq 0, \) \( \overline{u}(Z_1) \geq 0 \) and that the initial data are equal to zero, we can easily verify that

\[
\left\| \tilde{Z}(t) \right\|^2 \leq 2 \int_0^t |\overline{u}(Z_1(s)) - \overline{u}(Z_2(s))| \cdot \left( \langle P\tilde{Z}_2(s), \tilde{Z}(s) \rangle \right) ds.
\]

Since the map \( Z \mapsto \langle PZ, Z - Z^* \rangle \) is locally Lipschitz, we can easily verify that \( \overline{u} \) is a Lipschitz map, i.e. that there exists \( \alpha > 0 \) such that for any \( Z_1 \in X \) and \( Z_2 \in X, \) we have

\[
|\overline{u}(Z_1) - \overline{u}(Z_2)| \leq \alpha \left\| Z_1 - Z_2 \right\|.
\]

This can be obtained by truncating the smooth function \( Z \mapsto \rho^* - \langle PZ, Z - Z^* \rangle. \) Therefore, we have

\[
\left\| \tilde{Z}(t) \right\|^2 \leq 2\alpha \int_0^t \left\| Z_1(s) - Z_2(s) \right\| \cdot \left( \langle P\tilde{Z}_2(s), \tilde{Z}(s) \rangle \right) ds
\]

\[
\leq \alpha R \left\| P \right\| \int_0^t \left( \left\| Z \right\|^2_\infty + \left\| \tilde{Z}(s) \right\|^2 \right) ds
\]

\[
\leq \alpha R \left\| P \right\| \left\| Z \right\|^2_\infty T + \alpha R \left\| P \right\| \int_0^t \left\| \tilde{Z}(s) \right\|^2 ds.
\]

Gronwall lemma yields to

\[
\left\| \tilde{Z}(t) \right\|^2 \leq \alpha R \left\| P \right\| T e^{\alpha R \left\| P \right\| T} \left\| Z \right\|^2_\infty.
\]

We want that

\[
\alpha R \left\| P \right\| T e^{\alpha R \left\| P \right\| T} < 1
\]

together with (12), which is possible by taking \( T \) small enough.

The global existence follows from the fact that \( \left\| Z(t) \right\| \) does not blow up in finite time.

**Proof:** [Proof of Theorem 3] We assume that \( Z \) is the solution of (1) with initial state \( Z_0 \in X \) and \( u \) given by (5). We know that

\[
\left\| Z(t) - Z^* \right\| \leq \left\| Z_0 - Z^* \right\| \quad (t > 0),
\]

so that there exists \( C > 0 \) such that

\[
\left\| Z(t) \right\| \leq C \quad (t > 0).
\]

Therefore there exist a sequence \( t_n \to +\infty (n \to +\infty) \) and \( Z_{\infty} \in X \) such that \( Z(t_n) = S(t_n)Z_0 \to Z_{\infty} \) in \( X \) weakly.

The aim is to show that \( Z_{\infty} = Z^*. \)

We can easily verify that

\[
\left\| Z(t) - Z^* \right\| \leq \left\| Z_0 - Z^* \right\| + 2 \int_0^t g(Z(\sigma))d\sigma,
\]

where \( g \) is defined by (6), and then

\[
\int_0^T g(Z(\sigma))d\sigma \text{ converges as } t \to \infty.
\]

Using the fact that \( Z(\tau + t_n) = S(\tau + t_n)Z_0 = S(\tau)S(t_n)Z_0, \) we have

\[
0 = \lim_{n \to \infty} \int_{t_n}^{T+t_n} g(Z(t))dt = \lim_{n \to \infty} \int_0^T g(S(\tau)S(t_n)Z_0)dt.
\]

We have seen at the beginning of the proof that \( Z(t_n) = S(t_n)Z_0 \to Z_{\infty} \) weakly in \( X. \) Therefore, by assumptions (7) and (8),

\[
0 = \lim_{n \to \infty} \int_{t_n}^{T+t_n} g(Z(t))dt \geq \int_0^T g(S(\tau)Z_{\infty})d\tau \geq 0.
\]

Consequently we have

\[
\int_0^T g(S(\tau)Z_{\infty})d\tau = 0,
\]

i.e.

\[
g(S(\tau)Z_{\infty}) = 0 \quad (\tau \in [0, T]).
\]

Using assumption (9), we obtain

\[
S(\tau)Z_{\infty} = Z^* \quad (\tau \in [0, T]).
\]

Therefore \( Z(\tau + t_n) = S(\tau)S(t_n)Z_0 \to S(\tau)Z_{\infty} = Z^* \) weakly for every \( \tau > 0, \) which concludes the proof.

**III. APPLICATION TO BOOST CONVERTERS CONNECTED TO TRANSMISSION LINES**

The system of coupled partial and ordinary differential equations which will be studied in this section describes the coupling of a Boost converter and a transmission line which is described in Figure 1.
that the control to be designed is not the switching signal corresponding to the position of the switch in the circuit but the so called duty cycle, which is the average of this switching signal over a short time interval. Typically, this time interval is the switching period. The current \( I \) and the voltage \( V \) in the transmission line satisfy the telegraph equations
\[
\begin{align*}
\frac{\partial I}{\partial t} (x,t) &= - L_I^{-1} \frac{\partial V}{\partial x} (x,t) \quad (x \in (0,1), t \geq 0), \quad (15) \\
\frac{\partial V}{\partial t} (x,t) &= - C_I^{-1} \frac{\partial I}{\partial x} (x,t) \quad (x \in (0,1), t \geq 0), \quad (16)
\end{align*}
\]
\( V(0,t) = w(t), \ V(1,t) = R_L I(1,t) \quad (t \geq 0). \quad (17) \]
The system which is studied in this section is formed by equations (13)-(17), together with the initial conditions
\[
\begin{align*}
z(0) &= z_0, \ w(0) = w_0, \quad (18) \\
I(x,0) &= I_0(x), \ V(x,0) = V_0(x) \quad (x \in (0,1)). \quad (19)
\end{align*}
\]
In order to rewrite (13)-(19) in the abstract form (1) we set \( Z \) and \( Z_0 \) the state and the initial state of the system as
\[
Z = \begin{bmatrix}
z \\
w \\
I \\
V
\end{bmatrix}, \quad Z_0 = \begin{bmatrix}
z_0 \\
w_0 \\
I_0 \\
V_0
\end{bmatrix}
\]
and the Hilbert space \( X \) as
\[
X = \mathbb{R} \times \mathbb{R} \times L^2[0,1] \times L^2[0,1].
\]
We endow \( X \) with the norm (equivalent to the usual one)
\[
\|Z\|^2 = L|z|^2 + C|w|^2 + L_I \|I\|^2_{L^2[0,1]} + C_I \|V\|^2_{L^2[0,1]}.
\]
We define the operators \( A : \mathcal{D}(A) \to X, P : X \to X \) and \( F \in X \) by
\[
AZ = \begin{bmatrix}
0 \\
- \frac{I_t(0)}{L_I} \\
- \frac{1}{C_I} \frac{dV}{dx} \\
- \frac{1}{C_I} \frac{dI}{dx}
\end{bmatrix}, \quad PZ = \begin{bmatrix}
z \\
- \frac{V}{I}
\end{bmatrix}, \quad F = \begin{bmatrix}
e \\
0
\end{bmatrix},
\]
with
\[
\mathcal{D}(A) = \{ Z \in X \mid I, V \in H^1(0,1), \ V(0) = w, V(1) = R_L I(1) \},
\]
where \( H^1(0,1) \) stands, as usual practice, for the space of absolutely continuous functions on the interval \((0,1)\) whose derivative is in \( L^2[0,1] \).

Clearly \( P \in \mathcal{L}(X) \) and
\[
\langle PZ, Z \rangle = - wz + zw = 0.
\]
Moreover we have the following lemma.

**Lemma 4.** The operator \( A \) is \( m \)-dissipative.

**Proof:** To do that, we first note that for every \( Z \in \mathcal{D}(A) \) we have
\[
\langle AZ, Z \rangle = - (I(0)w - \frac{1}{2}) \left( \frac{dV}{dx} I + \frac{dI}{dx} V \right) dx.
\]
Using next the boundary condition (17) it easily follows
\[
\langle AZ, Z \rangle = - R_L \|I(1)\|^2 \leq 0,
\]
so that \( A \) is dissipative.

We next show that \( \lambda - A \) is onto for every \( \lambda > 0 \). Given \( (f_1, f_2, g_1, g_2)^t \in X \), we seek \( Z = (z, w, I, V)^t \in \mathcal{D}(A) \) solution of
\[
(\lambda I - A) \begin{bmatrix}
z \\
w \\
I \\
V
\end{bmatrix} = \begin{bmatrix}
f_1 \\
f_2 \\
g_1 \\
g_2
\end{bmatrix}.
\]
The above system writes
\[
\begin{align*}
\lambda z &= f_1, \\
\lambda w + \frac{I_t(0)}{L_I} &= f_2, \\
\lambda I + \frac{1}{L_I} \frac{dV}{dx} &= g_1, \\
\lambda V + \frac{1}{C_I} \frac{dI}{dx} &= g_2,
\end{align*}
\]
which leads to
\[
\begin{align*}
z &= \frac{f_1}{\lambda}, \\
w &= \frac{1}{\lambda} \left( f_2 - \frac{I_t(0)}{C_I} \right).
\end{align*}
\]
We have to solve the differential equations
\[
\begin{align*}
\lambda I + \frac{1}{L_I} \frac{dV}{dx} &= g_1, \\
\lambda V + \frac{1}{C_I} \frac{dI}{dx} &= g_2,
\end{align*}
\]
with the conditions \( V(0) = w \) and \( V(1) = R_L I(1) \). Differentiating the first equation, substituting \( \frac{dI}{dx} \) from the second one and using the fact that
\[
I = \frac{1}{\lambda} \left( g_1 - \frac{1}{L_I} \frac{dV}{dx} \right)
\]
lead to
\[
\begin{align*}
\lambda^2 V - \frac{1}{L_I C_I} \frac{d^2 V}{dx^2} &= \lambda g_2 - \frac{1}{C_I} \frac{dV}{dx} \in H^{-1}(0,1) \\
V(0) &= \frac{1}{L_I C_I} \frac{dV}{dx}(0) = \frac{1}{\lambda} \left( f_2 - \frac{I_t(0)}{C_I} \right) \\
V(1) + \frac{R_L}{L_I} \frac{dV}{dx}(1) &= 2 \frac{V}{I} g_1(1),
\end{align*}
\]
which admits a unique solution \( V \in H^1(0,1) \). Then \( I \) is given by (26), and \( w \) and \( z \) are given by (25) and (24). We clearly obtain that \( Z = (z, w, I, V)^t \in \mathcal{D}(A) \) is solution of (23), which finishes the proof of this lemma.

Consequently, applying Proposition 1, we obtain the following existence result.

**Proposition 5.** Assume that \( Z_0 \in X \) and \( u \in L^1_{\text{loc}}(0,\infty) \). Then (13)-(19) admits a unique solution
\[
Z \in C((0,\infty), X).
\]

Finally, if \( Z_0 \in \mathcal{D}(A) \) and \( u \in W_{\text{loc}}^{1,1}(0,\infty) \), the solution \( Z \) of (13)-(19) satisfies
\[
Z \in C((0,\infty), \mathcal{D}(A)) \cap C^1((0,\infty), X).
\]

Our main result concerns the design of a feedback control for (13)-(19), which steers the system to an equilibrium state.
Some simple calculations show that the equilibrium state \( Z^\ast \) of (13)-(19), which satisfies (4), is
\[
Z^\ast = \begin{bmatrix} z^\ast \\ w^\ast \\ I^\ast \\ V^\ast \end{bmatrix} = \begin{bmatrix} E \\ \rho^\ast R_L \\ E \\ \rho^\ast R_L \end{bmatrix},
\] (27)
for a given \( \rho^\ast \in (0,1) \).

According to (5), the saturated feedback that we consider is
\[
u(t) := \begin{cases} \rho^\ast + w^\ast z(t) - z^\ast w(t) & \text{if } \rho^\ast + w^\ast z(t) - z^\ast w(t) \in [0, 1] \\ 0 & \text{if } \rho^\ast + w^\ast z(t) - z^\ast w(t) < 0 \\ 1 & \text{if } \rho^\ast + w^\ast z(t) - z^\ast w(t) > 1. \end{cases}
\] (28)

Applying Theorem 2, we obtain the following existence result.

**Theorem 6.** For every solution \( z_0, w_0 \in \mathbb{R} \) and \( I_0, V_0 \in L^2[0,1] \) there exists a unique solution of (13)-(19), with \( u \) given by (28). Moreover, if \( Z^\ast \) is given by (27), then, for all \( t > 0 \),
\[
\| Z - Z^\ast \|_2 + 2R_L \int_0^t |I(1, \sigma) - I^\ast|_2^2 \, d\sigma = \| Z_0 - Z^\ast \|_2^2
\]
+ 2 \int_0^t (\rho^\ast - u(\sigma))(w^\ast z(\sigma) - w(\sigma)z^\ast) \, d\sigma.
\] (29)

Finally, to obtain the stability result, we have to verify the assumptions of Theorem 3.

**Lemma 7.** For every \( t > 0 \), \( S(t) \) is sequentially weakly continuous on \( X \), i.e., if \( Z_{0n} \to Z_\infty \) in \( X \) weakly, then \( S(t)Z_{0n} \to S(t)Z_\infty \) in \( X \) weakly.

**Proof:** Assume that \( Z_{0n} \to Z_\infty \) weakly in \( X \). Consequently there exists \( C > 0 \) such that
\[
\| Z_{0n} \| \leq C \quad (n \in \mathbb{N}^+).
\] (30)

Denote by \( Z_n \) the solution of
\[
\begin{cases} \dot{Z}_n(t) = A Z_n(t) + u_n(t) P Z_n(t) + F \\ Z_n(0) = Z_{0n}, \end{cases}
\]
where
\[
u_n(t) = \begin{cases} \rho^\ast + w^\ast z_n(t) - z^\ast w_n(t) & \text{if } \rho^\ast + w^\ast z_n(t) - z^\ast w_n(t) \in [0,1] \\ 0 & \text{if } \rho^\ast + w^\ast z_n(t) - z^\ast w_n(t) < 0 \\ 1 & \text{if } \rho^\ast + w^\ast z_n(t) - z^\ast w_n(t) > 1. \end{cases}
\] (31)

By (29), (30) and (31), there exists \( C > 0 \) such that
\[
\| Z_n(t) \| \leq C \quad (t > 0, \ n \in \mathbb{N}^+).
\]
Therefore there exists \( Z \in L^2([0,T], X) \) such that
\[
\begin{bmatrix} z_n \\ w_n \\ I_n \\ V_n \end{bmatrix} \to \begin{bmatrix} z \\ w \\ I \\ V \end{bmatrix} \quad \text{weakly.}
\] (32)

The aim is to prove that \( Z \) is solution of
\[
\begin{cases} \dot{Z}(t) = A Z(t) + u(t) P Z(t) + F \\ Z(0) = Z_{\infty}. \end{cases}
\] (33)

Using the facts that \( \dot{z}_n = -\frac{u_n(t)}{C} w_n + E, \ u_n(t) \in [0,1] \) and that \( \{w_n\} \) is bounded in \( L^2[0,T] \), it follows that \( \{z_n\} \) is bounded in \( H^1(0,T) \). Therefore \( z_n \to z \) in \( C[0,T] \) and
\[
z_n \to z \quad \text{in } H^1(0,T) \quad \text{weakly.}
\] (34)

Moreover, we can easily show that, for each \( n \in \mathbb{N} \),
\[
\| Z_n(t) \|_2^2 - \| Z_{0n} \|_2^2 = -2R_L \int_0^t |I(1, \sigma) - I^\ast|_2^2 \, d\sigma
\]
+ 2E \int_0^t z_n(\sigma) \, d\sigma \quad (t \in [0,T]).
\]
Therefore there exists a positive constant \( C \) with
\[
\int_0^T |I(1, \sigma) - I^\ast|_2^2 \, d\sigma \leq C \quad (n \in \mathbb{N}^+).
\] (35)

By the boundary condition \( V_n(1, t) = R_L I_n(1, t) \) we have
\[
\int_0^T |V(1, \sigma)|^2 \, d\sigma \leq C \quad (n \in \mathbb{N}^+).
\] (36)

On the other hand, we can easily verify that \( I_n + \frac{C}{L_l} V_n \) is constant along the characteristic \( t = \sqrt{C_l/L_l} x \) and that \( I_n - \frac{C}{L_l} V_n \) is constant along the characteristic \( t = -\sqrt{C_l/L_l} x \). Consequently for every \( t \geq \sqrt{C_l/L_l} \), we have
\[
\begin{cases} I_n(0, t) + \frac{C}{L_l} V_n(0, t) \\ I_n(0, t) - \frac{C}{L_l} V_n(0, t) \end{cases}
= I_n(1, \sqrt{C_l/L_l} t) + \frac{C}{L_l} V_n(1, \sqrt{C_l/L_l} t)
\]
\[
= I_n(1, \sqrt{C_l/L_l} t) - \frac{C}{L_l} V_n(1, \sqrt{C_l/L_l} t) + \frac{C}{L_l} V_n(1, \sqrt{C_l/L_l} t).
\]

and for every \( t < \sqrt{C_l/L_l} \), we have
\[
\begin{cases} I_n(0, t) + \frac{C}{L_l} V_n(0, t) \\ I_n(0, t) - \frac{C}{L_l} V_n(0, t) \end{cases}
= I_n(1, \sqrt{C_l/L_l} t) + \frac{C}{L_l} V_n(1, \sqrt{C_l/L_l} t)
\]
\[
= I_n(1, \sqrt{C_l/L_l} t) - \frac{C}{L_l} V_n(1, \sqrt{C_l/L_l} t) + \frac{C}{L_l} V_n(1, \sqrt{C_l/L_l} t).
\]

The above estimates, combined with (35) and (36) yield that \( (I_n(0, .)) \) and \( (V_n(0, .)) \) are bounded in \( L^2[0, T] \).

Then, by \( \tilde{w}_n = \frac{u_n(t)}{C} z_n - \frac{1}{C} I_n(0, t) \), we obtain that \( \{w_n\} \) is bounded in \( H^1(0,T) \) and then \( w_n \to w \) in \( C[0,T] \) and \( w_n \to w \) in \( H^1(0,T) \) weakly.

Since \( u_n(t) = \tilde{u}(z_n(t), w_n(t)) \), \( u(t) = \tilde{u}(z(t), w(t)) \) and \( \tilde{u} \) is a Lipschitz map, we have that
\[
u_n \to u \quad \text{in } C[0,T].
\] (38)

Combining (32), (34), (37) and (38) and the fact that \( I_n(1,. \) and \( V_n(1,. \) converge weakly in \( L^2[0,T] \), we obtain that \( Z \) is a weak solution of (33).

According to (6) some calculations show that the function \( g \) is given by
\[
g(Z(t)) = R_L |I(1,t) - I^\ast|^2 + |\rho^\ast - u(t)||w^\ast z(t) - z^\ast w(t)|.
\] (39)
Lemma 8. The function $g$ defined by (39) satisfies (8) and the unique continuation property (9).

Proof: To prove (8), it suffices to repeat step by step the estimates from the proof of Lemma 7 and to use a standard argument. The proof is left to the readers.

It remains to prove the unique continuation property (9). Assume that $g(Z(t)) = 0$ on $[0, T]$, i.e., assume that, for any $t \in [0, T]$, 

$$R_L |I(1, t) - I^*|^2 + |\rho^* - u(t)| |w^*(t) - z^* w(t)| = 0. \tag{40}$$

Consequently, for any $t \in [0, T]$, we have 

$I(1, t) = I^*$ and $(\rho^* - u(t))(w^* z(t) - z^* w(t)) = 0$.

Since, going back to (28), we see that $w^* z(t) - z^* w(t) = 0$ whenever $u(t) = \rho^*$, it follows that 

$$z(t) = \frac{z^* w(t)}{w^*} = \frac{w(t)}{\rho^* R_L} \quad (t \in [0, T]). \tag{41}$$

Moreover, using the facts that 

$$V(1, t) = R_L I(1, t) = R_L I^* \quad (t \in [0, T]),$$

and (27), we obtain that 

$$V(1, t) = V^* \quad (t \in [0, T]).$$

We now prove that $Z = Z^*$. Indeed, as previously, we can easily verify that $I + \sqrt{\frac{C}{L}} V$ is constant along the characteristic $t = \sqrt{\frac{C}{L}} t$ and $I - \sqrt{\frac{C}{L}} V$ is constant along the characteristic $t = -\sqrt{\frac{C}{L}} t$. Consequently for every $(x, t) \in A$, where 

$$A = \left\{ (x, t) \in [0, 1] \times [0, T]; t \geq \sqrt{\frac{C}{L}} (1 - x) \right\},$$

we have 

$$\left\{ \begin{array}{l}
I(x, t) + \sqrt{\frac{C}{L}} V(x, t) = I(1, \sqrt{\frac{C}{L}} t(1 - x) + t) \\
+ \frac{\sqrt{C}}{L} V(1, \sqrt{\frac{C}{L}} t(1 - x) + t) \\
\quad = I^* + \sqrt{\frac{C}{L}} V^* \\
I(x, t) - \sqrt{\frac{C}{L}} V(x, t) = I(1, (x - 1) \sqrt{\frac{C}{L}} t + t) \\
- \frac{\sqrt{C}}{L} V(1, (x - 1) \sqrt{\frac{C}{L}} t + t) \\
\quad = I^* - \sqrt{\frac{C}{L}} V^*.
\end{array} \right.$$ 

Then, for every $(x, t) \in A$, 

$$I(x, t) = I^* \text{ and } V(x, t) = V^*.$$ 

Since $V(0, t) = w(t)$ for every $t \geq \sqrt{\frac{C}{L}} t$, we have $w(t) = V^* = \frac{E}{\rho^*} = w^*$ so that, with (41), $z(t) = \frac{w(t)}{\rho^* R_L} = \frac{E}{\rho^* \sqrt{\frac{C}{L}}} = z^*$. We have obtained that for any $t \geq \sqrt{\frac{C}{L}} t$ and $x \in [0, 1]$, 

$$V(x, t) = V^*, \quad I(x, t) = I^*, \quad w(t) = w^* \text{ and } z(t) = z^*.$$ 

It is clear that $Z^*$ is solution of (13)-(19), which is equal to $Z$ for $t \geq \sqrt{\frac{C}{L}}$. Moreover, by (29) and (40), there exists $c \geq 0$ such that 

$$\|Z(t) - Z^*\| = c \quad (t > 0).$$

Since $Z(t) = Z^*$ for every $t \geq \sqrt{\frac{C}{L}}$, we obtain that 

$$\|Z(t) - Z^*\| = 0 \quad (t > 0),$$

which proves the unique continuation property (9).

The two previous lemma and Theorem 3 prove the following stability result.

**Theorem 9.** Under the assumptions of Theorem 6, the solution of (13)-(19), with $u$ given by (28), satisfies 

$$\lim_{t \to \infty} z(t) = z^*, \quad \lim_{t \to \infty} w(t) = w^*.$$ 

Moreover, we have 

$$\lim_{t \to \infty} I(t) = I^*, \quad \lim_{t \to \infty} V(t) = V^*,$$

in the weak topology of $L^2[0, 1]$.

**IV. Conclusion remarks**

We have shown that a control strategy similar to the one referred as "maximum descent" in [9] provides weak stabilization for a class of bilinear control systems. An interesting open question consists in studying other control strategies (for instance minimum switching strategy).

**References**


