A Solution to the Problem of Transient Stability of Multimachine Power Systems

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Abstract—In this paper it is shown that the controller design methodology recently introduced by the authors to stabilize non–globally feedback linearizable triangular systems solves the long–standing problem of transient stabilization of multimachine power systems with non–negligible transfer conductances. The 3n–dimensional (aggregated) model of the n–generator system, with lossy transmission lines, nonlinear loads and excitation control, is considered. A nonlinear dynamic state–feedback control law that ensures, under some conditions on the physical parameters, global asymptotic stability of the operating point is constructed. To the best of the authors’ knowledge only existence results—with the restrictive assumption of uniform inertia generators—are available to date for this problem. Simulation results of the New England benchmark system illustrate the performance of the proposed controller.

I. INTRODUCTION

Transient stability is concerned with the ability of a power system to reach a steady–state following a fault, e.g., a short circuit or a generator outage, that is later cleared by the protective system operation. The fault modifies the circuit topology—driving the system away from a stable operating point—and the question is whether the trajectory remains in the basin of attraction of this (or another) equilibrium after the fault is cleared. The key analysis issue is then the evaluation of the domain of attraction of the system’s operating equilibrium, while the control objective is the enlargement of the latter. See [2], [8], [9], [10], and the references therein, for more details and a literature review.

In this paper we consider the problem of excitation controller design for a power system consisting of n–generators, lossy transmission lines and loads, the mathematical description of which is given by the classical nonlinear 3n–dimensional reduced order model, see e.g., [2], [7]. In [8] this model is used to prove the existence of a nonlinear static state–feedback control law that ensures asymptotic stability of the operating point, with a well–defined estimate of the domain of attraction. Unfortunately, an explicit expression for the controller is given only for the single, two and three machine cases. Moreover, to establish the existence result for more than three machines the stringent assumptions that all generator inertias are equal, and that the line losses are sufficiently small, are made. In [3] the more natural and widely popular structure–preserving models are considered. A control law that ensures global attractiveness of the desired equilibrium point, provided the trajectories remain in the region where the model makes physical sense and a detectability assumption that is hard (if at all possible) to verify is satisfied, is then constructed.

To the best of our knowledge, no explicit globally asymptotically stabilizing controller, without these assumptions, has yet been reported—even in the lossless lines case. The contribution of this paper is a controller that achieves this objective, with a well–defined Lyapunov function, for the lossy transmission line case. The controller derivation is based on the recent methodology for stabilization of non–globally feedback linearizable triangular systems reported by the authors in [1]. Although in [1] the transient stabilization problem is treated as an example, the controller reported in the paper, which is a direct application of the methodology, suffers from serious drawbacks. First, it is extremely complex for its practical application. Second, as shown in [1], and further discussed here, the transient performance of the controller is far from satisfactory. Finally, the introduction of controller terms that do not comply with the natural topology of the system, which lives in a torus, destroys this important physical property. All these shortcomings are removed in the new control law reported in this paper, which more effectively exploits the particular structure of the power system model.

The remaining of the paper is organized as follows. In Section II we present the system model and the problem formulation. Section III contains some discussion regarding the main assumptions. The controller and main stability result are given in Section IV. Section V presents some simulation results of the New England benchmark system. The paper is wrapped–up with some concluding remarks and future research in Section VI.

Notation Unless stated otherwise, throughout the paper the subindex i takes values in the set \{1, \ldots, n\}, which will be omitted for brevity. Also, to simplify the notation, we define the n–dimensional column vector \(\mathbf{z}_i := [z_1, \ldots, z_n]^T\).

II. SYSTEM MODEL AND PROBLEM FORMULATION

The dynamics of the power system, consisting of n machines, loads and transmission lines, may be described by...
the aggregated reduced model equations [2], [8], namely
\[ \dot{\delta}_i = \omega_i, \]
\[ \omega_i = -D_i \omega_i + P_i - G_i E_i^2 - E_i \sum_{k=1, k \neq i}^n E_k Y_{ik} \sin(\delta_i - \delta_k + \alpha_{ik}) \]
\[ \dot{E}_i = -a_i E_i + b_i \sum_{k=1, k \neq i}^n E_k \cos(\delta_i - \delta_k + \alpha_{ik}) + \frac{1}{\tau_i} (E_{Fi_i} + \nu_i), \]
(1)
with \( \delta_i \in [0, 2\pi], \omega_i \in \mathbb{R} \) and \( E_i \in \mathbb{R}^+ \), the states, \( \nu_i \in \mathbb{R} \)
the control inputs, \( D_i, P_i, G_i, a_i, b_i, \tau_i \) and \( E_{Fi_i} \) positive constants depending
on the physical parameters of the \( i \)-th machine, and \( Y_{ik} \) and \( \alpha_{ik} \), with \( i \neq k \), constants depending
on the topology and physical properties of the network and
the loads. Consistent with physical constraints we assume
that the mechanical power verifies \( P_i > 1 \).

To formulate, and solve, the transient stability problem
we introduce the following assumptions.

Assumption 2.1: The equilibrium to be stabilized is
\[ \mathcal{E} := (\delta, 0, E) \in \mathbb{S} \times \mathbb{R}^n \times \mathbb{R}^n_+, \]
where \( \mathbb{S} := [0, 2\pi]^n, \overline{\delta} := \text{col}(\delta_i) \) and \( \overline{E} := \text{col}(E_i) \).
Moreover, \( \overline{\delta} \) is such that the second equation in (1), with
\( \omega_i = 0 \) and \( \delta_i = \overline{\delta}_i \), has a unique solution in \( E_i \).

Assumption 2.2: All states are measurable and all
parameters are exactly known.

To derive the proposed controller it is convenient to
express the system (1) in error coordinates, \( i.e. \)
\[ x_1 := \text{col}(\delta_i) - \overline{\delta}, \quad x_2 := \text{col}(\omega_i), \quad x_3 := \text{col}(E_i - \overline{E}), \]
and to apply a partial linearizing feedback, \( i.e., \)
\[ \nu_i = \tau_i [u_i + a_i E_i - b_i \sum_{k=1, k \neq i}^n E_k \cos(\delta_i - \delta_k + \alpha_{ik})] - E_{Fi_i}, \]
where \( u_i \) are the new control inputs. This yields the equations
\[ \dot{x}_1 = x_2, \]
\[ \dot{x}_2 = -\text{diag}(D_i)x_2 + P - \text{diag}\{x_{3,i} + \overline{E}_i\}h(x_1, x_3), \]
\[ \dot{x}_3 = u, \]
(2)
where
\[ P := \text{col}(P_i), \quad h(x_1, x_3) := \text{col}(h_i(x_1, x_3)), \quad u := \text{col}(u_i), \]
with
\[ h_i(x_1, x_3) := G_i(x_{3,i} + \overline{E}_i) + \sum_{k=1, k \neq i}^n (x_{3,k} + \overline{E}_k) Y_{ik} \sin(\mu_{ik}(x_1)), \]
(3)
where, for \( i \neq k \), we define
\[ \mu_{ik}(x_1) := x_{1,i} + \overline{\delta}_i - x_{1,k} - \overline{\delta}_k + \alpha_{ik}, \]
(4)
and, for \( j = 1, 2, 3 \), we denote
\[ x_j = [x_{j,1}, x_{j,2}, \ldots, x_{j,n}]^T. \]
As will become clear below, some conditions on the system
parameters must be imposed to ensure that the proposed
controller is well-defined. To express, in a compact form,
these conditions we find convenient to define the \( n \times n \)
matrixes (5) and (6) (given below), where
\[ s_{ik}(x_1) := \sin(\mu_{ik}(x_1)), \quad i \neq k \]
and \( \xi := \text{col}(\xi_i) \), is the state of the controller given below.
Notice that, given that \( P_i > 1 \), the matrix \( Q(x_1, \xi) \) is well-defined.

The aforementioned assumption on the system parameters
reads as follows.

Assumption 2.3: The parameters \( P_i, G_i, Y_{ik} \) and \( \alpha_{ik} \)
are such that the matrix \( L \) is invertible.

Problem Formulation Consider the system (2) verifying
Assumptions 2.1–2.3. Find a dynamic state–feedback control
law such that the origin is an asymptotically stable equilibrium
of the closed–loop with a well–defined domain of attraction.
Also, give conditions that ensure global asymptotic stability.

III. DISCUSSION

The following remarks regarding the problem formulation
and the assumptions are in order. To carry out the discussion,
it is convenient to recall the physical interpretation of some
of the parameters of the reduced model (1). (See [4], [7] for
additional details.) Namely, for \( k \neq i, \)
\[ Y_{ik} := \sqrt{G_{ik}^2 + B_{ik}^2}, \quad \alpha_{ik} := \arctan\left( \frac{G_{ik}}{B_{ik}} \right), \]
where \( G_{ik}, B_{ik} \) are the (normalized) conductance and
susceptance of the generator, respectively. If it is assumed that
the line is lossless, which is typically done in power system
studies, \( G_{ik} = 0 \)—and, consequently also \( \alpha_{ik} = 0 \). In this case
\( Y_{ik} = B_{ik} \), that is usually small compared to \( G_i \), which
is the self–conductance of the generator.

Assumption 2.1 requires that the equilibrium angles \( \bar{\delta}_i \) be
such that the quadratic equations
\[ G_i E_i^2 + E_i \sum_{k=1, k \neq i}^n E_k Y_{ik} \sin(\bar{\delta}_i - \overline{\delta}_k + \alpha_{ik}) - P_i = 0 \]
have a unique solution for \( E_i \). Under the assumption of lossless
line the second left–hand term will be small compared to
\( G_i \), particularly so if \( \bar{\delta}_i \approx \overline{\delta}_k \). Then, the equations take
the form \( G_i E_i^2 \approx P_i \) that, clearly, have unique (positive) solutions.

Knowledge of all states of Assumption 2.2, was practically
unreasonable a few years back. However, with the increasing
utilization of measuring and telemetering devices, in particular,
the Phasor Measurement Units [5], this is becoming a
\[
\mathbf{L} := \begin{pmatrix}
\frac{P_1}{E_1} + G_1 E_1 & Y_{12} E_2 \sin(\delta_1 - \delta_2 + \alpha_{12}) & \cdots & Y_{1n} E_n \sin(\delta_1 - \delta_n + \alpha_{1n}) \\
Y_{21} E_1 \sin(\delta_2 - \delta_1 + \alpha_{21}) & \frac{P_2}{E_2} + G_2 E_2 & \cdots & Y_{2n} E_n \sin(\delta_2 - \delta_n + \alpha_{2n}) \\
\vdots & \vdots & \ddots & \vdots \\
Y_{n1} E_1 \sin(\delta_n - \delta_1 + \alpha_{n1}) & Y_{n2} E_2 \sin(\delta_n - \delta_2 + \alpha_{n2}) & \cdots & \frac{P_n}{E_n} + G_n E_n
\end{pmatrix}
\]

(5)

Assumptions 2.3 holds if the \(Y_{ik}\)'s are sufficiently small compared with the \(G_i\)'s. Given the discussion above, this is the case for lossless line systems. Actually, a simple (conservative) sufficient condition for this assumption to hold stems from direct application of Gershgorin circle's theorem. Indeed, factoring from the right the full–rank matrix \(\text{diag}(E_i)\) and recalling that \(P_i > 0\), direct application of the theorem shows that if

\[
G_i \geq \sum_{k=1,k \neq i}^n Y_{ik} | \sin(\delta_i - \delta_k + \alpha_{ik})|,
\]

(7)

then the matrix \(\mathbf{L}\) has all its eigenvalues on the open left–hand plane, hence is full rank. Notice that a sufficient condition for (7) to hold is

\[
G_i \geq \sum_{k=1,k \neq i}^n Y_{ik}.
\]

(8)

The latter also guarantees that the matrix \(Q(x_1, \xi)\) is invertible for all \(x_1 \in \mathbb{S}\), \(\xi \in \mathbb{R}^n\)—a condition that is imposed in the proposition below for global stability.

Finally, as indicated in Assumption 2.1, because of physical constraints, the angle \(\delta_i\) and the quadrature axis internal voltage \(E_i\) are restricted to belong to \([0, 2\pi)\) and \(\mathbb{R}_+\), respectively [7]. Since our interest is to obtain global results, the latter restriction is not considered in the paper. Therefore we assume \(x_3 \in \mathbb{R}^n\). Note that, in contrast with the most of the controllers proposed in the literature, including the one given in [1], the restriction of \(x_1 \in \mathbb{S}\) will be enforced.

**IV. MAIN RESULT**

In this section we design a feedback control law and give conditions on the parameters of the machines and the network to ensure the desired stability properties of the closed-loop system. The interested reader is referred to [1] for further details on the rationale behind the design and a general methodology applicable for triangular non–globally feedback linearizable systems.

**Definition 4.1:** The function \(u : \mathbb{S} \times \mathbb{R}^{3n} \to \mathbb{R}^n\) is implicitly defined by

\[
u_i(x, \xi) = x_{3,i}^d + \beta_1 h_i(x_1, x_3) x_{2,i} - \beta_2 (x_{3,i} - x_{3,i}^0 + \bar{E}_i) + \epsilon_i \xi_i \sin(x_{1,i}),
\]

where \(\beta_1 > 0\), \(\beta_2 > 0\), \(\epsilon_i > 0\) and \(\xi_i\) is defined by

\[
\dot{\xi}_i = \beta_3 (h_i(x_1, x_3) - \xi_i) - \beta_4 (x_{3,i}^0 + \bar{E}_i) x_{2,i} + G_i u_i + \sum_{k=1,k \neq i}^n [u_k Y_{ik} \sin(\mu_{ik}(x_1)) + (x_{3,k} + \bar{E}_k) Y_{ik} (x_{2,i} - x_{2,k}) \cos(\mu_{ik}(x_1))] - \epsilon_i (x_{3,i} + \bar{E}_i) \sin(x_{1,i}),
\]

(10)

with \(\mu_{ik}(x_1)\) as in (4), \(\beta_3 > 0\), \(\beta_4 > 0\) and \(x_{3,i}^d\) given by

\[
x_{3,i}^d := \frac{1}{\xi_i} \left[ \sin(x_{1,i}) + P_i \right] - \bar{E}_i.
\]

**Proposition 4.1:** Consider system (2) verifying Assumptions 2.1–2.3.

i) The function \(u\) of Definition 4.1 is well-defined in a neighborhood of

\[
\mathcal{E}_0 := (0, h(0, 0)) \in \mathbb{S} \times \mathbb{R}^{3n}.
\]

ii) \((x, \xi) = \mathcal{E}_0\) is the unique equilibrium of the closed-loop system (2), (9), (10), and it is locally asymptotically stable if the tuning gains \(\epsilon_i\) are chosen such that

\[
0 < \epsilon_i < \frac{4D_i}{c_2(4 + D_i^2)}.
\]

(12)

where \(c_2 \geq 1\).

iii) The domain of attraction of \(\mathcal{E}_0\) contains all initial conditions such that, along the corresponding trajectories of the closed–loop system, the matrix \(Q(x_1(t), \xi(t))\), defined in (6), is invertible.

iv) If the matrix \(Q(x_1, \xi)\) is invertible for all \(x_1 \in \mathbb{S}\), \(\xi \in \mathbb{R}^n\), then \(\mathcal{E}_0\) is a globally asymptotically stable equilibrium.

Proof:

For instance, if (8) holds.
First, note that the control signal $u_i$ appears—through equations (3), (10) and (11)—on the right hand side of (9) and must, therefore, be made explicit. Computing $\dot{x}_{3,i}$ from (11), and substituting (3) and (10), we obtain

$$
\dot{x}_{3,i} = \frac{1}{\xi_i} g_{1,i} + \frac{1}{\xi_i} g_{2,i} \sum_{k=1}^{n} [H_{ik} u_k + K_{ik}] + \beta_3 (h_i - \xi_i) + \frac{1}{\xi_i} g_{3,i} + g_{4,i},
$$

(13)

where $g_{1,i}$, $g_{2,i}$, $g_{3,i}$, $g_{4,i}$, $H_{ik}$ and $K_{ik}$ are well-defined functions of $x$. Evaluating the second equation in (2) at $x = 0$ we conclude that $h_i(0, 0) > 0$. Now, from (11) we see that, at the equilibrium $\xi$ and $h$ coincide, hence $\xi_i > 0$, in some neighborhood of $\mathcal{E}_0$. Therefore, (9) can be rewritten as

$$u_i = g_{5,i} + \sum_{k=1}^{n} \frac{g_{2,i} H_{ik}}{\xi_i} u_k,$$

with $g_{5,i}$ well-defined in this neighborhood. Denoting $g_{5} := \text{col}(g_{5,1}, \ldots, g_{5,n})$, by $I_n$ the identity matrix of dimension $n$, and by $\overline{L}$ the matrix with entries

$$\overline{L}_{ik} = \frac{g_{2,i} H_{ik}}{\xi_i^2},$$

we rewrite the control equation (9) as

$$(I_n - \overline{L}) u = g_{5}.$$ 

Explicit computations yield

$$I_n - \overline{L} = \text{diag} \left\{ \frac{\sin(x_{1,i}) + P_i}{\xi_i} \right\} Q(x_1, \xi).$$

(14)

Therefore, since $P_i > 1$ and $\xi_i > 0$, $I_n - \overline{L}$ is invertible if and only if $Q(x_1, \xi)$ is invertible. Finally, since

$$Q(0, h(0, 0)) = \text{Ldiag} \{ \overline{E}_i \},$$

Assumption 2.3 implies that the control law is well-defined in $\mathcal{E}_0$ and, by continuity, also in a neighborhood of $\mathcal{E}_0$.

**ii)** Uniqueness of the equilibrium is immediately proven by substitution, as long as the control law $u$ is well-defined.

The proof of stability relies on the existence of a Lyapunov function. Consider the function $V : \mathcal{S} \times \mathbb{R}^{3n} \rightarrow \mathbb{R}$ defined by

$$V(x, \xi) = \sum_{i=1}^{n} \left( c_1 \int_{0}^{x_{1,i}} \sin(\eta) d\eta + c_2 \xi_i x_{2,i} \sin(x_{1,i}) \right) + \frac{1}{2} \left( c_3 |x_3|^2 + c_4 |\xi - h|^2 + c_5 |x_3 - x_{3,i}|^2 \right),$$

(15)

with $x_{3,i} := \text{col}(x_{3,i,1}, \ldots, x_{3,i,n})$ and $c_1$ to $c_5$ as positive constants. Note that $V$ can be rewritten as

$$V(x, \xi) = \frac{1}{2} c_4 |\xi - h|^2 + \frac{1}{2} c_5 |x_3 - x_{3,i}|^2 +$$

$$+ \sum_{i=1}^{n} \left( c_1 \int_{0}^{x_{1,i}} \sin(\eta) d\eta + \frac{1}{2} c_3 x_{2,i}^2 + c_2 \xi_i x_{2,i} \sin(x_{1,i}) \right).$$

If the terms in parenthesis are non-negative, $V$ is positive definite—with respect to the set

$$\{ x_1 = 0, x_2 = 0, x_3 = x_{3,i}, \xi = h(0, 0) \}.$$ 

(16)

Easy computations show that the Lyapunov function is positive definite, that its time derivative negative semidefinite, and that the trajectories converge to the set (16). Assumption 2.1 allows us to conclude that the latter set coincides with $\mathcal{E}_0$, hence the claim.

**iii)** The local result given by i) and ii) can be naturally extended to all trajectories along which $Q(x_1(t), \xi(t))$ is invertible, thus guaranteeing existence of the control law for all $t > 0$ and attractivity of $\mathcal{E}_0$.

**iv)** From (14), if $Q(x_1, \xi)$ is invertible for all $x_1 \in \mathcal{S}$, $\xi \in \mathbb{R}^n$, then also the control law $u$ is well-defined for all the points in the state–space. This, together with $V$ and properness of $V$, establish the global stability claim.

**Remark 4.1:** If the equilibrium $\mathcal{E}$ (see Assumption 2.1) is not unique, but it is isolated, we can conclude from the analysis above that it is locally asymptotically stable. Moreover, for all initial conditions such that $Q(x_1(t), \xi(t))$ is invertible, we have that $x_1(t) \rightarrow 0$ and $x_2(t) \rightarrow 0$.

**Remark 4.2:** The controller derived in Proposition 4.1 should be contrasted with the generic controller reported in [1] that, as explained in the introduction, suffers from the drawbacks of being too complex and destroying the physical structure of the system. Moreover, as shown in the next section, the transient performance of the new controller is significantly better.

V. Simulation Results

The controller of Proposition 4.1—with the terms $c_1 \sin(x_{1,i})$ replaced by $\tanh(x_{1,i})$—designed from the simplified model of a power system given by eq. (1) has been tested in simulations using the New England ten machine system [12]. This well known benchmark was built based on SimPowerSystems/Simulink blocks, which uses a more complex dynamics than (1). System parameters are given in [12], the initial conditions for the angular velocities are set to the values given below in (17), the initial conditions for $\delta$ and $E$ are the equilibrium points. In the simulations, the performance of the controller has been improved by suitable tuning of multiplicative constants in the Lyapunov function (15) and in the control function (9).

In Figures 1–4 the time-histories of the states of the ten machines and of the control inputs are depicted. It can be noted that the convergence of $\delta$ and $E$ is very rapid. On the other hand, the states $u$ and $\omega$ converge to their equilibrium values slowly, but remain within physically admissible ranges. Notice also that the behavior is very smooth, indicating that the controller injects low gains into the loop—a feature that is desirable in all practical applications.

These simulations should be contrasted with the ones reported in [1] that, for exactly the same conditions, exhibit
\[ \omega(0) = (0.2121 \ 1.2284 \ 1.3041 \ 0.5650 \ 0.5048 \ 0.1567 \ 1.0111 \ 0.0061 \ 0.2247 \ 0.4720)^\top. \] (17)

VI. CONCLUDING REMARKS

We have presented a solution to the longstanding problem of explicit derivation of a globally asymptotically stabilizing controller, with a \textit{bona fide} Lyapunov function, for multi-machine power systems with lossy lines—whose parameters verify the simple algebraic condition (7). It is clear that the assumption that all generators have excitation controllers—as well as the sensitivity to the system parameters of the feedback linearizing part, and the assumption of full state measurement—severely stymies the practical applicability of this result. Similarly to most developments reported by the control theory community on the transient stability problem, this kind of work pertains to the realm of fundamental research where basic issues, like solvability of the problem, are addressed. The present paper proves that, a globally asymptotically stabilizing controller can indeed be \textit{explicitly constructed}.

Current research is under way in the following directions.
- Explore the behavior of the controller in a realistic multi-machine simulation benchmark problem.
- Study the effect on performance of the tuning parameters \( \epsilon_i \) in Proposition 4.1.
- Extend the result to the more practically appealing structure preserving models.

REFERENCES

