Asymptotic Stability of Forced Equilibria for Distributed Port-Hamiltonian Systems

Alessandro Macchelli

Abstract—The main contribution of this paper is an energy shaping procedure for the stabilization of forced equilibria for linear, lossless, distributed port-Hamiltonian systems via Casimir generation. Once inputs and outputs have been properly chosen to have a well-posed boundary control system, conditions for the existence of Casimir functions in closed-loop are given, together with their relation with the controller structure. These invariants suggest how to select the controller Hamiltonian to introduce a minimum at the desired equilibrium. Such equilibrium can be made asymptotically stable via damping injection, if proper “pervasive” damping injection conditions are satisfied. The methodology is illustrated with the help of a Timoshenko beam with constant non-zero force applied at one side of the spatial domain, and full-actuation on the other one.

I. INTRODUCTION

This paper deals with the stabilization of forced (i.e., non-zero) equilibria via energy shaping of distributed port-Hamiltonian systems [1]. In recent works [2]–[5], this task has been accomplished by looking at or generating a set of Casimir functions in the closed-loop system that robustly (i.e. independently from the Hamiltonian function) relates the state of the infinite dimensional port-Hamiltonian system with the state of the controller. The controller has been usually modelled as a finite dimensional port-Hamiltonian system which has to be interconnected in a power conserving way to the boundary of the distributed parameter system. The shape of the energy function of the closed-loop system can be changed by properly choosing the Hamiltonian function of the controller in order to introduce a (possibly global) minimum in a desired configuration. This procedure is basically the generalization to the distributed parameter case of the control by interconnection via Casimir generation developed for finite dimensional port-Hamiltonian systems, [6]–[9].

In case of plants modelled as distributed parameter systems, it is relatively easy to shape the energy function. The main difficulties arise in proving that the new minimum of the closed-loop Hamiltonian function corresponds also to an asymptotically stable equilibrium point. Only stability can be verified by means of relatively simple techniques, as reported in [10]. This because, even if the extension to distributed parameter system of La Salle’s Invariance Principle exists, its application is not immediate due to several technical problems mostly related to the analysis of the solution of linear or nonlinear PDEs, [11]. It is possible to rely on stability results that hold for finite dimensional systems if a finite element approximation of the distributed parameter dynamics is used as starting point for the development of the regulator, as discussed in [12]. On the other hand, in this work the “full-order” dynamics (i.e., infinite dimensional) is considered and a systematic procedure for energy shaping and asymptotic stabilization via damping injection is provided.

The starting point is represented by the theory of boundary control systems in port-Hamiltonian form developed in [13]. Here, given a linear, lossless, distributed port-Hamiltonian systems with one-dimensional spatial domain, the boundary variables have been parametrized so that the \(C_0\)-semigroup associated with this system of PDEs is contractive or unitary. This parametrization is also used to properly define boundary inputs and outputs. Impedance energy-preserving systems and scattering energy-preserving systems are just particular cases of all the possible choices. This framework has been further extended in [14] for the static and dynamic feedback stabilization by means of boundary controllers of systems in impedance and scattering form, and in [15] where simple tools to check exponential stability are provided. These tools are based on the use of a generating function and an inequality condition on the boundary variables.

Once boundary inputs and outputs are chosen in such a way that the distributed port-Hamiltonian system is in impedance form, and a lossless controller is interconnected at the boundary port in power-conserving manner, conditions for the existence of Casimir functions in closed-loop are computed, and some of their properties are investigated. In particular, the relation between such invariants and the controller structure is illustrated. Such Casimirs are employed in the selection of the controller Hamiltonian to shape the energy function and introduce a minimum in the desired equilibrium configuration. Asymptotic stability is achieved via damping injection, under the fundamental hypothesis that dissipation does not “destroy” the structural invariants previously computed. If it is possible to have full dissipation at one side of the spatial domain, then asymptotic stability follows from [15]. The complete procedure is illustrated with reference to a Timoshenko beam with a non-zero force applied at one side of the spatial domain, and full actuation on the other one.

The paper is organized as follows. In Sect. II, the main results on the “correct” definition of a boundary control systems within the port-Hamiltonian framework, and on the exponential stability of such class of systems under the action of boundary control action are presented. The procedure for determining and/or creating a suitable set of Casimir...
functions that can be used to properly shape the energy function is then discussed in Sect. III. Here, also some remarks on how to inject damping to achieve asymptotic stability are also given. The Timoshenko beam example is reported in Sect. IV. Finally, conclusions and a discussion on possible future research directions are given in Sect. V.

II. BACKGROUND

In this paper, we refer to the class of distributed port-Hamiltonian systems that have been studied in [13], [15], i.e. to systems described by the following PDE:

\[
\frac{\partial x}{\partial t}(t, z) = P_1 \frac{\partial}{\partial z}(\mathcal{L}(z)x(t, z)) + P_0 \mathcal{L}(z)x(t, z) \tag{1}
\]

with \( x \in \mathbb{R}^n \) and \( z \in [a, b] \). Moreover, \( P_1 = P_1^T > 0 \), \( P_0 = -P_0^T \) and \( \mathcal{L}(\cdot) \) is a bounded and continuously differentiable matrix-valued function such that \( \mathcal{L}(z) = \mathcal{L}^T(z) \) and \( \mathcal{L}(z) \geq \kappa I \), with \( \kappa > 0 \), for all \( z \in [a, b] \). For simplicity, \( \mathcal{L}(z)x(t, z) \equiv \langle \mathcal{L}(z)x(t, z) \rangle \). The state space is \( X = L_2(a, b; \mathbb{R}^n) \), and is endowed with the inner product \( \langle x_1 | x_2 \rangle_L = \langle x_1 | L x_2 \rangle \) and norm \( \| x_1 \|^2_L = \langle x_1 | x_1 \rangle_L \), where \( \langle \cdot | \cdot \rangle \) denotes the natural \( L_2 \)-inner product. The selection of this space for the state variable is motivated by the fact that \( \| \cdot \|^2_L \) is proportional to the energy function. As a consequence, \( X \) is also called the space of energy variables and \( \mathcal{L}x \) is the co-energy variable. This class is quite general and includes models of flexible structures, traveling waves, heat exchangers, and bioreactors among others (if also dissipative effects are included, see again [15]).

To define a distributed port-Hamiltonian system, the PDE (1) has to be “completed” by proper boundary port. More precisely, given \( \mathcal{L}x \in H^1(a, b; \mathbb{R}^n) \), the boundary port variables associated to (1) are the vectors \( f_\partial, e_\partial \in \mathbb{R}^n \) defined by

\[
\begin{pmatrix} f_\partial \\ e_\partial \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} P_1 & -P_1 \\ I & I \end{pmatrix} \begin{pmatrix} (\mathcal{L}x)(b) \\ (\mathcal{L}x)(a) \end{pmatrix} \tag{2}
\]

The boundary port variables are just a linear combination of the restriction of the boundary variables, and simple integration by parts shows that

\[
\frac{1}{2} \frac{d}{dt} \| x(t) \|^2_L = e_\partial^T(t)f_\partial(t) \tag{3}
\]

The problem of determining the “right” boundary inputs and outputs for (1) to have a so-called boundary control system on \( X \), see e.g. [16], has been addressed in [13], [15].

Theorem 2.1: Let \( W \) be a \( n \times 2n \) real matrix. If \( W \) has full rank and satisfies \( W \Sigma W^T \geq 0 \), being

\[
\Sigma = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} \tag{4}
\]

then the system (1) with input

\[
u(t) = W \begin{pmatrix} f_\partial(t) \\ e_\partial(t) \end{pmatrix} \tag{5}
\]
is a boundary control system on \( X \). Furthermore, the operator

\[
A = P_1(\partial/\partial z)(\mathcal{L}x) + P_0 \mathcal{L}(x) \tag{6}
\]

generates a contraction semigroup on \( X \). Moreover, let \( \hat{W} \) be a full rank \( n \times 2n \) matrix such that \( \hat{W}^T \hat{W}^T \) is invertible and let \( P \) be given by

\[
P = \begin{pmatrix} W \Sigma W^T & W \Sigma \hat{W}^T \end{pmatrix}^{-1} \tag{7}
\]

Define the output as

\[
y(t) = \hat{W} \begin{pmatrix} f_\partial(t) \\ e_\partial(t) \end{pmatrix} \tag{8}
\]

Then, for \( u \in C^2(0, \infty; \mathbb{R}^n) \) and \( (\mathcal{L}x)(0) \in H^1(a, b; \mathbb{R}^n) \), the following energy balance equation is satisfied:

\[
\frac{1}{2} \frac{d}{dt} \| x(t) \|^2_L = \frac{1}{2} \begin{pmatrix} u(t) \\ y(t) \end{pmatrix}^T \begin{pmatrix} u(t) \\ y(t) \end{pmatrix} \tag{9}
\]

Proof: See [13].

The previous theorem addresses the problem of the existence of solutions for (1) and characterizes all the possible boundary inputs for which a boundary control system on \( X \) exists. The energy balance relation (9) turns out to be a good starting point for studying the stability of the semigroup in case \( u(t) = 0 \). This problem has been addressed in [15], and simple estimates to prove the exponential stability property have been provided.

Theorem 2.2: Consider a boundary control system as described in Theorem 2.1 with \( u(t) = 0 \), for all \( t \geq 0 \). If the energy of the system \( H(t) = \frac{1}{2} \| x(t) \|^2_L \) satisfies

\[
\frac{dH}{dt}(t) \leq -k \| (\mathcal{L}x)(t, b) \|^2 \quad \text{or} \quad \frac{dH}{dt}(t) \leq -k \| (\mathcal{L}x)(t, a) \|^2 \tag{10}
\]

where \( k > 0 \), then the system is exponentially stable.

Proof: See [15].

III. CONTROL BY ENERGY-SHAPING

Among all the possible choices for input and output suggested by Theorem 2.1, let us assume that \( u \) and \( y \) given by (5) and (8) are such that \( P = \Sigma \). In this way, the energy balance relation (9) reduces to

\[
\frac{1}{2} \frac{d}{dt} \| x(t) \|^2_L = y^T(t)u(t) \tag{11}
\]

From (7), it is easy to see that this requires that

\[
W \Sigma W^T = 0 \quad \hat{W} \Sigma \hat{W}^T = 0 \quad W \Sigma \hat{W}^T \tag{12}
\]

Let us consider a control system in port-Hamiltonian form:

\[
\begin{aligned}
\dot{x}_C(t) &= J_C \frac{\partial \mathcal{H}_C}{\partial x_C}(x_C(t)) + G_C u_C(t) \\
y_C(t) &= G_C^T \frac{\partial \mathcal{H}_C}{\partial x_C}(x_C(t))
\end{aligned} \tag{13}
\]
where \( x_C \in \mathbb{X}_C \equiv \mathbb{R}^{n_C}, u_C, y_C \in \mathbb{R}^n, J_C = -J_C^T \) is the (constant) interconnection matrix, \( G_C \) is the input matrix, supposed full-rank, and \( H_C \) is the Hamiltonian function. The system is linear if the Hamiltonian is quadratic. Note that dissipation is not taken into account.

The control system (13) is interconnected to the boundary of (1) in a power-conserving way through the input \( u \) and \( y \) defined in (5) and (8) as follows:

\[
\begin{pmatrix} u \\ y \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -I & 0 \end{pmatrix} \begin{pmatrix} u_C \\ y_C \end{pmatrix} + \begin{pmatrix} u' \\ 0 \end{pmatrix} \tag{14}
\]

where \( u' \in \mathbb{R}^n \) is a further control input. This is the standard feedback interconnection. The closed-loop system is characterized by a total Hamiltonian

\[
H_d(x, x_C) = \frac{1}{2} \|x(t)\|^2_{\mathcal{L}} + H_C(x_C) \tag{15}
\]

and it is evident that \( H_C \) can be chosen to properly shape the closed-loop Hamiltonian to achieve desired stability properties. The main idea of the control by energy-shaping is to select \( H_C \) to move the minimum of \( H_d \) in the desired equilibrium configuration, that can be reached once "sufficient" dissipation has been introduced into the system, e.g. via further control action. In case of distributed port-Hamiltonian systems this task has been always accomplished by relating the state variable of the controller to the state variable of the plant by means of a set of Casimir functions, [3]-[5]. For the class of boundary-controlled port-Hamiltonian systems treated in this paper, a possible definition of Casimir function can be the following:

**Definition 3.1:** Consider the (autonomous) port-Hamiltonian system resulting from the power-conserving interconnection (14) of (1) and (13), with \( u' = 0 \). A Casimir function is a function \( C : \mathbb{X} \times \mathbb{X}_C \rightarrow \mathbb{R} \), such that \( \dot{C} = 0 \) along the solutions for every possible choice of \( \mathcal{L}(\cdot) \) and \( H_C \).

In this paper, we will look for linear Casimir functions in the form

\[
C(x(t), x_C(t)) = \Gamma^T x_C(t) + \langle \Psi, x(t) \rangle
= \Gamma^T x_C(t) + \int_a^b \Psi(z)x(t, z)dz \tag{16}
\]

with \( \Gamma \in \mathbb{R}^{nc} \) and \( \Psi \in H^3(\mathbb{R}; \mathbb{R}^n) \). Since this function is invariant, for every possible choice of the controller Hamiltonian \( H_C \), a "structural" algebraic relation between state of the plant and of the controller is present and can be exploited to properly shape the closed-loop Hamiltonian. The characterization of the possible Casimir functions (16) in closed-loop is given in the following theorem.

**Theorem 3.1:** Consider the distributed port-Hamiltonian system with dynamics expressed by the PDE (1) and input and output \( u \) and \( y \) defined in (5) and (8), with \( W \) and \( \tilde{W} \) that satisfy (12), the controller (13), and the power-conserving interconnection (14), with \( u' = 0 \). Then, (16) is a Casimir function if and only if:

\[
P_1 \frac{\partial}{\partial z} \Psi(z) + P_0 \Psi(z) = 0 \tag{17}
\]

\[
J_C \Gamma - G_C \tilde{W} R \left( \frac{\partial}{\partial \alpha} \Psi(\alpha) \right) = 0 \tag{18}
\]

\[
G_C^T \Gamma - W R \left( \frac{\partial}{\partial \alpha} \Psi(\alpha) \right) = 0 \tag{19}
\]

**Proof:** From (1), (14) and (16) we have that

\[
\frac{d}{dt} C = \Gamma^T \dot{x}_C + \int_a^b \Psi \frac{\partial x}{\partial t} dz
= \Gamma^T \left( J_C \frac{\partial H_C}{\partial x_C} + G_C u_C \right) + \int_a^b \Psi^T \left( P_1 \frac{\partial}{\partial z} (Lx) + P_0 Lx \right) dz
\]

where the dependence on \( t \) and \( z \) has been omitted for simplicity. According to Definition 3.1, the Casimir function has to be independent from the Hamiltonian of the plant and of the controller. On the other hand, the interconnection introduces a constraint on the possible Hamiltonians that has to be properly managed. In this respect, it is convenient to "parametrize" the boundary variables \( (f_\delta, e_\delta) \) defined in (2) as follows. Given \( \gamma_1, \gamma_2 \in \mathbb{R}^n \), all the possible values of \( (f_\delta, e_\delta) \) can be described by

\[
\begin{pmatrix} f_\delta \\ e_\delta \end{pmatrix} = \Sigma \left( W^T \gamma_1 + \tilde{W}^T \gamma_2 \right) \tag{20}
\]

since, according to Theorem 2.1, \((W^T, \tilde{W}^T)\) is invertible. From the set of conditions (12) and the interconnection (14), we have that

\[
u_C = -y = -\tilde{W} \left( \begin{pmatrix} f_\delta \\ e_\delta \end{pmatrix} \right) = -\gamma_1 \tag{21}
\]

and that

\[
G_C^T \frac{\partial H_C}{\partial x_C} = y_C = u = W \left( \begin{pmatrix} f_\delta \\ e_\delta \end{pmatrix} \right) = \gamma_2 \tag{22}
\]

which implies that

\[
\begin{pmatrix} f_\delta \\ e_\delta \end{pmatrix} = \Sigma \left( W^T \gamma_1 + \tilde{W}^T G_C^T \frac{\partial H_C}{\partial x_C} \right) \tag{23}
\]

All the possible behaviors at \( (f_\delta, e_\delta) \) corresponding to Hamiltonian dynamics can be obtained by properly varying \( H_C \) and \( \gamma_1 \). Due to (2), this relation provides also all the values that the co-energy variables can assume on the boundary for every possible choice of the plant Hamiltonian, i.e. of \( \mathcal{L}(\cdot) \) in the linear case. Since

\[
\Psi^T \left( P_1 \frac{\partial}{\partial z} (Lx) + P_0 Lx \right) = \frac{\partial}{\partial z} \left( (Lx)^T P_1 \Psi \right) - (Lx)^T \left( P_1 \frac{\partial}{\partial z} \Psi + P_0 \Psi \right)
\]
and in spite of (21) and (23), we have that
\[
\frac{d}{dt} C = - \frac{\partial H_C}{\partial x_C} J_C \Gamma - \gamma_T G_C \Gamma - \int_a^b (L_x)^T (P_1 \frac{\partial}{\partial z} \Psi + P_0 \Psi) \, dz
\] (24)
The integral vanishes for all the \(L(\cdot)\) if and only if \(\Psi\) satisfies (17). Moreover, simple computations show that
\[
\begin{pmatrix} P_1 & 0 \\ 0 & -P_1 \end{pmatrix} = R^T \Sigma R
\]
which, from (2) and (23), implies that
\[
\begin{pmatrix} (L_x)(b) \\ (L_x)(a) \end{pmatrix} = \begin{pmatrix} P_1 & 0 \\ 0 & -P_1 \end{pmatrix} \begin{pmatrix} \Psi(b) \\ \Psi(a) \end{pmatrix} = \begin{pmatrix} \gamma_T W + \frac{\partial H_C}{\partial x_C} G_C \dot{W} \end{pmatrix} R \begin{pmatrix} \Psi(b) \\ \Psi(a) \end{pmatrix}
\]
since \(\Sigma \Sigma = I\). Then, substitution in (24) allows to obtain (18) and (19) by properly grouping all the terms multiplied by \(\frac{\partial H_C}{\partial x_C}\) and \(\gamma_T\), respectively.

The controller interconnection matrix \(J_C\) depends on the particular choice for the input \(u\) and output \(y\). In fact, since the transformation from \((f_\partial, e_\partial)\) to \((u, y)\) given by (5) and (8) has to be power preserving, i.e.:
\[
y^T u = e_\partial^T f_\partial
\] (25)
we have that
\[
\begin{pmatrix} f_\partial \\ e_\partial \end{pmatrix} \dot{W}^T W \begin{pmatrix} f_\partial \\ e_\partial \end{pmatrix} = \frac{1}{2} \begin{pmatrix} f_\partial \\ e_\partial \end{pmatrix}^T \Sigma \begin{pmatrix} f_\partial \\ e_\partial \end{pmatrix}
\] (26)
that implies
\[
\dot{W}^T W = \frac{1}{2} \Sigma - \tilde{J}
\] (27)
for some \(\tilde{J} = -J^T\). Now, let us assume that it is possible to relate all the state variables of the controller with the state of the plant through \(n_C\) independent Casimir functions \(C_i(x, x_C)\) in the form (16), and introduce the following matrices
\[
\tilde{\Gamma} = (\Gamma_1, \cdots, \Gamma_{n_C}) \quad \tilde{\Psi} = (\Psi_1, \cdots, \Psi_{n_C})
\] (28)
From (18), (19) and (27), we have that
\[
\tilde{\Gamma}^T J_C \tilde{\Gamma} = \begin{pmatrix} \tilde{\Psi}(b) \\ \tilde{\Psi}(a) \end{pmatrix}^T R^T W^T \dot{W} R \begin{pmatrix} \tilde{\Psi}(b) \\ \tilde{\Psi}(a) \end{pmatrix}
\] (29)
which implies that
\[
\tilde{\Gamma}^T J_C \tilde{\Gamma} = \begin{pmatrix} \tilde{\Psi}(b) \\ \tilde{\Psi}(a) \end{pmatrix}^T R^T \dot{R} R \begin{pmatrix} \tilde{\Psi}(b) \\ \tilde{\Psi}(a) \end{pmatrix}
\]
\[
0 = \begin{pmatrix} \tilde{\Psi}(b) \\ \tilde{\Psi}(a) \end{pmatrix}^T \begin{pmatrix} P_1 & 0 \\ 0 & -P_1 \end{pmatrix} \begin{pmatrix} \tilde{\Psi}(b) \\ \tilde{\Psi}(a) \end{pmatrix}
\] (30)
On the other hand, \(G_C\) can be computed directly from (19).

For control purposes, a common requirement is to find \(n_C\) independent Casimir functions such that \(\tilde{\Gamma} = -I\). In this way, in closed-loop we have that
\[
x_{C,i}(t) = \langle \Psi_i | x(t) \rangle = \int_a^b \Psi_i^T(z) x(t, z) \, dz + \kappa_i
\] (31)
for \(i = 1, \ldots, n_C\), being \(\kappa_i \in \mathbb{R}\) a constant that depend only on the initial conditions. Under this hypothesis, the Hamiltonian function of the controller is, in fact, a function on the plant state variable, and it can be directly chosen to obtain the desired stability property in closed-loop, namely a (possibly) global minimum at the desired equilibrium configuration.

In this respect, denote by \(x^* \in L_2(a, b; \mathbb{R}^n)\) this equilibrium configuration. To rely on Theorems 2.1 and 2.2 that provide conditions for the existence of solutions and a criterion for the asymptotic stability, it is necessary to have still a “quadratic” energy function in closed-loop. Consequently, let us assume that under the set of conditions (31) due to the constraints that the Casimir functions introduce, the closed-loop Hamiltonian (15) as a function of \(x\) only is given by
\[
H_C(x) = \frac{1}{2} \| x - x^* \|_{\mathcal{L}}^2
\] (32)
where \(\mathcal{L}'\) is a new bounded and continuously differentiable matrix-valued function with the same properties of \(\mathcal{L}\). It is easy to prove that, projected on the invariant spaces defined by the Casimir functions (31), the closed-loop dynamics is again port-Hamiltonian in the form (1), with the input \(u'\) introduced in (14) given by (5) and equal to 0.

At present stage, since both the plant and the controller are lossless, only stability can be deduced from energy considerations, i.e. from the energy balance relation (11) in which \(\mathcal{L}\) and \(u\) are replaced by \(\mathcal{L}'\) and \(u'\), respectively. Asymptotic stability requires that the boundary controller injects dissipation in the system. On the other hand, such dissipation must not “destroy” the Casimir functions that have been already introduced. This is exactly the so-called dissipation obstacle that has been already encountered in the control of finite dimensional port-Hamiltonian systems. Basically, we must be able to dissipate only in the directions along we did not shape the Hamiltonian function via the Casimir functions (31).

Independently from the applicability of the energy shaping procedure, Theorem 2.1 allows to understand that (1) generates a contraction semigroup on \(X\) when
\[
u = -\hat{\Xi} y, \quad \hat{\Xi} = \hat{\Xi}^T \geq 0
\] (33)
being \( u \) and \( y \) given by (5) and (8), with \( W \) and \( \tilde{W} \) that satisfy (12). In fact, (33) is equivalent to \( \hat{u} = 0 \) being \( \hat{u} = u \).

Asymptotic stability is achieved by adding dissipation on the momenta: in this way, the Casimir functions (39) are

\[
C_1(x, x_C) = x_{C,1} - (q_w - L q_\theta) - \int_0^L (\varepsilon_w - z \varepsilon_\theta) \, dz
\]

Moreover, let us assume that \( J_C = 0 \) and \( G_C = I \), while

\[
H_C(x_C) = H'_C(x_{C,1}, x_{C,2}) + F^* q_w
\]

where \( H'_C \) is the “true” Hamiltonian of the control system and the second term takes into account the constant force (and torque) applied in \( z = L \).

By solving the PDE (17) with boundary conditions (18) and (19), it is possible to verify that the closed-loop system is characterized by a pair of Casimir functions in the form (16), i.e.:}

\[
C_1(x, x_C) = x_{C,1} - (q_w - L q_\theta) - \int_0^L (\varepsilon_w - z \varepsilon_\theta) \, dz
\]

\[
C_2(x, x_C) = x_{C,2} - q_\theta - \int_0^L \varepsilon_\theta \, dz
\]

Note that these Casimir functions are independent from the momenta, so the Hamiltonian can be shaped only in the direction of the “displacements.” This is sufficient since the forced equilibrium that has to be made asymptotically stable is

\[
p_w^*(z) = p_\theta^*(z) = 0
\]

\[
\varepsilon_w^*(z) = \frac{F^*}{K}
\]

\[
\varepsilon_\theta^*(z) = \frac{F^*}{E I} (L - z)
\]

The idea is to find a boundary controller to be interconnected at \( z = 0 \) to asymptotically stabilize the beam in a deformed configuration due to application of a constant force \( F^* \) in \( z = L \), i.e. in case the boundary inputs in \( z = L \) are \( (F(L), \tau(L)) = (F^*, 0) \).

To follow the framework proposed in Sect. III, it is necessary to “embed” these non-zero boundary conditions in the regulator (13), whose state variable can now be written as

\[
x_C = (x_{C,1} \quad x_{C,2} \quad q_w \quad q_\theta)^T
\]

The Timoshenko beam equations can be written in the form (1) by defining the state variable as

\[
x = (p_w \quad p_\theta \quad \varepsilon_w \quad \varepsilon_\theta)^T
\]

where \( p_w \) and \( p_\theta \) denotes the translational and rotational momenta of the cross-section, \( \varepsilon_w \) the shear and \( \varepsilon_\theta \) the bending. Moreover,

\[
P_1 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}
\]

and

\[
P_0 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}
\]

where \( \rho, I_\rho, E \) and \( I \) are the mass per unit length, the mass moment of inertia of the cross section, Youngs modulus, and the moment of inertia of the cross section, respectively. The coefficient \( K \) is equal to \( k G A \), where \( G \) is the modulus of elasticity in shear, \( A \) is the cross sectional area, and \( k \) is a constant depending on the shape of the cross section, \( [3] \). Finally, assume that the spatial domain is \([0, L]\). The boundary port variables (2) are given by

\[
(f_\theta \quad \varepsilon_\theta) = -\sqrt{2} \begin{pmatrix} F(L) - F(0) \\ \tau(L) - \tau(0) \\ \varepsilon(L) + \varepsilon(0) \\ \varepsilon_\theta(L) + \varepsilon_\theta(0) \end{pmatrix}
\]

where \( v = \frac{p_w}{\rho} \) and \( \omega = \frac{p_\theta}{I_\rho} \) are the linear and rotational velocities of the cross-section, \( F = K \varepsilon_w \) the shear force, and \( \tau = E I \varepsilon_\theta \) the bending torque. The input \( u \) and output \( y \) are chosen according to (5) and (8) as follows:

\[
u = \frac{1}{\sqrt{2}} \begin{pmatrix} I & 0 & 0 & -I \\ I & 0 & 0 & I \end{pmatrix} \begin{pmatrix} f_\theta \\ \varepsilon_\theta \end{pmatrix} = -W \begin{pmatrix} -F(0) \\ -\tau(0) \end{pmatrix}
\]

\[
y = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & I & 0 & 0 \\ 0 & 0 & I & 0 \end{pmatrix} \begin{pmatrix} f_\theta \\ \varepsilon_\theta \end{pmatrix} = \begin{pmatrix} v(0) \\ \omega(0) \\ v(L) \\ \omega(L) \end{pmatrix} = \tilde{W} \begin{pmatrix} \varepsilon(0) \\ \varepsilon_\theta(0) \\ \varepsilon(L) \\ \varepsilon_\theta(L) \end{pmatrix}
\]

The existence of the semigroup follows from the fact that \( W \tilde{W}^T \geq 0 \) under the same hypothesis of Theorem 2.1. See also [14] for further details.
preserved. Given the auxiliary input \( u' \), and a \( 2 \times 2 \) symmetric and positive matrix \( \Xi \), let us define
\[
\begin{bmatrix}
\xi_1 & 0 \\
0 & \xi_2
\end{bmatrix}
\begin{bmatrix}
\xi_1 & \xi_2 > 0
\end{bmatrix}
\]  
from (43) the energy balance relation becomes
\[
\frac{d}{dt} H_{cl} = -\left(\xi_1 v^2(0) + \xi_2 \omega^2(0)\right)
\]
with
\[
F(0) = \xi_1 v(0), \quad \tau(0) = \xi_2 \omega(0)
\]
Consequently,
\[
\frac{d}{dt} H_{cl} = -\left(\frac{\xi_1}{2} v^2(0) + \frac{\xi_2}{2} \omega^2(0) + \frac{F^2(0)}{2\xi_1} + \frac{\tau^2(0)}{2\xi_2}\right)
\]
\[
\leq -\xi \left(v^2(0) + \omega^2(0) + F^2(0) + \tau^2(0)\right)
\]
being \( \xi = \frac{1}{2} \min\{\xi_1, \xi_2, \xi_1^{-1}, \xi_2^{-1}\} \). So, asymptotic stability follows from the second condition in (10).

V. CONCLUSIONS AND FUTURE WORKS

In this paper, an energy shaping procedure based on the existence of Casimir functions in closed-loop is presented for linear, lossless, distributed port-Hamiltonian systems. This contribution heavily relies on [13] as far as the definition of boundary control systems in the port-Hamiltonian setting is concerned, and on [15] for the proof of asymptotic stability of the damping injection (boundary) action. Here, conditions for the existence of Casimir functions are provided, together with their characterizations in terms of the regulator structure. The procedure is applied to the stabilization of a forced equilibrium of a Timoshenko beam, namely the case in which a constant force is applied at one extremity of the spatial domain, and fully actuation is present at the other side. Future research will deal with the applicability of the method for different choices of boundary inputs and outputs, e.g. when a scattering port is defined on the boundary of the distributed port-Hamiltonian systems. Moreover, the possible extension of the asymptotic stability criterion to relax the hypothesis of full dissipation at one side of the spatial domain will be investigated.

ACKNOWLEDGMENTS

Part of this work was carried out while the author was visiting the Institute of Automatic Control and Control Systems Technology at the Johannes Kepler University in Linz, Austria. The hospitality of this institution is gratefully acknowledged.

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