On the propagation of instability in interconnected networks

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Abstract—We consider how instability, when due to local interactions between agents in one part of a network, affects other parts of the network. In this initial work, we consider a stable bipartite system with homogeneous linear dynamics in each partition. The initially stable system is driven to the onset of instability by a local gain perturbation and we define a measure which indicates how the size of the resulting oscillations decays with nodal distance. For interconnections defined on $d = 1, 2$ dimensional lattices, we determine the asymptotic value of this measure, as the size of the network increases, using a Markov chain framework. In addition, approximate results are given for interconnection topologies described by a classical random graph model.

I. INTRODUCTION

Complex networks are typically made up of a large number of interacting dynamical systems. Examples include communication networks, economic networks and power networks. Substantial research effort has been devoted to establishing stability results for interconnected systems. However, little is known about how instability propagates spatially through an interconnection when it arises due to local interactions between agents in one part of the network. For example, the congestion control algorithm of the Internet may exhibit oscillatory behaviour due to local feedback interactions between users and routers. If it were just one user causing the problem, though, we would not expect this to be a problem for the network at large.

We consider a bipartite network of linearly interconnected systems. This is a very general setup, and applies whenever there are two classes of interconnected objects (eg generators/loads and transmission lines) and the interconnection is via physical variables that are summed or shared. For this initial study, the dynamics in each partition are assumed to be linear, time invariant and homogeneous. Oscillations may be induced in a stable network by the addition of new agents or by perturbation of the local dynamics. We are interested in studying how these oscillations distribute spatially across the network. Undoubtedly, the dynamics of the agents play a crucial role in determining the network response; but the structure of the underlying network is also a decisive factor. It is this interplay between the dynamics and the topology that underpins the development of this paper.

The paper is structured as follows. We begin in Section II with notation and preliminary results. The problem formulation is presented in Section III. In Section V, the decay rate for the aggregate amplitude of oscillations in $d = 1, 2$ dimensional lattices is determined using a Markov chain framework developed in Section IV. Our key result is that these decay rates are shown to approach the same asymptotic value as spatial distance from the perturbation increases. Furthermore, since lattices in higher dimensions have more nodes at any distance, it follows that the decay rate for the average amplitude of oscillations increases with the lattice dimension. Moreover, the asymptotic decay rate depends in a simple fashion on the margin of stability of the unperturbed system. Similar conclusions are observed to hold for classical random graphs considered in Section VII.

II. NOTATIONS AND PRELIMINARIES

The symbols $\mathbb{Z}, \mathbb{R}, \mathbb{C}$ denote the integer, real and complex numbers. The set of all positive reals is denoted by $\mathbb{R}_+$. Matrices of dimension $m$ by $n$ are denoted by $\mathbb{R}^{m \times n}$, with elements in the corresponding fields. The spectrum of a square matrix $A$ is denoted by $\sigma(A)$ and $\rho(A)$ is its spectral radius. A matrix $A$ is called nonnegative (positive), and denoted $A \geq (>0)$, if $a_{ij} \geq (>0)$ where $a_{ij}$ is the $(i,j)$ element of $A$. A nonnegative matrix is row stochastic if $\sum_j a_{ij} = 1$ and is primitive if there exists a natural number, $k$, such that $A^k$ is positive. The matrix with elements $x_i$ on the leading diagonal and zeros elsewhere is denoted by diag$(x_i)$ and $\text{Co}(S)$ denotes the convex hull of a set $S$. The space of transfer function of stable, linear, time invariant continuous time system with continuous frequency response is denoted by $\mathcal{H}_0$.

In this paper, a network is represented using a directed graph $G = (V,E)$ with the set of nodes $V = \{v_1, \ldots, v_N\}$ and edges $E \subset V \times V$. The adjacency matrix of $G$ is an $N \times N$ binary matrix $A$ where $a_{ij} = 1$ if $(v_i, v_j) \in E$ and is 0 otherwise. A graph is undirected if $(v_i, v_j) \in E \iff (v_j, v_i) \in E$. The neighbours of a node $v_i$ are defined as $J(v_i) = \{v_j \mid (v_j, v_i) \in E\}$ while the degree specifies the number of neighbours. $d(v_i) = |J(v_i)|$. A regular graph is a graph where each vertex has the same number of neighbours. The distance between $v_i$ and $v_j$, $\text{dist}(v_i, v_j)$, is the smallest $k$ for which the $(i,j)$ element of $A^k$ is positive. The diameter of a graph $\text{diam}(G)$ is the maximum distance between any two vertices. A directed graph is called strongly connected if there exists a path from each vertex in the graph to every other vertex following the edge direction.

III. PROBLEM SETUP & MAIN RESULTS

We consider a bipartite interconnected system with two disjoint sets of dynamics, $(f_i(s))_{i=1}^N$ and $(h_j(s))_{j=1}^M$, $f_i(s), h_j(s) \in \mathcal{H}_0$, that are represented by the vertex set $\bar{V} = \{v_i\}_{i=1}^N$ and $\{u_j\}_{j=1}^M$ respectively. Dynamics from one set are connected only to those from the other set and we assume
there are no self loops or multiple edges. The interconnection is defined by
\begin{align}
y(s) &= G(s)u(s) \quad (1) \\
u(s) &= D^{-1}A y(s)
\end{align}
where
\[ F(s) = \text{diag}(f_1(s), \ldots, f_N(s)) \quad \text{and} \quad H(s) = \text{diag}(h_1(s), \ldots, h_M(s)) \]
The matrix \( A \) is the adjacency matrix and \( D = \text{diag}(d_1, \ldots, d_{N+M}) \). \( d_j = \Sigma_j a_{ij} \) denotes the degree matrix. The return ratio of the interconnected system is \( L(s) = F(s)D_F^{-1}R^T H(s)D_H^{-1}R \).

**Definition 1.** The one-mode projection \( \hat{G} \) of a bipartite graph \( G \) is defined such that any two vertices \( v_i, v_j \in V \) is connected if they share a common neighbour. Specifically, \( \langle v_i, v_j \rangle \in E(\hat{G}) \) if and only if \( \langle v_j, u_k \rangle, \langle u_k, v_i \rangle \in E(G) \).

Suppose \( F(s) = f(s)I_M \) and \( H(s) = h(s)I_M \), Lemma 4 in the Appendix shows that the interconnection defined by (1)-(2) is stable, using the multivariable Nyquist stability criterion [5], if
\[ -1 \notin \text{Co}\{f(j\omega)h(j\omega), 0\}, \quad \forall \omega \in \mathbb{R}_+. \quad (3) \]
In other words, stability is guaranteed for any arbitrary interconnections defined in (2) if and only if this condition is satisfied, i.e., the gain margin of \( l(s) = f(s)h(s) \) is greater than one. Suppose that \( -1 < l(j\omega) < 0, \angle l(j\omega) = -\pi \) where \( \omega_0 \in \mathbb{R}_+ \), the stable interconnection is perturbed by a local gain perturbation imposed on \( v_1 \), without any loss of generality, such that \( -1 \in \sigma(L_p(j\omega_0)) \), \( \omega_0 \in \mathbb{R}_+ \) where \( L_p(s) = KL(s) \) and \( K = \text{diag}(k, 1, \ldots, 1) \). Proposition 1 in the Appendix shows that there exists such \( k > 1 \). The system exhibits oscillatory behaviour in steady state and we study the spatial distribution of these oscillations. Let \( \theta \) be the vector of input phasors in steady state for agents on the one-mode projection graph, then
\[ \theta = \lambda \theta K\tilde{A} \quad \text{where} \quad \lambda = -f(j\omega_0)h(j\omega_0) > 0, \quad (4) \]
\[ \theta = (\theta_1 \ldots \theta_N) \quad \text{and} \quad \tilde{A} = D_F^{-1}R^T D_H^{-1}R \]
where \( \theta_1 \in \mathbb{R}_+ \) and \( \theta_2 \in \mathbb{R}_+^{N-1} \) and \( \tilde{A} \) is partitioned conformally with \( \theta \).

Hence,
\[ \theta_2 = \theta_1 k \lambda \tilde{A}_{12} (I - \lambda \tilde{A}_{22})^{-1}. \quad (6) \]
The critical gain \( k \), which is given by
\[ k = (\lambda (\tilde{A}_{11} + \lambda \tilde{A}_{12} (I - \lambda \tilde{A}_{22})^{-1} \tilde{A}_{21}))^{-1}, \quad (7) \]
depends on the dynamics of the agents and the interconnection topology but is not required in order to state the results. Let \( \psi_i = \sum_{j; d_{ij} = 2(i-1)} \theta_j \) be the aggregate amplitude of oscillations at distance \( 2(i-1) \) from \( v_1 \), we write
\[ \psi_i = \psi_i^{(N)} \psi_{i-1}, \quad 2 \leq i \leq \text{diam}(G). \quad (8) \]
The asymptotic value of \( \psi_i^{(N)} \) in the limit of a large graph plays a central role in the analysis of the paper. The following theorem states one of the main results in this paper, the proof of which is relegated to Section V.

**Theorem 1.** Consider the interconnection given by (1)-(2). Suppose \( F(s) = f(s)I_M, H(s) = h(s)I_M \) and the interconnection is defined on \( d = 1, 2 \) dimensional integer lattice. The exponential decay \( \lim_{i \to N} \psi_i^{(N)} = \frac{1}{\lambda} (1 - \sqrt{1 - \lambda})^2 \) where \( \psi_i^{(N)} \) is given in (8) and \( \lambda \), defined in (4), is the inverse gain margin of the unperturbed system.

**Remark 1.** The decay rate for the average amplitude increases with the lattice dimension \( d \) since lattices in higher dimensions have more nodes at any given distance.

For interconnections defined on the classical random graphs with a Poisson degree distribution and mean degree \( k \), the asymptotic decay rate for the expectation of the aggregate amplitude of oscillations is closely approximated by \( (1 - \sqrt{1 - 4\lambda \mu(1 - \mu)})^2 / 4\lambda \mu^2, \mu = (1 - \exp(k))/k \) as \( N \) increases.

**IV. MARKOV CHAIN FRAMEWORK**

In this section we develop a Markov chain framework for the proof of Theorem 1 as well as more general results. Given a time homogeneous Markov chain with transition probability matrix \( P \), the stationary distribution \( \pi \) satisfies \( \pi P = \pi \). That is, \( \pi \) is a normalised left eigenvector of \( P \) associated with the eigenvalue 1. We show that the relative amplitude of input oscillations can be determined by the stationary distribution of the Markov chain with transition probability matrix \( P \) where
\[ p_{ij} = \lambda \tilde{a}_{ij} + (1 - \lambda) \delta_{ij}, \quad (9) \]
and \( 0 < \lambda < 1, \delta_{ij} = 1 \) if \( i = j \) and 0 otherwise. The Markov chain can be interpreted as the following: Given the current state \( i \), the Markov chain makes a transition to one of the states in \( \{ j : \tilde{a}_{ij} \neq 0 \} \) with probability \( 0 < \lambda < 1 \). Alternatively, it makes a transition to state 1 with a nonzero probability 1 - \( \lambda \). The stationary distribution \( \pi = (\pi_1 \pi_2) \) satisfies
\[ \lambda (\pi_1 \pi_2) \tilde{A} = (\pi_1 - (1 - \lambda) \pi_2). \quad (10) \]
\[ \text{Thus,} \quad \pi_1 = \frac{1 - \lambda}{1 - 1/k} \text{ and} \]
\[ \pi_2 = \pi_1 \lambda \tilde{A}_{12} (I - \lambda \tilde{A}_{22})^{-1} \quad (11) \]
where \( k \) is defined in (7). It follows that \( \theta_2 \) and \( \pi_2 \), given by (6) and (11), lie in the same subspace spanned by the vector \( \lambda \tilde{A}_{12} (I - \lambda \tilde{A}_{22})^{-1} \). More importantly,
\[ \theta_2 = k \pi_2 / \pi_1 \quad (12) \]
if \( \theta_1 = 1 \). This framework allows the vector of input phasors \( \theta \) to be computed in a Markov chain setting. Moreover, the ratio of the aggregate amplitude of oscillations \( \gamma_{\pi}^{(N)} \) for \( 3 \leq i \leq \text{diam}(\tilde{G}) \), as defined in (8), can be determined by

\[
\theta_i = \gamma_{\pi}^{(N)} \theta_{i-1}, \quad 3 \leq i \leq \text{diam}(\tilde{G}).
\]  

Lemma 1. Suppose \( \tilde{A} \) is irreducible. For \( 0 < \lambda < 1 \), the invariant distribution \( \pi = (\pi_1, \ldots, \pi_N) \) of \( P \) given by (9) is

\[
\pi_j = (1 - \lambda) \sum_{k=0}^{\infty} \lambda^k p_{ij}^k, \quad 1 \leq j \leq N
\]  

Proof: It can be easily shown that \( \sum_{j=1}^{N} \pi_j = 1 \) and \( \sum_{j=1}^{N} \pi_j p_{ij} = \pi_j \).

Let the stationary distribution \( \pi \) be partitioned conformally with \( P \), i.e. \( \pi = (\pi_1, \pi_2, \ldots, \pi_N) \), where \( \pi_i \) is a row vector of length \( n_i \) and \( n_i \) is the number of states in block \( i \). It can be easily shown that the stationary distribution \( \pi \) of the aggregate matrix \( S \) with

\[
s_{ij} = \frac{\pi_i}{||\pi||_1} p_{ij}
\]  

is given by \( \tilde{\pi}_i = \pi_i \mathbf{1} \) [14]. The element \( s_{ij} \) represents the transition probability from any states in block \( i \) to any states in block \( j \). If the Markov chain is given a random walk interpretation starting from \( v_1 \), then \( \tilde{\pi}_i \) represents the aggregate probability that the walker ends up at a distance \( 2(i-1) \) from \( v_1 \).

V. FINITE DIMENSIONAL LATTICES

In this section, we derive asymptotic decay rate for the oscillations in \( d = 1, 2 \) dimensional lattices stated in Theorem 1. Each network is bipartite and nodes are ordered in ascending order of their distances from the origin. We assume for simplicity that \( F(s) = f(s)I_N \) and \( H(s) = h(s)I_M \). The asymptotic rate has an important dependence on the lattice dimension.

A. One-dimensional Lattice

Consider a bipartite network where each partition consists of \( N \) identical agents arranged in an one-sided one-dimensional string. We study the instability propagation problem as the size of the network increases. The interconnection is specified by

\[
R = \begin{pmatrix} 1 & 1 & 0 & \ldots & 0 \\ 0 & 1 & 1 & \ldots & 0 \\ 0 & 0 & 1 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \ldots & 1 \end{pmatrix}.
\]  

Using the Markov chain framework, the stationary distribution of \( P \) defined in (9), where \( \Lambda = D_F^{-1}R^T D_P^1 R \), satisfies

\[
\frac{1}{4} \lambda \pi_{i+1} - (1 - \frac{1}{4} \lambda) \pi_i + \frac{1}{4} \lambda \pi_{i-1} = 0
\]  

for \( 3 \leq i \leq N - 1 \) and similarly,

\[
\frac{1}{4} \lambda \pi_{N-1} - (1 - \frac{1}{4} \lambda) \pi_N = 0.
\]  

Using (13), simple substitution yields

\[
\gamma_{\pi}^{(N)} = \begin{cases} \frac{1}{4} \lambda (1 - \frac{3}{4} \lambda)^{-1} & i = N \\ \frac{1}{4} \lambda \left(1 - \frac{1}{4} \lambda - \frac{1}{4} \lambda \gamma_{\pi}^{(N)}\right)^{-1} & 3 \leq i \leq N - 1 \end{cases}
\]  

The asymptotic solutions of (19), in the limit as \( N \to \infty \), are

\[
\gamma(\lambda) = \frac{1}{\lambda} \left(1 \pm \sqrt{1 - \lambda}\right)^2.
\]  

We will now prove Theorem 1 for the one-dimensional lattice by showing

\[
\lim_{N \to \infty} \gamma_{\pi}^{(N)} = \frac{1}{\lambda} \left(1 - \sqrt{1 - \lambda}\right)^2, \quad \forall i > 2.
\]  

Proof. For \( 0 < \lambda < 1 \), it can be easily shown that \( \frac{1}{4} \lambda \left(1 + \sqrt{1 - \lambda}\right)^2 > 1 \). On the other hand, \( 0 < \frac{1}{4} \lambda \left(1 + \sqrt{1 - \lambda}\right)^2 < 1 \) as it is a monotonically increasing function bounded above and below by 1 and 0 respectively. By definition, \( \pi \mathbf{1} = 1 \). Hence, the asymptotic value of \( \gamma_{\pi}^{(N)} \) satisfies (21) in the limit of large \( N \) for all \( i > 2 \).

We conclude that the amplitude of oscillations decreases monotonically as the distance from the perturbed node increases. Fig. 1 shows the asymptotic decay rate for \( \lambda \in [0,1] \). It clearly indicates that instability decays faster with a smaller \( \lambda \) as the unperturbed network has a greater stability margin. Note also that the asymptotic decay rate increases linearly with \( \lambda \) when \( \lambda \) is small. That is, \( \gamma(\lambda) \to \frac{1}{\lambda} \) as \( \lambda \to 0 \) for the one-dimensional lattice. In other words, when \( \lambda \) is small, the decay rate is determined predominantly by the preceding node in the forward path.

B. Two-dimensional Lattice

It is assumed that the perturbed node \( v_1 \) resides at the origin of a two-dimensional lattice. The decay rate for the aggregate amplitude of oscillations will be used here as a performance measure and we study how this quantity behaves asymptotically as the size of the network increases. For the one-dimensional lattice, the oscillations decay at a uniform rate as the distance distribution of a given vertex is symmetric, i.e. the fraction of neighbours located closer and further away from the origin is equal. However, for the two-dimensional lattice, vertices along the axes have greater proportion of neighbours located further away from the origin. We intend to show that the effect of these vertices becomes vanishingly small as spatial distance from the origin increases.
increases and the asymptotic decay rate for the aggregate amplitude of oscillations is identical for the one- and two-dimensional lattices. In other words, for the two-dimensional lattice, the aggregate one-step transition probabilities satisfy

\[
\lim_{N \to \infty} \lim_{i \to \infty} s_{i,j+1} = \lim_{N \to \infty} \lim_{i \to \infty} s_{i,j} + \frac{1}{4} \lambda, \quad \lim_{N \to \infty} \lim_{i \to \infty} s_{i,j} = \frac{1}{2} \lambda.
\]

(22)

The Markov chain with transition matrix \( P \) in (9) specifies a folding\(^2\) on the one-mode projection of the two-dimensional lattice due to its symmetry, as shown in Fig. 2. Let the vertices be referred to by their Cartesian coordinates \( v(x,y), x, y \in \mathbb{Z} \). On the one-mode projection graph, the aggregate distribution satisfies \( \pi = \pi S \) where \( \pi_i = \sum_{(x,y) \in C_i} \pi(x,y) \) and \( C_i = \{ v(x,y) : \text{dist} (v(x,y), v(0,0)) = 2(i-1) \} \). Exploiting the symmetry of the lattice, the elements of the aggregate transition matrix \( S \) are given by

\[
s_{ij} = \begin{cases} \frac{\lambda}{2} - \frac{3\lambda}{16} (4\pi_1) - \frac{\lambda}{16} (8\pi_2), & j = i - 1 \\ \frac{\lambda}{2} + \frac{3\lambda}{16} (8\pi_2), & j = i \\ \frac{\lambda}{2} + \frac{3\lambda}{16} (4\pi_1), & j = i + 1. \end{cases}
\]

(23)

for \( 2 \leq i \leq \text{diam}(\tilde{G}) - 1 \) where

\[
\pi_k = \frac{\pi(k-1,d_i-k+1)}{\sum_{(x,y) \in C_i} \pi(x,y)}, \quad d_i = 2(i-1)
\]

(24)

is defined for \( 1 \leq k \leq i \) and \( \pi(x,y) \) denotes the long-run proportion of visits of the Markov chain to vertex \( v(x,y) \). In other words, \( \pi_{ik} \) denotes the relative probability of visits to a vertex in the subgraph \( C_i = \{ v(k,d_i-k) : d_i = 2(i-1), 0 \leq k \leq i-1 \} \) given that the random walk visits any vertex in \( C_i \). A detailed derivation of (23) is given in [9].

We introduce a path counting technique to show that the transition probabilities, given by (23), satisfy (22). For \( P \) in (9), Lemma 1 indicates that the stationary distribution is described by \( \pi_j = (1 - \lambda) \sum_{n=0}^{\infty} \lambda^n d_{ij}^{(n)}, v_j \in V \). For the finite two-dimensional lattice, it can be easily established that \( d_{ij}^{(n)} = (1/4)^n q_{ij}^{(n)}, v2n \leq \text{diam}(G) \) where \( d_{ij}^{(n)} \) denotes the number of paths of length \( 2n \) that connects \( v_1 \) to \( v_j \) in an infinite lattice. Therefore,

\[
\lim_{N \to \infty} \pi_j = (1 - \lambda) \sum_{n=0}^{\infty} (\lambda/16)^n q_{ij}^{(n)}.
\]

(25)

It can be shown that the number of paths of a fixed length \( l \) from the origin to a vertex \( v(x,y) \) is given by

\[
q_l(x,y) = \left( \frac{1}{n} \right) \left( \frac{1}{n+l} \right)
\]

(26)

where \( l = 2n + x + y \) and \( q_l(x,y)/q_l(x+1,y-1) < 1 \) if \( x+1 < y \). It follows that at an even distance \( d \) from the origin, \( q_l(k,d-k), 0 \leq k \leq d/2 \) is strictly monotonically increasing. From (25), we conclude that \( \pi_{ik} \), defined in (24), is monotonically increasing with \( k \) in the limit of a large graph.

**Lemma 2.** Consider the modified random walk on the two-dimensional lattice with \( P \) defined by (9). Given \( \pi_{ij} \) defined by (24), \( \lim_{d \to \infty} \lim_{N \to \infty} \pi_{ik} = 0 \), for \( k = 1, 2 \).

**Proof.** By definition, \( 4\pi_1 + \sum_{2 \leq k \leq 1 - 8\pi_k + 4\pi_{ii} = 1} \). Since the graph is strongly connected, \( \pi_{ij} > 0, \forall i, 1 \leq j \leq i \). Suppose \( \pi_{ik} \) is monotonically increasing with \( k \), then

\[
8(i-1)\pi_{ii} < 1 \quad \text{and} \quad \pi_{ii} < 1/(8(i-1)).
\]

(27)

Hence, given any \( \varepsilon > 0, \pi_{ii} < \varepsilon, \forall i > 1/\varepsilon \) and thus \( \lim_{d \to \infty} \lim_{N \to \infty} \pi_{ij} = 0 \). Using a similar argument, \( \pi_{ij} < \varepsilon, \forall i > 1/\varepsilon \) and \( \lim_{d \to \infty} \lim_{N \to \infty} \pi_{ii} = 0 \).

The proof of Theorem 1 for the two-dimensional lattice is given next using the Markov chain framework.

**Proof.** The aggregate distribution of the modified random walk on the two-dimensional lattice can be determined by

\[
0 = s_{i+1,j} \pi_{i+1,j} + s_{i,j} \pi_{i-1,j} - s_{i,j} \pi_{i,j} - s_{i,j} \pi_{i,j+1} + s_{i,j} \pi_{i,j-1} + s_{i,j} \pi_{i,j+1} + s_{i,j} \pi_{i,j-1}
\]

(28)

for \( 2 \leq i \leq \text{diam}(G) - 1 \) where \( s_{ij} \), given by (23), satisfy (22) as a direct consequence of Lemma 2. Let the ratio of the aggregate distribution be as defined in (13). The Poincaré’s theorem [11] indicates that \( \lim_{d \to \infty} \lim_{N \to \infty} \gamma_i^{(N)} = \gamma \) where \( \gamma \) is one of the roots of the characteristic equation

\[
z^2 - \frac{4}{\lambda} \left( 1 - \frac{1}{2} \lambda \right) z + 1 = 0,
\]

(29)

which are \( \gamma(\lambda) = \frac{1}{2} \sqrt{1 - \lambda} \). Given that \( \gamma(1) > 1 \) whereas \( 0 < \gamma < 1 \) for \( 0 < \lambda < 1 \), we conclude that \( \gamma = \gamma(1) \), since \( \sum_i \pi_i = 1 \).

**VI. Example**

In this section, we consider the following example:

\[
f(s) = h(s) = \frac{k_1}{s^2 + 2s + 1}.
\]

(30)

For \( k_1 = 1 \), Fig. 3a shows the output response of a general network when the system is driven to the verge of instability by a local gain perturbation. It is clear that the amplitude of oscillations decays monotonically as node distance from \( v_1 \) increases. For \( d = 1, 2 \) dimensional lattices, Fig. 3b shows that the aggregate amplitude of oscillations decays asymptotically at the same rate as distance from \( v_1 \) increases. This agrees with the results established in Theorem 1.

\[\text{Fig. 2: (a) A two-dimensional lattice and (b) its image under the folding.}\]
VII. RANDOM GRAPHS

Thus far, we have considered finite dimensional regular lattices. However, complex networks such as the Internet and the power networks are better modelled by random graphs. We consider the classical bipartite random graphs \( \mathcal{G}(N, p) \) with two sets of vertices \( V \) and \( U \) where \( |V| = |U| = N \). Edges between vertices from one set to another are selected independently and with probability \( p \). The degree distribution \( k \) of such graphs is binomial and can be approximated by a Poisson distribution for large \( N \). The decay rate for the aggregate mean of oscillations, averaged over all possible graphs, is studied for random graphs with sparse connections using the Markov chain framework.

It can be shown that the neighbourhood within a fixed distance \( d \) of a randomly chosen vertex of a random graph \( G(N, c/N) \) converges asymptotically in distribution, in the limit as \( N \) goes to infinity, to the first \( d \) generations of the Galton-Watson branching process with a Poisson offspring distribution \([4]\). The process is constructed as follows: start with a single node at generation 0 and each node in generation \( n \) gives rise to a Poisson number of offspring in generation \( n+1 \) with mean \( c \). For the modified random walk on the random graph with transition matrix \( P \) given by (9), it follows from (15) that

\[
\mathbb{E}(\pi_l 1) = \sum_{m, |m-l| \leq l} \lambda \mathbb{E}(\pi_m \bar{A}_{ml} 1) 
\]  

for \( l \geq 2 \). Let \( C_l = \{v_j : \text{dist}(v_j, v_1) = 2l\} \) be the set of vertices at an even distance \( 2l \) from the vertex \( v_1 \) in the bipartite graph. From (14), we can also write

\[
\mathbb{E}(\pi_m \bar{A}_{ml} 1) 
= (1 - \lambda) \sum_{n=0}^m \lambda^n \mathbb{E}\left( \sum_{v_i \in C_m} \bar{a}_{ii}^{[n]} \sum_{v_j \in C_l} \bar{a}_{ij} \right) 
= (1 - \lambda) \sum_{n=0}^m \lambda^n \mathbb{E}\left( \sum_{v_i \in C_m} \bar{a}_{ii}^{[n]} \right) \mathbb{E}_m\left( \sum_{v_j \in C_l} \bar{a}_{ij} \right) + \epsilon_{m,l} 
\]

\[
= \mathbb{E}(\pi_m 1) \mathbb{E}_m\left( \sum_{v_i \in C_l} \bar{a}_{ii} \right) + \epsilon_{m,l} 
\]  

(32)

where \( \mathbb{E}_m(\sum_{v_j \in C_l} \bar{a}_{ij}) \) denotes the expectation of the aggregate one-step transition probabilities, conditioned on \( v_i \in C_m \), and \( \epsilon_{m,l} \) is a covariance term. If it were true that \( \epsilon_{m,l} = 0, \forall m, l \), simple substitution into (31) would yield a

three-term recurrence equation on the expected aggregate distribution. Furthermore, given that the random graphs are locally tree-like, we define \( s_{ml} = \mathbb{E}_m(\sum_{v_j \in C_l} \bar{a}_{ij}) \) as for a Galton-Watson branching processes,

\[
s_{i-1,l} = (1 - \mu)^2, \quad s_{i,l} = 2\mu(1 - \mu), \quad s_{i+1,l} = \mu^2 \]  

(33)

where \( \mu = \mathbb{E}(1/(k+1)) \) denotes the expected inverse degree of a randomly chosen vertex and \( k \) is Poisson distributed. Consider the recurrence equation

\[
x_i = \lambda(1 - \mu)^2 x_{i-1} + 2\lambda(1 - \mu)x_i + \lambda\mu^2 x_{i+1} \]  

(34)

for \( i > 1 \). The characteristic equation of (34) is

\[
0 = \lambda(1 - \mu)^2 - (1 - 2\lambda\mu(1 - \mu))s + \lambda\mu^2 s^2. \]  

(35)

Suppose \( \gamma = x_i/x_{i-1} \) and \( x_i < 1, \forall i \). Then, \( \gamma = \gamma, \forall i \geq 2 \) where

\[
\gamma = (1 - \sqrt{1 - 4\lambda\mu(1 - \mu)})^2/4\lambda\mu^2, \]  

(36)

which is the stable root of (35). For the random graphs, numerical simulations show that the asymptotic decay rate for the expected aggregate distribution of the modified random walk on the one-mode projection is closely approximated by (36) where \( \mu = \mathbb{E}\left(\frac{1}{k+1}\right) = \frac{1}{d} \left(1 - \exp(-c)\right) \) since \( k \) is Poisson distributed with mean \( c \). This is shown in Fig. 4. Furthermore, \( \gamma \) increases with the mean degree \( c \). As we have shown earlier, the asymptotic decay rate for the expected aggregate distribution is only given by (36) if \( \epsilon_{m,l} = 0, \forall m, l \) in (32) and the tree approximation is valid in the limit of a large graph. Given that \( \bar{a}_{ij} \) is the product of the inverse degree of vertices on a walk of length \( 2n \) connecting \( v_1 \) and \( v_j \) on the bipartite graph, summed over all possible walks of length \( 2n \), vertices may be traversed more than once on a walk of length \( 2n, n > l \), the term \( \epsilon_{m,l} \) is not identically zero in general. Readers are referred to [9] for more details. Equation (36) only provides a close approximation for the decay rate as suggested by the simulation results.

VIII. DISCUSSION

Consider the example in Section VI. For \( k_1 = 1.99 \), Fig. 5 compares the average amplitude of oscillations in \( d = 1, 2 \) dimensional lattices and classical random graphs with mean degree \( \langle k \rangle \). For regular lattices, the decay rate increases with the lattice dimension. For illustrative purposes, the average amplitude of oscillations for random graphs is shown based on the close approximation provided by (36). The average amplitude of oscillations decays more rapidly in random graphs compared to that of the regular lattices. A one-dimensional lattice is a deterministic tree where each vertex has \( k = 1 \) offspring and the asymptotic decay rate can be determined by (36) with \( \mu = \frac{1}{k+1} \). Given that \( \mathbb{E}\left(\frac{1}{k+1}\right) > \frac{1}{d(k+1)} \), the average amplitude of oscillations decays more rapidly for the the random graphs compared to a deterministic tree of the same growth. For the random graphs with mean degree \( \langle k \rangle = 1.5936 \), such that \( \mathbb{E}\left(\frac{1}{k}\right) = \frac{1}{\langle k \rangle} \), the average amplitude of oscillations decays more rapidly for the random graphs compared to the regular lattices as the number of nodes at a given distance from a randomly chosen node grows exponentially with the mean degree of the vertex.
which correspond to the eigenvector $\mathbf{1}$ for all graphs. Hence, $\rho(\tilde{A}) \leq 1$ since $\min \sum_j a_{ij} \leq \rho(\tilde{A}) \leq \max \sum_j a_{ij}$ [8] and the magnitude of all other eigenvalues of $\tilde{A}$ is strictly less than 1 by the Perron-Frobenius theorem if $\tilde{A}$ is primitive. To prove iii), $\sigma(\tilde{A}) \cup 0 = \sigma(\tilde{R}^T \tilde{R}) \cup 0$ where $\tilde{R} = \tilde{D}_H^{-1/2} \tilde{D}_F^{-1/2}$. This completes the proof since $\tilde{R}^T \tilde{R}$ is positive semidefinite. \qed

**Lemma 4.** Suppose $F(s) = f(s)I_n, H(s) = h(s)I_n$, the interconnected system described in (2) is stable if

$$-1 \notin \text{Co}(\{f(j\omega)h(j\omega)\cup 0\}), \forall \omega \in \mathbb{R}_+. \quad (37)$$

**Proof.** The multivariable Nyquist stability criterion states that the interconnection is stable if and only if the eigenloci of the return ratio does not encircle the critical point, $(-1,0)$, on the complex plane [5]. Since $L(s) = f(s)h(s)\tilde{A}$ and $\sigma(\tilde{A}) \subset [0,1]$ (Lemma 3), $\sigma(L(j\omega)) \subset \text{Co}(\{f(j\omega)h(j\omega)\cup 0\}), \forall \omega \in \mathbb{R}_+$. Thus, the system is stable for any arbitrary interconnection if and only if the Nyquist locus of $l(s) = f(s)h(s)$ does not encircle $(-1,0)$. \qed

**Proposition 1.** Let $\tilde{A} \in \mathbb{C}^{n \times n}$ be a nonnegative stochastic matrix and $0 < \lambda < 1$, there exists $k > 1$ such that $\rho(\lambda \tilde{A}) = 1$ where $K = \text{diag}(k,1,...,1)$.

**Proof.** This follows immediately from the fact that $\rho(\tilde{A}) = 1$ if $\tilde{A}$ is stochastic and eigenvalues of a matrix are continuous functions of the matrix elements [8]. \qed

**IX. CONCLUSIONS AND FUTURE WORKS**

We have studied instability propagation problem in interconnected bipartite networks and showed that oscillations in a network, due to a local gain perturbation, decay away from the perturbation under homogeneity assumptions on the agents dynamics. Using a Markov chain framework, the decay rates for the aggregate amplitude of oscillations in $d = 1, 2$ dimensional lattices are shown to converge asymptotically to the same value as distance from the perturbation increases. Specifically, the asymptotic decay rate is characterised by the stability robustness measure and the distance distribution of the nearest neighbours of a given vertex, i.e. the fraction of neighbours located at a smaller or greater distance from the perturbed node. Similar conclusions are observed to hold for classical random networks using a local tree approximation. Further extension to networks with nonlinear dynamics, and particularly on the propagation of nonlinear oscillations in communication networks, has been investigated and the results will be published elsewhere.

**APPENDIX**

Proofs for the lemmas encountered earlier in this paper are now presented.

**Lemma 3.** Let $G$ be a strongly connected bipartite graph. $\tilde{A}$ satisfies the following properties:

i. $\tilde{A}$ is row stochastic, nonnegative and primitive.

ii. $\rho(\tilde{A}) = 1$ and it is a simple eigenvalue of $\tilde{A}$.

iii. All eigenvalues of $\tilde{A}$ are real and nonnegative.

**Proof.** For i), $\tilde{A}\mathbf{1} = D_H^{-1}R^TD_H^{-1}\mathbf{1} = D_F^{-1}R^T\mathbf{1} = \mathbf{1}$ where $\mathbf{1} = (1,1,...,1)^T$. Thus, $\tilde{A}$ always has an eigenvalue $1$

**REFERENCES**


