Change of controller based on partial feedback linearization with time-varying function

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Abstract—This paper considers the problem to transfer the state from one zero dynamics submanifold to another one in finite time, for a time-invariant nonlinear system. The usage of time-varying zero dynamics submanifold is proposed to accomplish the transfer. The feature of this paper is focus on keeping the state on zero dynamics submanifold during the transfer. Main contribution is to develop the condition for doing this for the case that the two zero dynamics submanifolds have the same dimension. The validity of the controller that is designed to satisfy the condition is demonstrated via a numerical simulation of mono-rotor unmanned aerial vehicle (UAV) system.

I. INTRODUCTION

Mechanical systems with restricted numbers of actuators, called underactuated systems, has gathered much attention since there are a lot of meaningful applications such as locomotion [1], [2], [3], [4], human like motion [5], [6], [7], etc. Generally, an underactuated mechanical system is not feedback linearizable and non-minimum phase. For such systems, nonlinearities in dynamics can not be removed by feedback transformation, therefore, it is challenging task to control underactuated systems such as motion planning and motion stabilization.

One of the major approaches to controlling underactuated mechanical systems is partial feedback linearization. This method, summarized in [8], [9] and [10], divides a system into linear and nonlinear subsystems. The dynamics of these subsystems are called external and internal dynamics respectively. Some motion control problems whose desired motion can be expressed by linearizing coordinates are solved by this method since the states of linear subsystem easily converge to zero by simple linear controller. After zeroing the states of linear subsystem, the whole system’s dynamics is dominated by only internal dynamics, and it is called zero dynamics. Thus, it is significantly important to take zero dynamics into consideration for design of linearizing coordinate.

In order to achieve several tasks, the change of controller is required. In the previous researches, controller switching via partial feedback linearization has been applied to stabilize a system. For example, Spong [11] achieved to stabilize a robot system via a switching a nonlinear controller based on partial feedback linearization and a linear controller via linear approximation. However there are few researches that discuss the switching between nonlinear controllers constructed by partial feedback linearization.

In this paper, we consider controller switching based on partial feedback linearization, and we focus on the switching after zeroing the states of linear subsystem. Moreover, we focuses on the condition to keep zero states of linearized subsystem during a controller switching. This condition is required since it is difficult to analyze the dynamics outside the zero dynamics submanifold.

This paper is organized as follows. Section II gives mathematical preliminary and introduces the relative degree structure of a system. After that, we discuss the property of zero dynamics and the problem statement is provided. Section III provides main result of this paper: analysis of time depend functions with time invariant nonlinear system and expression of the condition for smooth change of time-varying linearizing coordinate. In Section IV, proposed controller changing scheme is applied to a mono-rotor UAV system. Section V concludes the paper.

II. PRELIMINARY

A. Notations and Definitions

In this paper, we consider an input-affine time-invariant system

$$\dot{x} = f(x) + \sum_{i=1}^{m} g_i(x)u_i = f(x) + G(x)u$$ (1)

where $x = [x_1, \ldots, x_r]^T \in M$ is state, $u = [u_1, \ldots, u_m]^T$ is input, and $f, g_1, \ldots, g_m$ are smooth vector fields on the state space $M \subset \mathbb{R}^n$. The space spanned by smooth vector fields $X_1, \ldots, X_r$ are called distribution and denoted by span $\{X_1, \ldots, X_r\}$. The important property of a distribution is involutiveness. A distribution said to be involutive if it is closed under Lie bracket, where Lie bracket is the operation between two vector fields $X_1, X_2$ such that $\frac{\partial X_2}{\partial x}\cdot X_1 - \frac{\partial X_1}{\partial x}\cdot X_2$ and denoted by $[X_1, X_2]$. The smallest involutive distribution which includes the distribution $\Delta$ is called involutive closure of $\Delta$ and referred to as $\bar{\Delta}$. dim $\Delta$ denotes the dimension of the distribution $\Delta$. We suppose that all distributions defined below are smooth and regular, that is, their dimensions are constant over $M$, and we sometimes omit the coordinate representation $(x)$ for simplicity.

For the system (1), the following distributions are defined.

$$G^1 = G = \text{span} \{g_1, \ldots, g_m\},$$
$$G^i = \text{span} \{\text{ad}_f^{-i+1}g, G^{i-1}\}, \quad i \geq 2, \quad (2)$$

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where \( \text{ad}_f^g = [f, \text{ad}_f^{-1} g], \text{ad}_f g = [f, g], \text{ad}_f g = \{ \text{ad}_f g \mid g \in \mathcal{G} \} \) and \( \text{ad}_f^{-1} g = \{ \text{ad}_f^{-1} g \mid g \in \mathcal{G} \} \). The equation (2) means that there are nested structures \( \mathcal{G}^i \subset \mathcal{G}^{i+1} \) and \( \mathcal{G}^i \subset \mathcal{G}^{i+1} \).

\[ \mathcal{L}_f h \] denotes the Lie derivative of a scalar function \( h \) along with a vector field \( f \), and the function \( \mathcal{L}_f h \) denotes the recursion \( \mathcal{L}_f \left( \mathcal{L}_f^{-1} h \right) \) with \( \mathcal{L}_f h = h \). The definition of a relative degree is described by Lie derivative as follows.

**Definition 1 (Relative degree of function [10]):** The relative degree \( r \) of a scalar function \( h(x) \) is a natural number satisfying

\[
\mathcal{L}_{g_i} \mathcal{L}_{f}^{j} h = 0, \quad i \in \{0, \ldots, r - 2\}, \quad k = 1, \ldots, m
\]

\[
\{ \mathcal{L}_{g_i} \mathcal{L}_{f}^{j} h, \ldots, \mathcal{L}_{g_m} \mathcal{L}_{f}^{j} h \} \neq 0.
\]

If there is no \( r \) satisfying (3), the relative degree is defined as \( \infty \).

Following definition is the relative degree for a system.

**Definition 2 (Vector relative degree of system [10]):** The system (1) with functions \( h = \{h_1, \ldots, h_m\} \) has the vector relative degree \( \{k_1, \ldots, k_m\} \) on \( M \) if \( k_i \) is a relative degree of \( h_i \) for \( i = 1, \ldots, m \), and the matrix \( \mathcal{L}_g \mathcal{L}_f^{k-1} h \) in (4) is nonsingular on \( M \).

\[
\mathcal{L}_g \mathcal{L}_f^{k-1} h = \begin{bmatrix}
\mathcal{L}_{g_j} \mathcal{L}_{f}^{k-j} h_1 & \ldots & \mathcal{L}_{g_m} \mathcal{L}_{f}^{k-m} h_1 \\
\mathcal{L}_{g_j} \mathcal{L}_{f}^{k-j} h_m & \ldots & \mathcal{L}_{g_m} \mathcal{L}_{f}^{k-m} h_m
\end{bmatrix}
\]

(4)

The matrix (4) is often called decoupling matrix. If a system with functions \( \{h_1, \ldots, h_m\} \) has the vector relative degree \( \{k_1, \ldots, k_m\} \), then the linear subsystem via partial feedback linearization has the controllability indices \( \{k_1, \ldots, k_m\} \). The functions \( \{h_1, \ldots, h_m\} \) whose relative degrees correspond to the vector relative degree are called linearizing coordinate.

The notion of the system structure based on the relative degree is defined as follows.

**Definition 3 (Relative Degree Structure [12]):** The relative degree structure of the system with \( n \) states and \( m \) inputs is defined as the pair of two sets of indices.

\[
(r_1^*, \ldots, r_n^*) = \{k_1^*, \ldots, k_m^*\},
\]

where each index is defined as below.

\[
r_i^* = r \quad \text{s.t.} \quad \sum_{j=1}^{r} \sigma_j = n - i + 1 \leq \sum_{j=1}^{r} \sigma_j,
\]

\[
k_i^* = 2\{k \mid k \geq i, k \in \mathcal{N}\},
\]

where \( 2\{\} \) denotes the cardinality of the set \( \{\} \) and \( \mathcal{N} \) denotes the natural number \( \{1, 2, \ldots\} \). The indices \( \sigma^i \) and \( \gamma^i \) are

\[
\sigma^i = \dim \mathcal{G}^i - \dim \mathcal{G}^{i-1}, \quad \sigma^0 = 0,
\]

\[
\gamma^i = \dim \mathcal{G}^i - \dim \mathcal{G}^{i-1}.
\]

These indices means that there exists a following coordinate and input transformation for a system with \( (r_1^*, \ldots, r_n^*) = \{k_1^*, \ldots, k_m^*\} \):

\[
\frac{d}{dt} \begin{bmatrix} x_l \\ x_n \end{bmatrix} = \begin{bmatrix} A_x x_l \\ \xi(x_l, x_n) \end{bmatrix} + \begin{bmatrix} B_c \\ \eta(x_l, x_n) \end{bmatrix} v,
\]

where \( (A_x, B_c) \) is Brunovsky canonical form with controllability indices \( \{k_1^*, \ldots, k_m^*\} \), and \( [u_1, \ldots, u_m]^T = \alpha(x_l, x_n) + \beta(x_l, x_n) v \). Moreover \( \{x_l, x_n\} \) have relative degree \( \{r_1^*, \ldots, r_n^*\} \). For more details of the relative degree structure, see [12].

The state of linear subsystem can be zero using the transformed input \( v \), and we call this input zeroing controller. After applying zeroing controller, a dynamics of whole system is trapped into following submanifold:

\[
Z^* = \{ x \in M \mid x_l = 0 \}.
\]

Remaining dynamics on \( Z^* \) is \( \dot{x}_n = \xi(0, x_n) \) and called zero dynamics. The input \( \alpha(0, x_n) \) ties the trajectory to \( Z^* \) and is called equivalent input. In this paper, for clear discussion zero dynamics submanifold with respect to a linearizing coordinate \( x_l \) is denoted by \( Z^*(x_l) \).

Based on the relative degree structure, there exists a new coordinate \( \{\phi_1, \phi_2, \ldots, \phi_n\} \) whose relative degrees are \( \{r_1^*, \ldots, r_n^*\} \). At the end of this section, we introduce the theorem about the parameterization of a function using \( \{\phi_1, \ldots, \phi_n\} \).

**Theorem 1:** [12] Let the coordinate functions \( \{\phi_1, \ldots, \phi_n\} \) have relative degree \( \{r_1^*, \ldots, r_n^*\} \). A scalar function \( h \) has relative degree \( i \) if and only if \( h \) is parameterized as follows:

\[
h = h(\phi_1, \ldots, \phi_j),
\]

where \( j \) is the largest integer such that relative degree of \( \phi_j \) is \( i \), and there exists an integer \( k \) such that \( \frac{\partial h}{\partial \phi_k} \neq 0 \), where relative degree of \( \phi_k \) is \( i \).

This theorem indicates that the coordinate based on the relative degree structure parameterizes an arbitrary scalar function with respect to its relative degree. Note that the parameters \( \phi_1, \ldots, \phi_j \) in (10) are the functions with relative degree \( i \) or higher than \( i \), and the condition \( \frac{\partial h}{\partial \phi_k} \neq 0 \) guarantees that the relative degree of \( h \) is \( i \).

**B. Problem Statement**

The partial feedback linearization is applied to meet various purposes. These purposes are categorized with respect to their tasks:

- to realize virtual constraints, or
- to control whole dynamics including zero dynamics.

The first task is easy to achieve by partial feedback linearization if the number of virtual constraints is smaller than \( m \). In this case, the linearizing coordinate is chosen to describe the virtual constraints, and zeroing control for linearized subsystem achieves the task. This task is often considered
for full-actuated systems, exactly linearizable systems or minimum phase systems since there is no need to consider nonlinear subsystem. The second is challenging task since we can not control the zero dynamics directly, but there are a lot of demand such as most under actuated mechanical systems. This paper also discusses this control purpose.

Next let us consider the property of zero dynamics. If zero dynamics has desired global property, such as globally asymptotically stable, then only remaining problem is to achieve zeroing of linearized subsystem. For instance, the Acrobot system can be stabilized to upright position via linear controller for linearized subsystem [13], [14]. However, there is no systematic design method of zero dynamics with proper global property. Most zero dynamics submanifold has some areas such that stable area, cyclic area, and unstable area, etc. A configuration of an initial state on such zero dynamics submanifold is important to achieve control objective. Once a dynamics is trapped into $Z^*$, there are no available inputs. Thus we sometimes consider controlling an internal dynamics until a trajectory reaches zero dynamics submanifold to get a proper initial state on $Z^*$. However, it is complicated to analyze internal dynamics because of the interactions between internal and external dynamics.

In order to avoid this difficulty, we do not consider the internal dynamics but changing a controller with keeping linear state zero. Note that the controller change based on the partial feedback linearization is achieved via changing the linearizing coordinates since we only consider the controller that converges the states of linear subsystem to zero. In this paper we discuss not only instantaneous switch but also smooth change of linearizing coordinate, and we propose the following controller changing scheme.

Consider the change of linearizing coordinate from $x_{t_1}$ to $x_{t_2}$.

1) Apply a zeroing controller to a linear subsystem (8),
2) design the time-varying function which connects $x_{t_1}$ and $x_{t_2}$, and
3) apply time-varying function with equivalent input.

In general, a time-varying linearizing coordinate is formalized as follows:

$$x_t(t) = \begin{cases} x_{t_1} & t < t_s, \\ x_{t_1}(t) & t \in [t_s, t_e], \\ x_{t_2} & t > t_e, \end{cases}$$

(11)

where $[t_s, t_e]$ is the time interval when the linearizing coordinate is a function of time. In the case of $t_s = t_e$, the change of linearizing coordinate is achieved instantaneously. A time-varying function $x_{t_1}(t)$ is designed to connect two trajectories, and the key point of this scheme is to keep the trajectory on zero dynamics submanifold throughout the change of linearizing coordinate to avoid getting into the internal dynamics. How to design a time-varying function that maintains zero state of linear subsystem is the main problem. In the next section we provide the conditions for the trajectory to maintain the zero dynamics submanifold. Hereinafter, the zeroing phase in the changing scheme is not discussed, that is, we assume that the linearizing coordinate is zero at the initial time.

The image of the time evolution of a trajectory based on the proposed scheme is depicted in Fig. 1. Green layers describe the zero dynamics submanifolds with respect to the linearizing coordinate $x_{t_1}$ and $x_{t_2}$ respectively. Red surface represents the zero dynamics submanifold with respect to time-varying function $x_{t_1}(t)$. The trajectory from red initial point, denoted by black line, evolves to $Z^*(x_{t_1})$ by zeroing controller, and then it goes from $Z^*(x_{t_1})$ to $Z^*(x_{t_2})$ through $Z^*(x_{t_1}(t))$.

**III. MAIN RESULTS**

In this section, the property of a time-varying function is discussed to introduce a time-varying linearizing coordinate, and then the condition to maintain the zero state of linear subsystem is given.

A. Time-Varying Function

This subsection formulates the property of time-varying functions, and some results for time-invariant functions are generalized to be applied to time-varying functions.

Let us consider a time-varying function $h(t, x)$, and its time derivative:

$$\dot{h} = \frac{\partial h}{\partial t} + \frac{\partial h}{\partial x}(f(x) + g(x)u)$$

(12)

Only input terms are concerned for calculating relative degree of $h(t, x)$, and then we introduce the following system representation:

$$\dot{X} := \frac{d}{dt} \begin{bmatrix} t \\ x \end{bmatrix} = \begin{bmatrix} 1 \\ f(x) \end{bmatrix} + \begin{bmatrix} 0 \\ g(x) \end{bmatrix} u$$

$$= F(x) + G(x)u,$$

(13)

where $F, G$ are vector fields defined on extended state space $M_e$.

**Definition 4**: Lie derivative of a time-varying function $h(t, x)$ along with vector field $F(x)$ and $G(x)$ are

$$\mathcal{L}_F h = \frac{\partial h}{\partial x} F = \frac{\partial h}{\partial t} + \frac{\partial h}{\partial x} f(x),$$

$$\mathcal{L}_G h = \frac{\partial h}{\partial x} G = \frac{\partial h}{\partial x} g = \mathcal{L}_g h.$$
Using Lie derivative of a time-varying function, the time derivative (12) is rewritten as follows:
\[ \dot{h}(t, x) = \mathcal{L}_f h + \cdots = \mathcal{L}_G L_{k-2} f h = 0, \mathcal{L}_G L_{k-1} f h \neq 0 \]

**Lemma 1:** Lie operator along with a time invariant vector field \( f(x) \) and partial derivative about \( t \) are commutative, that is, these operators satisfy the following equation
\[ \mathcal{L}_f \left( \frac{\partial h}{\partial t} \right) = \frac{\partial}{\partial t} (\mathcal{L}_f h), \]
where \( h \) is a scalar function.

**Proof:**
\[ \frac{\partial}{\partial t} (\mathcal{L}_f h) = \frac{\partial}{\partial t} \left( \frac{\partial h}{\partial x} f \right) = \frac{\partial^2 h}{\partial t \partial x} f + \left( \frac{\partial h}{\partial x} \right) \frac{\partial f}{\partial t} = \left( \frac{\partial}{\partial t} \frac{\partial h}{\partial x} \right) f = \mathcal{L}_f \left( \frac{\partial h}{\partial t} \right), \]
where \( \frac{\partial f}{\partial t} = 0 \) since \( f \) is time invariant.

Note that this commutativity is false if the vector field \( f \) is time variant since the third equal of (15) fails, and of course, two Lie operators \( \mathcal{L}_f \) and \( \mathcal{L}_g \) are not commutative. This lemma allows recursive expression of extended Lie derivative as follows.

**Lemma 2:**
\[ \mathcal{L}_f^k h = \left( \frac{\partial}{\partial t} + \mathcal{L}_f \right)^i h \]
(16)

**Proof:** Assume the equation (16) is true with \( i = k \).
\[ \mathcal{L}_f^{k+1} h = \frac{\partial (\mathcal{L}_f^k h)}{\partial X} F = \left( \frac{\partial}{\partial t} + \mathcal{L}_f \right) \left\{ \left( \frac{\partial}{\partial t} + \mathcal{L}_f \right)^k h \right\} = \left( \frac{\partial}{\partial t} + \mathcal{L}_f \right)^{k+1} h, \]
where the last equality is based on the commutativity of \( \frac{\partial}{\partial t} \) and \( \mathcal{L}_f \).

**Theorem 2:** A function \( h(t, x) \) has relative degree \( r \) if and only if \( \mathcal{L}_g h = \cdots = \mathcal{L}_g \mathcal{L}_f^{r-2} h = 0 \) and \( \mathcal{L}_g \mathcal{L}_f^{r-1} h \neq 0 \).

**Proof:** Assume \( \mathcal{L}_g h = \cdots = \mathcal{L}_g^{(i-2)} h = 0 \), then the \( i \)-th time derivative of \( h(t, x) \) is
\[ h^{(i)} = \mathcal{L}_f^{(i-1)} + \cdots + \mathcal{L}_g^{(i-1)} h = \mathcal{L}_f^{(i-1)} h + \cdots + \mathcal{L}_g^{(i-1)} h = \mathcal{L}_f^{i-1} h + \cdots + \mathcal{L}_g^{i-1} h. \]
Let a function \( h(t, x) \) has relative degree greater than \( r > 0 \), then all we have to do is to check
\[ \mathcal{L}_g \mathcal{L}_f^{i-1} h = \mathcal{L}_g \mathcal{L}_f^i h \quad i = 0, \ldots, r - 1. \]

Assume the equation (17) is true with \( i = k - 1 < r - 1 \) and \( \mathcal{L}_g h = \cdots = \mathcal{L}_g \mathcal{L}_f^{k-1} h = 0 \), then
\[ \mathcal{L}_g \mathcal{L}_f^k h = \mathcal{L}_g \left\{ \left( \frac{\partial}{\partial \theta} + \mathcal{L}_f \right)^k \right\} (h) = \mathcal{L}_g \left( \frac{\partial^k h}{\partial \theta} + \frac{\partial^{k-1}}{\partial \theta^{k-1}} \mathcal{L}_f h + \cdots + \mathcal{L}_f^k h \right), \]

**Lemma 1** means
\[ \mathcal{L}_g \mathcal{L}_f^k h = \mathcal{L}_g \mathcal{L}_f^k h, \]
where second equality comes with the assumption of induction. Since a function \( h \) has relative degree \( r \), \( \mathcal{L}_g \mathcal{L}_f^r h \) have to be zero with \( k < r - 1 \), and \( \mathcal{L}_g \mathcal{L}_f^{r-1} h \neq 0. \]

**Remark 1:** Theorem 2 means that there is no difference between time-invariant and time-varying functions with respect to relative degree. Thus the relative degree structure and the parameterization in Theorem 1 are also works for time-varying functions with the same condition for time-invariant functions.

**Lemma 3:** Let \( e \) be a time-invariant vector field on \( M \) and \( E \) be a vector field on \( M_e \) with form \([0, e]^T\). Then Lie bracket of \( F \) and \( E \) has following form:
\[ [F, E] = \begin{bmatrix} 0 \\ [f, e] \end{bmatrix}, \]
(18)

where \( e \) is an orthogonal projection of \( E \) onto \( M \). Moreover, the iterative Lie bracket of \( F \) and \( G \) is
\[ \text{ad}_f^i G = \begin{bmatrix} 0 \\ \text{ad}_f^i g \end{bmatrix}. \]
(19)

**Proof:**
\[ [F, E] = \frac{\partial E}{\partial X} F - \frac{\partial F}{\partial X} E = \begin{bmatrix} 0 & 0 \\ \frac{\partial e}{\partial x} & f \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & \frac{\partial e}{\partial x} \end{bmatrix} \begin{bmatrix} 0 \\ e \end{bmatrix} = \begin{bmatrix} 0 \\ [f, e] \end{bmatrix}. \]

Thus the equation (19) is a straightforward consequence of (18).

**Corollary 1:** Let a relative degree of a time-varying function \( h(t, x) \) be \( k \). Then the four sets of conditions
\[
(i) \mathcal{L}_g h = \cdots = \mathcal{L}_g \mathcal{L}_f^{k-2} h = 0, \quad \mathcal{L}_g \mathcal{L}_f^{k-1} h \neq 0 \\
(ii) \mathcal{L}_g h = \cdots = \text{ad}_{f}^{i-2} g h = 0, \quad \text{ad}_{f}^{i-1} g h \neq 0 \\
(iii) \mathcal{L}_g h = \cdots = \mathcal{L}_g \mathcal{L}_f^{k-2} h = 0, \quad \mathcal{L}_g \mathcal{L}_f^{k-1} h \neq 0 \\
(iv) \mathcal{L}_g h = \cdots = \text{ad}_{f}^{i-2} g h = 0, \quad \text{ad}_{f}^{i-1} g h \neq 0
\]
are equivalent.
Proof: The equivalence between (i) and (ii) is proved at Lemma 4.1.2 of [10], and 2 shows (i) if and only if (iii). Lemma 3 represents that (ii) is equivalent to (iv).

Corollary 2: A decoupling matrix $L_g L_p^{-1} h(t, x)$ is equal to $L_g L_p^{-1} h(t, x)$.
Proof: This is a direct consequence of Corollary 1.

Thus the vector relative degree is also defined as follows.

Definition 5: The time-invariant system (1) with time-varying functions $h_1(t, x), \ldots, h_m(t, x)$ has the vector relative degree $[k_1, k_m]$ if and only if the decoupling matrix $L_g L_p^{-1} h(t, x)$ is nonsingular throughout the control.

For an extended system representation (13), a zero dynamics submanifold is redefined as follows.

Definition 6: The zero dynamics submanifold $Z^*$ corresponding to a time-varying linearizing coordinate $x_i(t) = h_1(t, x), \ldots, h_{k_i-1}(t, x), \ldots, h_m(t, x), \ldots, h_{k_i-1}(t, x)$ is defined as follows:

$$Z^*(x_{12}) = \{ x \in M \mid \mathcal{L}_p^k h_i(t, x) = 0, 1 \leq k \leq k_i - 1, 1 \leq i \leq m \},$$

where $k_i$ is relative degree of $h_i$.

B. Changing of linearizing coordinate

First let us consider single input system and a linearizing coordinate

$$x_i(t) = [h(t, x), \ldots, h^{(r-1)}(t, x)]^T,$$  \hspace{1cm} (20)

where $r$ is relative degree of $h(t, x)$. Equations (17) denote that the input transformation for time-varying linearizing coordinate (20) is defined as follows:

$$u = -\left( L_g L_p^{-1} h \right)^{-1} L_p h.$$  \hspace{1cm} (21)

Theorem 3: Let a relative degree of a time-varying function $h(t, x)$ be $r$. A time-varying process maintains zero state of linear subsystem by a continuous input if $h(t, x)$ is a function of class $C^r$ with respect to time and an equivalent input (21) is well-defined.

Proof: To realize keeping zero state of linear subsystem via a continuous input means that all time evolutions happened in time derivatives are canceled by continuous input. Thus the equivalent input (21) have to be well-defined during all controlled time.

Time derivatives of $h(t, x)$ is written as follows:

$$\frac{d^k h}{dt^k} = \mathcal{L}_p^k h^{(k-1)} = \sum_{i=0}^{k} \frac{\partial^i}{\partial t^i} L_p^k h + u L_g L_p^{-1} h,$$

where $L_g L_p^{-1} h$ is zero for $k < r$ and $L_g L_p^{-1} h \neq 0$. If $h(t, x)$ is not a function of class $C^r$ with respect to time, then there exists $k \leq r$ such that $h^{(k)}(t, x)$ is discontinuous. The equivalent input (21) affects $h^{(r-i)}(t, x)$ with $i$-th time delay and can not cancel a discontinuous change occurring at lower degree than $r$. Thus the degree of continuity have to be greater than $r$ in order for a continuous input to cancel the change of output.

Remark 2: If we can utilize a discontinuous input, then the condition for $h(t, x)$ is weaken to $C^{r-1}$ since the discontinuity of $h^{(r)}(t, x)$ can be canceled by a discontinuous input (21).

Multi-input version is direct extension of single input case.

Theorem 4: Let a vector relative degree of a system (1) with time-varying functions $\{ h_1(t, x), \ldots, h_m(t, x) \}$ be $\{ r_1, \ldots, r_m \}$. A time-varying process maintains zero state of linear subsystem by a continuous input if $h_1(t, x), \ldots, h_m(t, x)$ are functions of class $C^{r_1}, \ldots, C^{r_m}$ with respect to time and a corresponding decoupling matrix is nonsingular.

Proof: The proof comes from by applying the result Theorem 3 to decoupled systems.

Next let us consider time functions $c_{k1}(t)$ and $c_{2i}(t)$ defined as

$$c_{ki}(t) = \begin{cases} c_{ki1}, & t < s_{ki1}, \\ c_{ki2}, & t > s_{ki2}, \end{cases} \quad \text{for } k = 1, 2, i = 1, \ldots, m,$$  \hspace{1cm} (22)

where $c_{ki1} = 1$, $c_{ki2} = 0$, and $c_{2i1} = 0$, $c_{2i2} = 1$, and a transient function $c_{ki}(t)$ is of class $C^{r_i}$ in time interval $(s_{ki1}, s_{ki2})$ and satisfies

$$\dot{c}_{ki1}(s_{ki1}) = \cdots = c_{ki1}(s_{ki1}) = 0,$$
$$\dot{c}_{ki2}(s_{ki2}) = \cdots = c_{ki2}(s_{ki2}) = 0.$$  \hspace{1cm} (23)

Thus $c_{ki}(t)$ is function of class $C^{r_i}$. Using these time functions, the condition for linearizing coordinate (11) is formalized as follows:

Corollary 3: Assume a system (1) with two set of time-varying functions $h^1_i(x) = \{ h_1^1(x), \ldots, h_m^1(x) \}$ and $h^2_i(x) = \{ h_1^2(x), \ldots, h_m^2(x) \}$ has the same vector relative degree $r = \{ r_1, \ldots, r_m \}$. The continuous change from $h^1_i(x)$ to $h^2_i(x)$ with zero state of linear subsystem is achieved if a decoupling matrix $L_g L_p^{-1} h(t, x)$ of a function

$$h(t, x) = C_1(t) h^1(x) + C_2(t) h^2(x)$$  \hspace{1cm} (24)

is nonsingular for all controlled time, where $C_1(t)$ and $C_2(t)$ are diagonal matrix whose diagonal elements $c_{1i}$ and $c_{2i}$ are $C^{r_i}$ continuous functions defined by (22).

Proof: This corollary is a special case of Theorem 4.

The transient interval of proposed output function (23) is $[\min(t_{s11}, t_{s21}), \max(t_{e11}, t_{e21})]$ and the decoupling matrices has following form:

$$L_g L_p^{-1} h(t, x) = C_1(t) L_g L_p^{-1} h^1(x) + C_2(t) L_g L_p^{-1} h^2(x),$$

where $L_g L_p^{-1} h^1(x)$ and $L_g L_p^{-1} h^2(x)$ are decoupling matrices with respect to $h^1(x)$ and $h^2(x)$.

IV. EXAMPLE: MONO-ROTOR UAV SYSTEM

Consider mono-rotor unmanned aerial vehicle(UAV) system shown in Fig. 2, where $z$, $\psi$, and $\theta_p$ are physical variables, and $m$ denotes the mass of the body. The control objective is a motion that goes desired height $z_d$ and attitude $\psi_d$ with zero velocity periodically, and we call this motion “semi-hovering”. This model has countering between angular velocity of body $\dot{\psi}$ and propeller $\dot{\theta}_p$ as follows:

$$\ddot{\psi}(t) = -\frac{J_c + 2 J_p}{2 J_p} (\dot{\psi}(t) - \dot{\psi}(t_k)) + \dot{\theta}(t_k),$$  \hspace{1cm} (24)
where \( J_r \) and \( J_p \) are physical parameter, and \( t_k \) is initial time. We assume that the propulsion force is proportional to a square of angular velocity. As a result the dynamics of the system is derived by Euler-Lagrange Method as follows:

\[
\dot{x} = f(x) + g(x)\tau, \quad f(x), \quad g(x) \in \mathbb{R}^{4\times 1} \quad (25)
\]

\[
x = [\theta, \psi, \dot{\psi}]^T \in \mathbb{R}^3 \times \mathbb{S}, \quad (26)
\]

where \( \theta_p \) is omitted since it does not affect the dynamics, and \( \tau \) is input torque.

A. Controller Design

The control strategy for semi-hovering is divided into the following two part.

1) : Stop the posture angle at \( \psi_d \) periodically, and
2) : synchronize position \( x \) with posture angle \( \psi \).

Step 1) : Control based on Partial Feedback Linearization

As a control of \( \psi \), we apply zeroing control to the following linearizing coordinate:

\[
x_k = \begin{bmatrix} h(\psi, t), \dot{h}(\psi, t) \end{bmatrix}^T \quad (27)
\]

\[
h(\psi, t) = \psi - \psi_r(t), \quad (28)
\]

\[
\psi_r(t) = \sin(2\pi f(t - t_0)) - 2\pi f(t - t_0), \quad (29)
\]

where we choose the reference attitude \( \psi_r \) as 0, and \( t_0 \) is initial time. The graph of \( \psi_r(t) \) is depicted in Fig. 3.

Step 2) : Zero Dynamics Control

The relation between \( f \) and \( z \) is described in Fig. 4, and this figure shows that \( z \) increases (decreases) when \( f \) is large value (small value). This means that the zero dynamics can be controlled using \( f \). Therefore, we switch the zeroing controller to change the frequency \( f \). Theorem 3 means that a time-varying function \( h(t, x) \) have to be of class \( C^2 \) since the relative degree of \( h \) defined in (28) is 2. \( h \) is, of course, a function of class \( C^2 \). Time derivatives \( \dot{\psi}_r \) and \( \ddot{\psi}_r \) of reference function are 0 when \( f(t - t_0) \) is integer. That is, \( \psi_r(t_1) = -2\pi \) where \( t_1 = 1/f + t_0 \). At that time, \( \psi_r \), \( \dot{\psi}_r \) and \( \ddot{\psi}_r \) are not function of \( f \). Thus, \( f \) can be changed as a virtual input for zero dynamics.

As a virtual control input, \( f \) is designed every time \( t \) when \( f(t - t_0) \) is integer. Let \( f^{(k)} \) be the constant frequency in time span \( [t_k, t_{k+1}] \), where \( t_{k+1} = t_k + 1/f^{(k)} \). In order to control the zero dynamics, reference attitude \( \psi_r \) is modified as follows:

\[
\psi_r(t) = \sin(2\pi f^{(k)}(t - t_k)) - 2\pi f^{(k)}(t - t_k) - 2\pi k \quad (30)
\]

This reference attitude is a function of class \( C^2 \) regardless of \( f^{(k)} \). The zero dynamics in time span \( [t_k, t_{k+1}] \) with (30) is derived explicitly by integrating \( \dot{z} \) in (24) as follows:

\[
z(t - t_k) = a_2(t - t_k)^2 - a_1(t - t_k) + a_0, \quad (31)
\]

where \( a_i (i = 0, 1, 2) \) are constant depending on designed parameter \( f^{(k)} \) and initial conditions \( z(t_k), \dot{z}(t_k), \psi(t_k) \), and \( \psi(t_k) \). The virtual controller \( f^{(k)} \) is designed based on the analysis on the Poincare section satisfying \( \psi = 0, \dot{\psi} = 0 \).

\[
f^{(k)} = f_0^{(k)} + f_1^{(k)}(1 - a), \quad (32)
\]

where \( f_0^{(k)} \) is the frequency satisfying \( \dot{z}(t_{k+1}) = 0 \), and \( f_1^{(k)} \) is the frequency satisfying \( \dot{z}(t_{k+1}) = 0 \) at \( t_{k+1} = t_k + 1/f^{(k)} \).

Figures 5-8 are the result of numerical simulation via zeroing controller and virtual input (32). Periodical motion of \( z \) and \( \psi \) are confirmed by Fig. 5 and Fig. 6. Moreover the zero state of linear subsystem is kept independent from the virtual input \( f^{(k)} \). Portrait on the Poincare section is depicted in Fig. 8 and shows that both \( z \) and \( \psi \) are synchronized with each other.

V. CONCLUSIONS

In this paper, we discussed the transfer the state from one zero dynamics submanifold to another one in finite time, for a time-invariant nonlinear system. We proposed the state transfer through a time-varying submanifold that connects the two zero dynamics submanifolds.

Extended system representation of a time-invariant nonlinear system was considered to discuss a time-varying zero dynamics submanifold, and we revealed that any time-varying function \( h(t, x) \) has the same relative degree with a time invariant function \( h(c, x) \) where \( c \) is constant value substituted for \( t \).

Moreover, we developed the conditions to accomplish the transfer for the case that the two zero dynamics submanifolds were the same dimension. The condition requires the continuity of time-varying zero dynamics submanifold with respect to a relative degree of a linearizing coordinate so that
the trajectory on the zero dynamics submanifold is not apart from the submanifold during the transfer. That is, a time-varying linearizing coordinate had to be smoother than its relative degree to keep zero state of linear subsystem, and we demonstrated the controller design based on the proposed scheme via mono-rotor UAV system.

REFERENCES