Conditional integrator for non-minimum phase nonlinear systems

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Abstract—In this paper we propose a conditional integrator for solving the regulation problem of a class of nonlinear systems that probably possess unstable zero dynamics. The control scheme mainly takes advantage of the auxiliary controller proposed in [3] and a slow integrator. We present that the solution of the regulation problem can be cast as two auxiliary problems. Furthermore it is shown that the design of the conditional integrator requires the design of a feedback control that solves two stabilization problems simultaneously, one inside the boundary layer of a continuously-implemented sliding model control, and one outside it. Hence the design of the auxiliary controller can be independent of the internal model. We approach this issue by using a two-time-scale method. The effectiveness of the controller is illustrated through a linear example.

I. INTRODUCTION

The nonlinear output regulation problem has received a lot of attentions since the pioneering work [7]; see the recent advances in [8], [9] and [10]. Most of the existing literature assumes that the system is minimum-phase. In the stabilization paper [3], a strategy for dealing with unstable zero dynamic is proposed by introducing an auxiliary controller. This idea was extended to regulation of nonlinear nonminimum-phase systems in [1], [2], [4], [11] and [12]. In [2], [11] and [12], the considered system has a definite high frequency gain, or at least measurable. [11] is an improvement over [2] and [12] in that firstly proposed that the design of auxiliary controller can be independent of the choice of internal model. A small gain method is used to impose the gain restriction on the inner loop that consists of auxiliary plant and controller only. [1] further improved the result in [11] in that it deals with the case when there exist uncertainties in the high-frequency gain by using periodically varying controllers. In [4], an extended high gain observer is used to estimate the driving signal of the auxiliary controller in addition to the system output and its \( \rho \) derivatives (system relative degree). It analyzes the effect of uncertainties in the high-frequency gain.

The conditional integrator is introduced in [5] for a class of minimum-phase systems. It is shown in [5] that the inclusion of the conditional integrator can have better transient performance without losing asymptotic regulation. In the current paper we still use a conditional integrator for a class of nonlinear nonminimum-phase system expecting a better transient performance. Moreover we use a slow integrator such that the "sufficiently small parameter" restriction in multi-time-scale theory is satisfied. The slow integrator was used in [6] but in [6] the controlled nonlinear plant is globally exponential stable with any constant input and its governing function is globally Lipschitz. These conditions are relaxed in the current paper. The comparison with [6] can be found in the conclusion section.

The result in the current paper is most related to [11] since we consider the same problem and the high-frequency gain is simply chosen as 1. One of the contributions of [11] is a dissipativity-based method where the product of the gains of an inner (auxiliary controller and auxiliary plant) and outer (internal model) loops is required to be less than one. We define two auxiliary problems to achieve regulation and require them to be solved simultaneously. It is assumed in this paper that the exosystem is \( \dot{w} = 0 \). In other words the unknown signal is a constant.

II. PROBLEM FORMULATION

Consider system

\[
\begin{align*}
\dot{\eta} &= \phi(\eta, \xi, w, \theta) \\
\dot{\xi}_i &= \xi_{i+1}, \quad 1 \leq i \leq \rho - 1 \\
\dot{\xi}_\rho &= b(\eta, \xi, w, \theta) + u - \chi(w, \theta) \\
e &= \xi_1
\end{align*}
\]  

(1)

where \( w \) and \( \theta \) are unknown constants that belong to compact sets \( W \) and \( \Theta \), respectively, \( \chi(w, \theta) \) is the steady-state control that maintains the trajectories on the zero-error manifold \( \{ \eta = 0, \xi = 0 \} \), \( \phi \) and \( b \) are smooth functions that vanish on this manifold, and \( e \) is the regulation error. In [5] the origin of the system \( \dot{\eta} = \phi(\eta, 0, w, \theta) \) is required to be asymptotically stable so that the system is minimum phase, but in the current paper we do not have such assumption.

We propose the following controller for the system (1):

\[
\begin{align*}
\dot{\sigma} &= -\lambda \sigma + \mu \lambda \text{sat}(s/\mu) \\
\dot{\xi} &= L(z, \xi_1, \cdots, \xi_{\rho-1}) + M(z, \xi_1, \cdots, \xi_{\rho-1}) k \text{sat}(\frac{s}{\mu}) \\
u &= -b_0(\xi) + N(z, \xi_1, \cdots, \xi_{\rho-1}) - k \text{sat}(s/\mu) \\
s &= \sigma + \xi_{\rho} - N(z, \xi_1, \cdots, \xi_{\rho-1}) - \chi(w, \theta)
\end{align*}
\]  

(2)

where the functions \( b_0 \), \( L(\cdot \cdot \cdot) \) and \( N(\cdot \cdot \cdot) \) vanish when their respective arguments are zero and \( N \) is the total derivative of \( N \). The controller (2) assumes that the whole vector \( \xi \) is available for feedback.
The closed-loop system can be written as
\[ \dot{x}_a = \phi_a(x_a, \xi_p) \]
\[ \dot{\sigma} = -\lambda \sigma + \lambda \mu \text{sat}(s) \]
\[ \dot{z} = L(z, \xi_1, \cdots, \xi_{p-1}) + M(z, \xi_1, \cdots, \xi_{p-1}) - \text{sat}(\frac{s}{\rho}) \]

where \( x_a = [\eta \xi_1 \cdots \xi_{p-1}]^T \) and \( \phi_u(x_a, \xi_p) = [\phi(\cdot) \xi_2 \cdots \xi_{p-1}]^T \).

For convenience we drop \((w, \theta)\) from notation. We introduce a change of variables
\[ \psi = z + M(\xi_p - N) \]
then the closed-loop system (3) rewrites as
\[ \dot{x}_a = \phi_u(x_a, s + N - \sigma) \]
\[ \dot{\sigma} = -\lambda \sigma + \lambda \mu \text{sat}(s) \]
\[ \dot{\psi} = b(x_a, \xi_p) - b_0(\xi) + N(z, \xi_1, \cdots, \xi_{p-1}) - \text{sat}(\frac{s}{\rho}) \]

(4)

III. MAIN RESULTS

In this section we study the convergence of system (4). We divide the analysis into two phases, a reaching phase where the goal is to show that \( s \) reaches the boundary layer \( \{ |s| < \mu \} \) in finite time, while all state variables are bounded, followed by analysis of the system inside the boundary layer. Because \( \chi \) is bounded by assumption and \( s, \sigma \) are bounded by design, to ensure boundedness of \( x_a \) and \( \psi \) we, essentially, need to design the triplet \((L, N, M)\) to stabilize the origin of the \((\dot{x}_a, \dot{\psi})\)-equations when \((\chi, \sigma, s) = 0\). Bound-input-bounded-state stability will follow then from some reasonable assumptions. Noting that the \((\dot{x}_a, \psi)\)-equations can be represented as a feedback connection, we arrive at the following design task:

**Auxiliary Problem 1:** Design a feedback controller of the form
\[ \dot{\psi} = L(\psi, \xi) + M(\psi, \xi) y_a \]
\[ u_a = N(\psi, \xi) \]
(5)
to stabilize the origin of the system
\[ \dot{x}_a = \phi_u(x_a, u_a), \quad y_a = h_u(x_a, u_a) \]
(6)

where \( h_u = b - b_0 \) and \( \xi = [\xi_1 \cdots \xi_{p-1}]^T \).

Then we make the following assumption.

**Assumption 1:** The triplet \((L, M, N)\) is chosen such that there exists a \( C^1 \) function \( V_1(\mathcal{X}, w) \), in which \( \mathcal{X} = (\psi^T, x_a^T)^T, \mathcal{X}_\infty \) functions \( \alpha_1 \) and \( \alpha_2 \) and \( \mathcal{X} \) functions \( \alpha_3 \) and \( \alpha_4 \), independent of \( w \), such that
\[ \alpha_1(\|\psi\|) \leq V_1(\mathcal{X}, w) \leq \alpha_2(\|\mathcal{X}\|) \]
\[ \frac{\partial V_1}{\partial \mathcal{X}} \phi(\mathcal{X}, w_d, w) \leq -\alpha_3(\|\mathcal{X}\|) \]
for all \|\mathcal{X}\| \geq \alpha_4(\|w_d\|), \) in which \( w_d = (w_a, w_b)^T \) and \( w_a = s - \sigma, w_b = \chi + \frac{k}{\mu} \sigma \), and \( \phi(\mathcal{X}, w_d, w) \) is the RHS of the equation
\[ \dot{x}_a = \phi_u(x_a, s + N - \sigma) \]
\[ \dot{\psi} = L(\cdot) + M[b - b_0 - \chi - \frac{k}{\mu} \sigma] \]

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With the result of Proposition 1, the system (4) can be written as
\[ \begin{align*}
\dot{x}_a &= \phi_a(x_a, s + N - \sigma) \\
\sigma &= \lambda (s - \sigma) \\
\psi &= L(\cdot) + M[b - b_0 - \chi - \frac{k}{\mu} \sigma] \\
\dot{s} &= b - b_0 - \chi - \frac{k}{\mu} s - \lambda \sigma + \lambda s 
\end{align*} \] (8)

insider the boundary layer. It can be shown that system (8) has a zero-error manifold of the form
\[ \{s = \sigma, x_a = 0, \psi = 0\} \quad \text{where} \quad \sigma = \mu \chi / k \]

On this manifold \( \eta = 0 \) and \( \xi = 0 \). A crucial factor in ensuring the existence of this manifold is the driving term \( \text{sat}(s/\mu) - \lambda/\mu \). Defining
\[ \nu = \frac{\sigma - \sigma}{\mu} \quad \text{and} \quad \gamma = \frac{s - \sigma}{\mu} \]

The system (8) can be represented in the singularly perturbed form
\[ \begin{align*}
\dot{x}_a &= \phi_a(x_a, N + \mu \gamma) \\
\nu &= \lambda \gamma \\
\psi &= L(\cdot) + M[b - b_0 - k \nu] \\
\mu \dot{\gamma} &= b - b_0 - k \gamma - k \nu 
\end{align*} \] (9)

Setting \( \mu = 0 \), the quasi-steady state of \( \gamma \) is \( -\nu + \frac{b - b_0}{k} \) and we obtain the slow model
\[ \begin{align*}
\dot{\nu} &= \lambda (-\nu + \frac{b - b_0}{k}) \\
\dot{x}_a &= \phi_a(x_a, N) \\
\psi &= L(\cdot) + M[b - b_0 - k \nu] 
\end{align*} \] (10)

This system has an equilibrium point at the origin. To ensure convergence to the zero error manifold we need to design the compensator functions \{L(\cdot), M(\cdot), N(\cdot)\} to stabilize the origin. This is equivalent to the following problems.

**Auxiliary Problem II:** Design a feedback controller of the form
\[ \begin{align*}
\psi &= L(\psi, \zeta) + M(\psi, \zeta)\bar{y}_a \\
u &= N(\psi, \zeta) \\
u_a &= N(\psi, \zeta) 
\end{align*} \] (11)

to stabilize the origin of the system
\[ \begin{align*}
\dot{\nu} &= -\lambda \nu + \lambda \bar{u}_a / k \\
\dot{x}_a &= \phi_a(x_a, u_a) \\
\bar{y}_a &= h_a(u_a, x_a) - k \nu 
\end{align*} \] (12)

The two problems are related to each other. In particular, the plant in the second problem includes the plant in the first one in cascade with the transfer function
\[ H(s) = \frac{s}{s + \lambda} \]

Hence the block diagram corresponding to equation (11) and (12) is shown in Fig. 2.

In [1] and [11], a system corresponding to (11) and (12) is considered as well. They first applied a dissipativity-based approach to obtain the stability of the origin. Then an improved result is achieved by choosing specifically the free parameters of internal model. Roughly speaking, the idea of the improved result is to divide the system corresponding to (11) and (12) into inner loop (auxiliary plant+auxiliary controller) and outer loop (internal model), and then require the norm of the product of inner loop and outer loop to be smaller than 1. Hence the design of triplet \{L(\cdot), M(\cdot), N(\cdot)\} is independent of internal model.

Our challenge here is to design the controller to solve the two auxiliary problems simultaneously. We plan to approach this problem using multiple-time scales. In the following we will present our main idea. We introduce a new time scale \( \tau = \lambda t \) and choose \( \lambda \) as a small parameter, then in this new scale the system (12) can be transformed to
\[ \begin{align*}
\dot{\nu} &= -\nu + \frac{b - b_0}{k} \\
\lambda \dot{x}_a &= \phi_a(x_a, N) \\
\lambda \psi &= L(\cdot) + M[b - b_0 - k \nu] 
\end{align*} \] (13)

where \( \cdot \) now denotes the derivative with respect to \( \tau \). Setting \( \lambda = 0 \) results in
\[ \begin{align*}
0 &= \phi_a(x_a, N) \\
0 &= L(\cdot) + M[b - b_0 - k \nu] 
\end{align*} \] (14)

Let \( g(\nu) \) be the steady-state gain from input \( \nu \) to \( b - b_0 \) of the closed-loop system obtained in Auxiliary Problem I. Then the slow system writes as
\[ \dot{\nu} = -\nu + \frac{g(\nu)}{k} \] (15)

For this slow system to be stable, \( g(\nu) \) should be less than \( k \nu \).

Thus, it is shown that the two auxiliary design problems simultaneously can be achieved by choosing \( \lambda \) small enough and solving the Auxiliary Problem I under a constraint which is brought by the following assumption.

**Assumption 2:** With reference to the closed-loop system of Figure 1, assume that the steady-state gain of the input-output map from \( w_B \) to \( y_a \) is less than one. Before we state the main result of the paper, an additional assumption is needed as follows.

**Assumption 3:** The triplet \( (L, M, N) \) is chosen such that for the closed-loop system in Fig. 1 with \( w_{ax} = 0 \) and \( w_B \) is constant, then for every \( w_B \) in a certain range the system has an equilibrium point, which is exponentially stable uniformly in \( w \) and \( w_B \).

We summarize the aforementioned analysis in the following theorem.
Theorem 1: For the system (8), choose the triplet \( \{L(\cdot), M(\cdot), N(\cdot)\} \) such that Assumptions 1, 2 and 3 are satisfied, then there exist \( \lambda^* \) and \( \mu^* \) such that for all \( 0 < \lambda < \lambda^* \) and \( 0 < \mu < \mu^* \), the origin of system (8) is asymptotically stable.

Proof: The proof of Theorem 1 is not included due to space limitation, but its idea is similar to the proof of Theorem 11.4 of [14], except that we have a three-time-scale system compared to a two-time-scale system in [14]. Therefore, the steps of the proof in [14] are applied iteratively.

IV. SIMULATION

Consider the system
\[
\begin{align*}
x_1 &= x_1 + 4x_2 \\
x_2 &= -x_1 - 3x_2 + u - w
\end{align*}
\]
(16)

where \( w = 0 \). Throughout the simulations, the initial conditions for \( x \) and \( w \) are set as \( x_1(0) = 0, x_2(0) = 1 \) and \( w(0) = 1 \).

In the first step we design the controller (2) as follows.
\[
\begin{align*}
u &= -2x_2 + \sigma - 5 \text{sat}(\frac{x_2 - z + \sigma}{0.1}) \\
\sigma &= -0.1\sigma + 0.01\text{sat}(\frac{x_2 - z + \sigma}{0.1}) \\
\dot{z} &= 2z + 5(\text{sat}(\frac{x_2 - z + \sigma}{0.1}) - \sigma)
\end{align*}
\]

which means we choose \( L(\cdot) = 2z, M(\cdot) = 1, N(\cdot) = z, b(\cdot) = 2x_2, \mu = \lambda = 0.1 \) and \( k = 5 \).

We design the controller in [11] (using its notation) as
\[
\begin{align*}
u &= \phi + \eta + \nu \\
\phi &= -5[\xi - \varphi] \\
\eta &= -4[\xi - \varphi] \\
\varphi &= -5\psi + 5.5\xi
\end{align*}
\]
which means \( L(t) = 0.5t, M = 1, N(t) = t \) and \( k = 5 \).

Finally we illustrate the robustness of the proposed conditional integrator and consider the following uncertain system
\[
\begin{align*}
\dot{x}_1 &= x_1 + ax_2 \\
\dot{x}_2 &= -x_1 - 3x_2 + u - w
\end{align*}
\]
(17)

where \( a \in [4, +\infty) \). We still choose \( L(\cdot) = 2z, M(\cdot) = 1, N(\cdot) = z, b(\cdot) = 2x_2, \mu = \lambda = 0.1 \) and \( k = 5 \). The real value for \( a \) is chosen as 10, 15 and 20, respectively. We can see from the Fig. 3 that the convergence is still achieved. But it can be seen that the speed of convergence is lowered when the value of the parameter \( a \) is increased. When \( a \) is set as 10, 15 and 20, the steady state gain of (15) from \( kv \) to \( y_u \) is 0.7134, 0.8333 and 0.9259, respectively. Actually the proposed controller also works when there exist unknown parameters in the \( x_2 \) equations, i.e., \( \dot{x}_2 = -bx_1 - cx_2 + u - w \) where \( b \) and \( c \) are in some range but we do not show them all here.

Fig. 3. Comparison of the conditional integrator with the controller in [11]

Fig. 4. Comparison of the conditional integrator with the controller in [4]

Fig. 5. Illustrations of robustness of conditional integrator
V. CONCLUSION

In this paper we have proposed a tentative solution to the regulation problem for a class of non-minimum phase nonlinear system. The introduction of a conditional integrator is meant to achieve better transient performance which is illustrated in the simulations. Inside the boundary layer the multi-scale method is used to derive the stability conditions for the triplet $L(\cdot),M(\cdot),N(\cdot)$ that are in terms of two auxiliary problems which can be solved simultaneously. Moreover it can be seen that the design of auxiliary controller can be independent of internal model. It is worthy to be mentioned that the idea of slow integrator can be traced back to [6]. Furthermore in [6] they had a requirement that the steady-state input-output gain be strictly increasing. However there are two differences. First the plant in [6] is globally uniformly exponentially stable (uniformly in the constant input) while this is not the fact in our case. Second [6] required the governing functions of the plant to be globally Lipschitz while this is not the case in our work.

In [5], the conditional integrator is introduced to achieve better transient performance. This is meant to prevent the integral of tracking error from entering into the control input for a transient time so as not to destroy the transient performance. But in the non-minimum phase case the conditional integrator is introduced to freeze the integral of $\xi_p - N$ for a transient time. It has nothing to do with the idea of ”prevent the integral of tracking error from entering into the control input for a transient time”. Moreover in the case of minimum phase systems it is possible to design a robust controller that makes the error arbitrarily small. Adding a conditional integrator preserves the transient behavior of the robust controller and achieves regulation. In the case of non-minimum phase systems it is not possible to design a controller to make the error arbitrarily small. This is a fundamental limitation of non-minimum phase systems and there is no way around it. Furthermore because the error under the conditional integrator will have to reach the quasi-steady-state value before it moves slowly towards zero, we can not guarantee better performance over a traditional integrator.

REFERENCES