A PID tuning method based on matching between one-shot experimental data and filtered desired closed-loop responses

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Abstract—We consider a problem of direct controller tuning by using a single set of measured experimental closed-loop data. In this paper, we introduce a shaping filter for the desired reference input which is used in tuning so that the desired closed-loop response can be easily achieved even if an inadequate reference reference input is given. Two examples illustrate the effectiveness of the method.

Index Terms—Controller tuning, data-driven control, model-free design

I. INTRODUCTION

PID control is most widely utilized in many actual applications. For PID control, how the proportional gain, the integral gain and the differential gain should be tuned plays a key role for the achievement of the desired performance.

In general, the performance of feedback control systems by model-based designs depend on the quality of the plant model to be used in the design. Therefore, it is important to precisely identify the plant model. However, in system identification, experiments to obtain the precise model require significant costs.

PID control is superior in simpleness to any other model-based control. PID control does not explicitly require the plant model to be used in the design. Although iterative feedback tuning (IFT) by [4] and VRFT (Virtual Reference Feedback Tuning) by [5], in view of that they can tune controller without using any plant models, and reduce the experimental costs for the controller tuning. Although iterative feedback tuning (IFT) (see for example [6], [7] etc.) gives a fine tuning result, it requires repeating experiments for controller tuning.

The reference response which controller tuning aims to achieve should be given as an ideal model of the closed-loop system (called a reference model). However, it is difficult to find an adequate reference model, because we do not have enough information on the plant. So, it is also difficult to obtain a controller attaining a response close to such an ideal one. It is desirable to tune the ideal reference model as well.

In this paper, we propose a new method that makes the closed-loop response close to the reference response to introduce a reference response shaping filter based on the method by [1] and [2]. By the method, we can achieve the desired closed-loop response close to the reference response even if we are given an unachievable reference model.

Notation: For a vector \(g\), its Euclidean norm is denoted by \(\|g\|\). For a discrete-time signal \(f(0), f(1), f(2), \cdots\), its z-transform representation is denoted by

\[
f(z) = \sum_{k=0}^{\infty} f(k)z^{-k}.
\]

For a \(f(z)\), we use the norm defined by

\[
\|f(z)\|_2 = \left(\sum_{k=0}^{\infty} \|f(k)\|^2\right)^{1/2}.
\]

II. PROBLEM FORMULATION

We consider a unity feedback system which consists of a SISO linear time-invariant discrete-time system \(P\) and a controller \(C\) as shown in Fig. 1. Then, we assume that the closed-loop system is stabilized by \(C\). Our problem is to tune \(C\) such that the transfer function from the reference input \(r\) to the output \(y\) becomes the desired \(T\) and/or the transfer function from the disturbance input \(d\) to output \(y\) does the desired \(M\). The transfer function \(T\) and \(M\) represent the complementary sensitivity and the disturbance rejection, respectively. When \(T\) and/or \(M\) are given as the desired one, we call it a reference model.

In this paper, we consider a problem in which we tune the controller \(C\) to achieve the desired \(T\) or \(M\) without identifying any models of the plant \(P\). This is,

Find \(C\) s.t. \(\min_{C} \|Tr - \frac{PC}{1+PC}r\|^2\) \(\leq 1\) \tag{1}

or

Find \(C\) s.t. \(\min_{C} \|Md - \frac{P}{1+PC}d\|^2\) \(\leq 1\). \tag{2}
In general, there does not always exist a $C$ satisfying $PC/(1+PC) = T$ and/or $P/(1+PC) = M$. Additionally, if we have no information on $P$, we cannot determine whether $T$ and $M$ are suitable for the feedback system with $P$ and $C$. Thus, it is desirable that the reference models $T$ or $M$ is tuned together with $C$.

In this paper, instead of direct tuning $T$ and/or $M$, we introduce a reference input shaping filter $F$ that shapes $Tr$ and/or $Md$. Hence, in our problem, we find a $C$ and $F$ minimizing

$$J(F,C) := \left\| \frac{FT - PC}{FT - Tr} \right\|_2^2$$

for the complementary sensitivity or

$$J(F,C) := \left\| \frac{FMd - P}{FMd - Md} \right\|_2^2$$

for the disturbance response.

III. DATA-DRIVEN TUNING

A. Algorithm

Since we cannot minimize (3) or (4) without using a model of the plant $P$, we will rewrite them to use the relationship between the input and the output data. If data $u_{sr}$ and $y_{sr}$ are obtained when $r$ is applied to the feedback system under the assumption that $d = 0$, they satisfy $y_{sr} = Pu_{sr}$. In this case, we can rewrite the first element of the right hand side in (3) as

$$FT \left( \frac{1}{1+PC} + \frac{PC}{1+PC} \right) r - \frac{PC}{1+PC} r$$

Furthermore, we obtain

$$FT \left( \frac{1}{1+PC} r + y_{sr} \right) - y_{sr}$$

Hence, we can use a new cost function

$$\tilde{J}(F,C,\alpha) := \left\| \frac{FTu_{sr} + TCy_{sr} - Cy_{sr}}{\alpha(FT - Tr)} \right\|_2^2$$

instead of (3). Here, $\alpha \geq 0$ is a tunable parameter.

Similarly, if $u_{sd}$ and $y_{sd}$ are obtained when $d$ is applied to the feedback system for $r = 0$, they satisfy $y_{sd} = P^d u_{sd}$. Hence, in this case, we can rewrite the first element of the right hand side in (4) as

$$FM \left( \frac{1}{1+PC} + \frac{PC}{1+PC} \right) d - \frac{P}{1+PC} d$$

Furthermore, we obtain

$$FM \left( \frac{1}{1+PC} d + y_{sd} \right) - \frac{P}{1+PC} d$$

Hence, we can use a new cost function

$$J(F,C,\alpha) := \left\| \frac{FMu_{sd} + MCy_{sd} - y_{sd}}{\alpha(FMd - Md)} \right\|_2^2$$

B. PID controller tuning

Any PID controller can be represented as

$$C(z,\theta) = \theta^T \beta(z),$$

where

$$\beta(z) = \begin{bmatrix} 1 \\ z \\ z-1 \\ 1/z \end{bmatrix}$$

is a transfer function with a proportional, an integral and a differential element and

$$\theta = \begin{bmatrix} K_p \\ K_i \\ K_d \end{bmatrix}$$

is a PID parameter vector. Since (16) is linearly dependent on $\theta$, the minimization problem in Step 2 to obtain $\hat{\theta}_n := C(z,\hat{\theta}_n)$ can be cast into a convex optimization problem according to (3)). First, (15) is equivalent to

$$\hat{\theta}_n = \arg \min_{\theta} \| \theta^T \xi(z) - v(z) \|_2^2,$$

where

$$\xi(z) := \beta(z)(1 - \hat{F}_n T) y_{sr}, v(z) := \hat{F}_n Tu_{sr}$$

for the complementary sensitivity,

$$\xi(z) := \beta(z) \hat{F}_n M y_{sd}, v(z) := y_{sd} - \hat{F}_n M u_{sd}$$

for the disturbance response.

We can solve (19) as a quadratic programming problem

$$\hat{\theta}_n = \arg \min_{\omega} \omega$$

subject to $$\omega - \| \theta^T \xi(z) - v(z) \|_2^2 0 > 0.$$
The constraint of the (1,1)-element in (22) can be simplified as
\[ \omega - \|T(z) - v(z)\|_2^2 > 0 \]
\[ \Leftrightarrow \omega - (\Xi\theta - \bar{v})^T(\Xi\theta - \bar{v}) > 0 \]  \hspace{1cm} (23)
Furthermore, by using the Schur complement, we see that (26)
\[ \Xi = X\Sigma Y, \Sigma = \begin{bmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \\ 0 & 0 & \sigma_3 \end{bmatrix} \]  \hspace{1cm} (24)
where \( X \) and \( Y \) are unitary matrices; \( \sigma_i \)'s are non-zero singular values; and by defining
\[ \bar{v} := X^Tv, \Xi := \Sigma Y, \]  \hspace{1cm} (25)
we see that (23) is equivalent to
\[ \omega - (\Xi\theta - \bar{v})^T(\Xi\theta - \bar{v}) > 0 \]  \hspace{1cm} (26)
Furthermore, the Schur complement, we see that (26) is equivalent to
\[ \begin{bmatrix} \omega & (\Xi\theta - \bar{v})^T \\ \Xi\theta - \bar{v} & I \end{bmatrix} > 0. \]  \hspace{1cm} (28)
By adding a constraint condition on \( K_p, K_1 \) and \( K_d \), we can rewrite (22) as
\[ \hat{\theta}_n = \arg \min_{\theta_n} \omega \text{ subject to } \begin{bmatrix} \omega & (\Xi\theta - \bar{v})^T \\ \Xi\theta - \bar{v} & I \end{bmatrix} > 0 \]  \hspace{1cm} (29)
\[ \begin{bmatrix} X \Sigma Y \end{bmatrix} \]
\[ \Xi = X\Sigma Y, \Sigma = \begin{bmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \\ 0 & 0 & \sigma_3 \end{bmatrix} \]  \hspace{1cm} (24)
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Since the reference input shaping filter \( F \) should be stable, \( \hat{F}_n \) in Step 4 is assumed to be a finite impulse response approximation of \( F \). If \( F \) is stable and we take a sufficiently large number of steps, the approximation error becomes smaller because the impulse response of \( F \) converges to zero.
We will show the method to obtain \( \hat{F}_n \) as follows. By using an impulse response representation
\[ F(z) = \sum_{k=0}^{\infty} f(k)z^{-k}, \]  \hspace{1cm} (30)
we can express the output \( y(z) \) of the transfer function \( F(z) \) with the input \( u(z) \) as the product of the infinite dimensional Toeplitz matrix \( T(f) \)
\[ T(f) = \begin{bmatrix} f(0) & 0 & \cdots \\ f(1) & f(0) & \ddots \\ \vdots & \ddots & \ddots \end{bmatrix} \]  \hspace{1cm} (31)
and the infinite dimensional vector \( \nu(u) \)
\[ \nu(u) = \begin{bmatrix} u(0) \\ u(1) \\ \vdots \end{bmatrix}. \]  \hspace{1cm} (32)
That is,
\[ y(z) = \sum_{k=0}^{\infty} y(k)z^{-k} \]  \hspace{1cm} (33)
Furthermore,
\[ T(f)\nu(u) = T(u)\nu(f) \]  \hspace{1cm} (35)
is satisfied. Then, by defining
\[ a(z) := Tu_{SR} + T\hat{C}_{n,y_{SR}}, \quad b(z) := \hat{C}_{n,y_{SR}}, \quad c(z) := Tr \]  \hspace{1cm} (36)
or
\[ a(z) := Mu_{SD} + M\hat{C}_{n,y_{SD}}, \quad b(z) := y_{SD}, \quad c(z) := M d; \]  \hspace{1cm} (37)
we can rewrite (9) and (14) as
\[ \left\| \frac{T(a)}{\alpha T(c)} \nu(f) - \left[ \begin{array}{c} \nu(b) \\ \alpha \nu(c) \end{array} \right] \right\|^2. \]  \hspace{1cm} (38)
If we truncate the impulse response of \( F \) by a finite step \( N \), we can get
\[ \left\| \frac{T_N(a)}{\alpha T_N(c)} \nu_N(f) - \left[ \begin{array}{c} \nu_N(b) \\ \alpha \nu_N(c) \end{array} \right] \right\|^2. \]  \hspace{1cm} (39)
where
\[ T_N(f) = \begin{bmatrix} f(0) & 0 & \cdots & 0 \\ f(1) & f(0) & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ f(N) & \cdots & f(1) & f(0) \end{bmatrix}, \]  \hspace{1cm} (40)
\[ \nu_N(f) = \begin{bmatrix} f(0) \\ f(1) \\ \vdots \\ f(N) \end{bmatrix}. \]  \hspace{1cm} (41)
Then, \( \nu_N(f) \) minimizing (39) is given by
\[ \nu_N(f) = \begin{bmatrix} T_N(a) \alpha T_N(c) \\ T_N(a) \alpha \nu_N(c) \end{bmatrix}^{-1} \begin{bmatrix} T_N(a) \\ \alpha T_N(c) \end{bmatrix}^T \]  \hspace{1cm} (42)
By the elements \( f(k) \) of \( \nu_N(f) \), we obtain
\[ \hat{F}_n = \sum_{k=0}^{N} f(k)z^{-k}. \]  \hspace{1cm} (43)
IV. NUMERICAL EXAMPLES

A. Responses for the step reference input

We show a numerical example in which we tune PID controller (16) so that the closed-loop response closer to the reference model response.

We assume that we have

\[ P(z) = \frac{0.01(z + 0.5)}{(z - 1.06)(z - 0.1)} \]  \tag{44}  

as a plant (but the exact information on \( P \) is unknown). The plant \( P \) is assumed to be stabilized by a PID controller (16) with the initial parameter

\[ \theta_0 = \begin{bmatrix} 5 \\ 1 \\ 0 \end{bmatrix} . \]

When the unit step signal was applied to \( r \) in Fig. 1, we observed the response of \( u_{sr} \) and \( y_{sr} \) of \( P \). By using \( u_{sr} \) and \( y_{sr} \) and the response \( T_r \) of

\[ T(z) = \frac{0.2}{z - 0.8}, \]  \tag{45}  

we obtained the optimal parameter

\[ \theta := \begin{bmatrix} 25.0 \\ 1.01 \\ 43.2 \end{bmatrix} . \]  \tag{46}  

We used \( \alpha = 10 \) in Step 4. Fig. 2 shows the convergence of \( \theta_n \). Additionally, we show the impulse response of the obtained optimal reference input shaping filter in Fig. 3. The response converging to zero shows that the filter is stable.

To compare with the conventional method without using any filters, we obtained the optimal parameter

\[ \tilde{\theta} := \begin{bmatrix} 11.0 \\ 0.001 \\ 44.0 \end{bmatrix} . \]  \tag{47}  

by minimizing (9) where \( F_n = 1 \) in Step 4. We show the comparison in Fig. 4. The response of the closed-loop system before controller tuning, the response of the closed-loop system after conventional controller tuning and the proposed method, are depicted by the broken line, the dashed line, and the solid line, respectively. The dotted line in Fig. 4 shows the reference response \( T_r \). It shows that \( y(\tilde{\theta}) \) is not close to \( T_r \). On the other hand, \( y(\theta) \) is close to \( T_r \). Their relative errors are

\[ \frac{\| y(\tilde{\theta}) - T_r \|_2}{\| T_r \|_2} = 0.46 \]  \tag{48}  

and

\[ \frac{\| y(\theta) - T_r \|_2}{\| T_r \|_2} = 0.084. \]  \tag{49}  

Fig. 2. Convergence of Ki

Fig. 3. The impulse response of the obtained filter \( F \)

Fig. 4. Comparison of step responses of the closed-loop system before tuning (the broken line), after tuning by [2] (the dashed line), after tuning by the proposed method (the solid line). The dotted line is \( T_r \).

Fig. 5. Step response of the closed-loop system by \( \hat{C}_5 \) (the broken line), \( \hat{C}_{15} \) (the dashed line), and \( \hat{C}_{1000} \) (the solid line). The dotted line is \( T_r \).
Next, in Fig. 5, we show \( y(\hat{\theta}_5), y(\hat{\theta}_{15}) \), and \( y(\hat{\theta}_{1000}) \) by the broken line, the dashed line, and the solid line, respectively. The dotted line in Fig. 5 is \( Tr \). And, in Fig. 6, we show \( y(\hat{F}_5Tr), y(\hat{F}_{15}Tr) \), and \( y(\hat{F}_{1000}Tr) \) by the broken line, the dashed line, and the solid line, respectively. The dotted line in Fig. 6 is \( Tr \). By an iterative calculation, the closed loop system response close to the reference response and 

\[
\lim_{t \to \infty} \| y(\hat{\theta}_n) - Tr \|_2 / \| Tr \|_2 = 0.084,
\]

it can be seen from Fig. 7.

**B. Disturbance rejection**

To verify the effectiveness of the proposed method, we also show an experimental result in which we used a uniaxial type two-wheeled inverted robot called e-nuvo WHEEL (Fig. 8) for the plant. The input \( u \) is the command input to the wheel drive motor torque, and the output \( y \) is the angle of the body from the vertical direction.

The goal is to tune a PD controller (16) (in which \( K_i \equiv 0 \)) stabilizing e-nuvo WHEEL and to improve its disturbance rejection. The initial parameter

\[
\theta_0 = \begin{bmatrix} 2.0 \\ 0 \\ 0.2 \end{bmatrix}
\]

of the PD controller (16) stabilized the closed-loop system in which \( r \equiv 0 \). When the impulse-like signal was applied to \( d \) in Fig. 1 we observed the response of \( u_{sd}, y_{sd} \). By using the input/output data of e-nuvo WHEEL and the reference response \( Md \) of

\[
M(z) = \frac{0.03z}{z - 0.97},
\]

we obtained the optimal parameter

\[
\hat{\theta} = \begin{bmatrix} 2.1 \\ 0 \\ 0.43 \end{bmatrix}.
\]

We used \( \alpha = 1/3 \) in Step 4. Fig. 9 shows the convergence of \( \hat{\theta}_n \). Additionally, we show the impulse response of the obtained optimal reference input shaping filter in Fig. 10. The response converging to zero, the filter is stable.

We show the comparison in Fig. 11. The response of the closed-loop system before controller tuning, the response of the closed-loop system after the proposed controller tuning, are depicted by the broken line and the solid line, respectively. The dotted line in Fig. 11 shows the reference response \( Md \).

Next, in Fig. 12, we show \( y(\hat{\theta}) \) and \( FMd \) by the solid line and the broken line, respectively. The dotted line in Fig. 12 is \( Tr \). Since \( Md \) is properly shaped by \( F \), \( FMd \) is close to \( y(\hat{\theta}) \) than \( Md \). Indeed,

\[
\| y(\hat{\theta}) - Md \|_2 = 0.178 > \| y(\hat{\theta}) - FMd \|_2 = 0.114.
\]
V. CONCLUSION

In this paper, we have proposed the method to tune both a controller and a reference model so that we obtain the desired performance even if we are given an imperfect reference model. As the desired performance, we have considered the complementary and the disturbance response property.

REFERENCES