Game Theoretic Analysis of Customer Subscription Decisions in Networks with Positive Externality

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Abstract—In this paper we build a game-theoretic model that captures the strategic behavior of customers facing a service subscription purchasing decision. We focus on a class of products with positive network effects. Customers interact according to a network structure through which the positive externality is conveyed. We use a one-shot simultaneous incomplete information game to model the interaction of customers and provide an equilibrium in which a customer’s subscription decision depends on her network centrality. Using Banach Fixed Point Theorem, we prove the existence and the uniqueness of the equilibrium and provide an algorithm to compute the equilibrium. Finally, we apply our results to real network data to illustrate the customers’ decisions. We also discuss the convergence of our equilibrium computing algorithm and explain how the steps of algorithm relate to customers’ behavior in reaching the equilibrium in reality.

I. INTRODUCTION

Network effects are prevalent in product purchasing decisions where the utility a user gains by consuming the good changes with other people’s consumption. The term externality is used to denote this change in utility [1]. The externality of a service product is often negative when the firm’s capacity to serve its customers is limited: A service user causes congestion to the system, incurring waiting cost to other customers [2], [3]. Under the existence of negative externality, performance of firms selling service products in the form of subscription has been well analyzed by researches in the field of operations management [4]. In those settings, one customer incurs disutility to all other customers waiting in line, i.e., the interaction among customers is all to all.

In this paper, we analyze the customers’ subscription behavior when the service firm’s capacity is non-binding and positive externality is exerted locally by subscribers interacting according to a social network structure. Examples of this setting can be such as cell phone service and golf-club membership, where customers gain higher utility by purchasing a membership from the same service carrier as more of her immediate friends.

More specifically, cell phone carriers in many countries charge their members per-minute fee on top of a fixed monthly membership fee. Most of the carriers give discount to in-network calls, i.e., calls between members of the same carrier, and charge higher per-use price for out-of-network calls. A high per-use fee is charged to people who pay-per-use without purchasing the subscription plan. AT&T’s GoPhone or “Pay As You Go” plan for mobile phone service carriers are examples of the pay-per-use option. A phone user then estimates the amount of phone calls she will make to her friends, and decides whether to purchase the membership or to pay-per-use. The decision depends both on the estimation of her usage and her friends’ subscription decisions, which depend on the friends’ usage.

Before we move on to the specifics of the model, it is worth mentioning that customers’ actions depend on information that is not necessarily known to them: their friends’ usage. This is because a customer’s decision depends on her friend’s decision, which is a function of that friend’s usage amount. Unobservable information leads the customers’ decisions to be modeled as an incomplete information game. Modeling this as an incomplete information game has a benefit: games played on networks usually have multiple equilibria but introducing uncertainty can resolve the multiplicity [5]. Customers process the incomplete information by constructing a belief about their friends’ usage as a form of a probability distribution function.

To analyze the customers’ decisions we build a one-shot simultaneous incomplete information game that captures the strategic behavior of customers. We show that the equilibrium subscription decisions follow a threshold policy where the customers subscribe only when their usage is greater than some threshold values. We find that there is a linkage between the threshold value of a customer and a network centrality measure of the user.

Our work lies on the stream of literature that studies network games in which players interact locally according to a network structure. In particular, our problem is related to [6], [7], [8], and [9] in that the decisions of players incur externality and the equilibrium of the game relates to Bonacich Centrality. However, our model differs from previous researches in two aspects. 1) Context-wise, our paper models the customers choosing between two different service packages that are using different pricing schemes, subscription versus pay-per-use. Yet previous models assume pricing scheme to be exogenous to the agents playing the game. This difference in the level of customers’ decisions fundamentally distinguishes our problem from others. For example, the setting in [9] is similar to ours as they also look into customers purchasing service goods with positive network externality. Nevertheless, customers in their model decide how much service to purchase given the pay-per-use pricing scheme, whereas in our model they decide whether
they pay-per-use or subscribe given the amount of service they will use. 2) Theoretically, our model analyzes binary decisions of players, while others study decisions of agents over continuous action spaces. Binary decision games are in general tedious to analyze due to existence of multiple equilibria, which may require a combinatorial approach to compute the complete set. However, by introducing uncertainty to players’ knowledge about their neighbors, our game yields a unique threshold type equilibrium. The unique equilibrium turns out to have a linkage to Bonacich centrality as in [6], [7] [8], and [9].

II. THE MODEL

A. The Setting

A monopolistic service firm sells two different types of service plans: subscription and pay-per-use. From now on we use “subscribers” and “members” interchangeably to refer to customers who subscribe and use “non-subscribers” or “non-members” to denote customers who pay-per-use. Subscribers are charged a fixed membership fee, $A$, by default, and additional per-use fee of $P_M > 0$ when they consume the service with a non-member. $P_M$ can be thought of as the per-minute fee a cell phone service carrier charges to its subscribers per-use fee when they use the service with non-members in addition to the membership fee, while non-members incur per-use cost for all the service they use. The utility function of each customer is then assumed to be $u_i(x_i, x_{i-1}; G) = -q_i(x_i, x_{i-1}; G)$, and each customer chooses action $x_i$ to maximize her own utility given what others do.

We further assume that customer-$i$ knows her own degree and usage, but knows neither those nor the subscription decision of her neighbors. However, customer-$i$ forms a belief about the total usage of her neighbor-$j$, according to a cumulative distribution function $F(t) = \Pr(\sum_k \omega_{jk} \leq t)$ for all $j \in N(i)$. We assume that the distribution is identical and independent for all neighbors of all customers.

III. THE EQUILIBRIUM

We now characterize a Bayesian Nash equilibrium of the simultaneous, incomplete information game described in section II and prove the existence and the uniqueness of the equilibrium.

A. Characterizing the Equilibrium

As a procedure to solve for an equilibrium of our game, we will first postulate that a certain type of policy is an equilibrium. Then we prove that the policy is indeed a Nash equilibrium by showing that there is no incentive for anybody to deviate.

**Theorem 1 (Characterization of the Equilibrium).** The following threshold policy:

$$x_i = \begin{cases} 1, & \text{if } \sum_j \omega_{ij} \geq T_i \\ 0, & \text{if } \sum_j \omega_{ij} < T_i \end{cases}$$

with the threshold, $T_i$, characterized by

$$P_N T_i - P_M \sum_j \omega_{ij} F(T_j) - A = 0$$

is a Bayesian Nash Equilibrium of the membership game defined in Section II.

**Proof:** Suppose that everybody besides customer-$i$ is following the postulated threshold policy defined in (2). Note that customer-$i$’s objective is to maximize her utility, $u_i(x_i, x_{i-1}; G)$, which is identical to minimizing the cost, $q_i(x_i, x_{i-1}; G)$. Customer-$i$’s problem then becomes:

$$\min_{x_i} \left\{ A + P_M \sum_j \omega_{ij} \cdot (1-x_j) \right\} \cdot x_i + \left\{ P_N \sum_j \omega_{ij} \right\} \cdot (1-x_i),$$

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which reduces to
\[ \max_{x_i} \left\{ P_N \sum_j \omega_{ij} - P_M \sum_j \omega_{ij}(1 - x_j) - A \right\} x_i. \]
Then the utility maximizing action for customer-\(i\) is:
\[ x_i = \begin{cases} 1, & \text{if } P_N \sum_j \omega_{ij} - P_M \sum_j \omega_{ij}(1 - x_j) - A \geq 0 \\ 0, & \text{if } P_N \sum_j \omega_{ij} - P_M \sum_j \omega_{ij}(1 - x_j) - A < 0. \end{cases} \]
Since customer-\(i\) does not know the edge-weights of her neighbors, she does not know \(x_j\). However, \(i\) has a belief about the sum of edge-weights of \(j\), and can calculate the expected value of \(x_j\) as
\[ E\left[ P_N \sum_j \omega_{ij} - P_M \sum_j \omega_{ij}(1 - x_j) - A \right] = P_N \sum_j \omega_{ij} - P_M \sum_j \omega_{ij} \Pr\left( \sum_k \omega_{jk} < T_j \right) - A. \]
The above equality follows from the assumption that all customers besides \(i\) are following the threshold policy (2), and customer-\(i\)'s belief about \(j\) choosing \(x_j = 0\) is with probability \(\Pr(\sum_k \omega_{jk} < T_j)\). Now one can see that the best response for customer-\(i\) is to play \(x_i = 1\) if and only if
\[ P_N \sum_j \omega_{ij} - P_M \sum_j \omega_{ij} F(T_j) - A \geq 0. \]
Hereby, we showed that when everybody except for \(i\) follows the threshold policy (2), then to follow the same threshold policy with the threshold characterized by (3) is the best response for customer-\(i\). Therefore, there is no incentive for anybody to unilaterally deviate, and this shows that (2) is indeed a Nash equilibrium of the game.

**B. Computing Thresholds**

So far, we have shown that a threshold type policy is an equilibrium of the customers’ membership decision game and have characterized the thresholds. Now we prove that under reasonable belief of customers about their neighbors’ usage, this equilibrium always exists and that the existing threshold equilibrium is unique. Then we also provide a constructive method to compute the equilibrium thresholds.

**Theorem 2** (The Existence and the Uniqueness of the Equilibrium). Suppose that customers’ belief about their neighbors’ service usage is represented by cumulative distribution function \(F : \mathbb{R} \to [0, 1]\) that is Lipschitz continuous:
\[ \|F(t_1) - F(t_2)\| \leq L \|t_1 - t_2\| \quad (4)\]
with coefficient \(L < P_N/(P_M \sigma_1(W))\), where \(\sigma_1(\cdot)\) denotes the maximum singular value of \(W\). Then there exists the equilibrium threshold vector, \(\bar{T}_i = T_i\), satisfying (3) for each \(i\) and the threshold vector is unique.

**Proof**: We define a vector form of the c.d.f., \([F(\bar{T})]_i = F(T_i)\), and write (3) as a vector equation,
\[ P_N \bar{T} - P_M W F(\bar{T}) = A \bar{I}, \quad (5) \]
where \(\bar{I}\) denotes the vector of ones. Normalizing by \(P_N\) characterizes \(\bar{T}\) with a fixed point equation
\[ \bar{T} = P_M W F(\bar{T}) + A/P_N \bar{I}. \quad (6) \]
We now define the mapping \(H : \mathbb{R}^N_+ \to \mathbb{R}^N_+\) as
\[ H(\bar{T}) = P_M W F(\bar{T}) + A/P_N \bar{I}. \quad (7) \]
Then the fixed point equation (6) can be written as
\[ \bar{T} = H(\bar{T}). \]
We see that
\[ ||H(\bar{T}_1) - H(\bar{T}_2)|| = ||P_M W F(\bar{T}_1) - F(\bar{T}_2)|| \leq \sigma_1(W) P_M ||F(\bar{T}_1) - F(\bar{T}_2)|| \leq \sigma_1(W) P_M L ||\bar{T}_1 - \bar{T}_2||, \]
where the first inequality follows from the matrix norm \(||W||\) being bounded above by the maximum singular value \(\sigma_1(W)\) and the second comes from the assumption of \(F(t)\) being Lipschitz continuous. Our premise of \(L < P_N/(P_M \sigma_1(W))\) guarantees that there exists \(\delta = \sigma_1(W) P_M / P_N L < 1\) such that
\[ ||H(\bar{T}_1) - H(\bar{T}_2)|| \leq \delta ||\bar{T}_1 - \bar{T}_2||, \quad (8) \]
i.e., mapping \(H(\bar{T})\) is contracting. Since \(H\) is a contraction mapping on a non-empty complete metric space, Banach Fixed Point Theorem [10][Theorem B.1] concludes that the equilibrium (2) with \(\bar{T}\) satisfying (5) is the unique fixed point of mapping \(H\).

Theorem 2, on the one hand, answers the theoretical question of when the equilibrium exists and whether it is unique if it exists. On the other hand, it sheds light to the practical issue of computing the threshold vector \(\bar{T}\) as shown in the following corollary.

**Corollary 1** (Computing the Thresholds). Suppose that \(F\) is Lipschitz continuous as defined in (4). Then for \(\bar{T}^0 \in \mathbb{R}^N_+\), the sequence \(\{\bar{T}^k\}_{k=0}^{\infty}\) defined as
\[ \bar{T}^k = P_M W F(\bar{T}^{k-1}) + A/P_N \bar{I}, \quad (9) \]
converges to the unique equilibrium threshold vector that satisfies (6).

Note that in order to compute an individual’s equilibrium threshold, iteration (9) seems to use the global information, \(W\), about the entire network, which is not available to that individual. However, we can break it down to a component-wise equation that uses only the local network information, \(\omega_{ij}\), and the threshold \(T_j\) of the immediate neighbors of \(i\) to calculate customer-\(i\)’s threshold as the following corollary states.

**Corollary 2** (Locality of the Thresholds). The fixed point iteration equation (9) that can be written as a component-wise equation for customer-\(i\) that uses local usage information, \(\omega_{ij}\), and the threshold value, \(T_j\), for \(j \in N(i)\) as follows.
\[ T_i^k = P_M \sum_{j \in N(i)} \omega_{ij} F(T_j^{k-1}) + A/P_N \bar{I}, \quad (10) \]
Computing thresholds using (10) in practice can be thought of as customers communicating their strategies with their immediate neighbors and adjusting their responses before they make their decisions. Because the mapping \(H(\bar{T})\)
is contracting and the sequence \( \{ \mathbf{T}_k \}_{k=0}^{\infty} \) defined in (9) converges, rounds of such communication guarantees the convergence of customers’ action to the true equilibrium of the game. Although it seems to require global network information to solve an individual optimization problem, customers can make a near optimal decision by communicating their subscription decisions with their immediate friends.

So far, we have shown that a unique equilibrium exists for the customers’ game and it can be computed using an iterative process when the belief of customers about their neighbors’ usage is well-behaved. Now we mention that the customers’ “belief” about their neighbors’ usage, \( F \), can be considered as the perceived degree distribution of the service network. Recall that the edge-weights of the network are the service usage amount between two people, and the degree of a node is the total usage of that customer. Thus, the result in this section relates the graph structure with the existence, uniqueness, and computability of the equilibrium of our game played on the network. More specifically, if the customers believe the network to have a smooth degree distribution and the maximum singular value satisfying \( \sigma_1(\mathbf{W}) < \frac{P_M}{P_N(P_M L)} \), then our game exhibits a unique equilibrium.

C. Centrality-Based Thresholds

In this section, we assume a linear functional form for the belief distribution and determine a closed-form expression for the threshold values. We then relate the expression of the thresholds to Bonacich centrality measure defined in [6].

**Theorem 3.** Given a uniform belief distribution with \( \sum_k \omega_{jk} \in [0, M] \) and \( M = \max \{ \mathbf{W} \} \), the equilibrium threshold vector defined in (6) is given as

\[
\mathbf{T} = \left( \mathbf{I} - \frac{P_M}{P_N M} \mathbf{W} \right)^{-1} \frac{A}{P_N} \mathbf{1}. \tag{11}
\]

**Proof:** The uniform cumulative distribution function is

\[
F(t) = \begin{cases} 
0, & \text{if } t < 0 \\
\frac{t}{M}, & \text{if } 0 \leq t \leq M \\
1, & \text{if } t > M.
\end{cases}
\]

Since \( 0 \leq F(T_j) \leq 1 \) for all \( j \),

\[
T_i = \frac{P_M}{P_N} \sum_{j \in N(i)} \omega_{ij} F(T_j) + \frac{A}{P_N} \in \left[ \frac{A}{P_N}, \frac{P_M}{P_N} \omega_i + \frac{A}{P_N} \right],
\]

where \( \omega_i = \sum_{j \in N(i)} \omega_{ij} \). Then for the parameter values satisfying \( \frac{P_M}{P_N} \omega_i + \frac{A}{P_N} \leq M \) for all \( i \), we can write \( F(\mathbf{T}) \) as

\[
F(\mathbf{T}) = \frac{\mathbf{T}}{M}. \tag{12}
\]

Substituting (12) into the fixed point equation (6), we have

\[
\left( \mathbf{I} - \frac{P_M}{P_N M} \mathbf{W} \right) \mathbf{T} = \frac{A}{P_N} \mathbf{1}. \tag{13}
\]

Since \( P_M < P_N \) by assumption and \( \sigma_1(\mathbf{W}) \leq M \) by [11][Section 2.2], the matrix \( \left( \mathbf{I} - \frac{P_M}{P_N M} \mathbf{W} \right) \) is positive definite and therefore invertible, proving (11).

Theorem 3 shows that when the belief distribution is uniform, the equilibrium threshold vector follows the form of Bonacich centrality vector defined as

\[
\mathbf{C}(\mathbf{W}, a) = (I - a \mathbf{W})^{-1} \mathbf{1} = \sum_{k=0}^{\infty} a^k \mathbf{W}^k \mathbf{1}. \tag{14}
\]

Note that \( \sum_{k=0}^{\infty} a^k \mathbf{W}^k \) counts the number of paths in the graph that start from node-\( i \) and end at \( j \). \( a \) is an attenuation factor determining how the influence of walks drops off with their length and \( \mathbf{W}^k \) represents the \( (i, j) \)-component of \( k \)-th power of \( \mathbf{W} \). Thus, Bonacich centrality for node-\( i \)-counts the total number of paths that starts from \( i \), and measures how influential the node is within a social network.

Note that our equilibrium threshold values can be written in terms of the Bonacich centrality \( \mathbf{C} \) with \( a = P_M / (P_N M) \) as

\[
\mathbf{T} = \frac{A}{P_N} \mathbf{C}(\mathbf{W}, P_M / P_N M). \tag{15}
\]

Therefore, the equilibrium threshold values for customers that are more influential are higher than less influential customers. Even though the customers’ real usage distribution may not follow uniform distribution, the analysis is worth studying for expository purpose. For a general belief distribution \( F(t) \), although we may not be able to express the thresholds using Bonacich centrality, the fixed point iteration (9) implies that we can still express them as centrality-like measure, specific to the distribution analogously as for the uniform distribution.

Recall that Bonacich centrality measures the number and the strength of paths between nodes. This can still be expressed in thresholds with general \( F(t) \) by breaking one step local iteration (10) into two pieces:

\[
y_{k-1} = F(T_{k-1}) \]

\[
T_i^k = \frac{P_M}{P_N} \sum_{j \in N(i)} \omega_{ij} y_{j-1} + \frac{A}{P_N} \tag{15}
\]

From (15) we see that the threshold for customer-\( i \) is determined by summing the “scores” of its immediate neighbors, while the score, \( y_{k-1} \), is the distribution function value of \( j \)-’s threshold, \( F(T_{k-1}^j) \), i.e., the probability that \( j \) will not purchase the membership. Thus, if your neighbor’s probability of not purchasing the membership is higher, then you have higher threshold value. The scores are weighted by the strength of the relationship. \( A/P_N \) is a constant factor added to all of the nodes, so if the fixed membership fee, \( A \), is high compared to the non-members’ per-use fee, \( P_N \), then customers tend to have higher threshold values.

Note that each step of iteration (15) introduces the effect of one hop longer paths in the graph to customer-\( i \)-’s threshold. Therefore, the result of \( k \)-th iteration, \( T_i^k \), measures the effect of \( k \)-hop neighbors’ threshold values on customer-\( i \)-’s membership decision. To illustrate, one customer’s non-purchasing probability factors into her neighbors’ threshold values and their non-purchasing probabilities. Then the probabilities of those neighbors further affect their neighbors’ thresholds, and one can see how this chain of action can be extended to further neighbors on the network. Note that the effect of longer paths are discounted by the attenuation factor,
$P_M/P_N$, and when members have to pay higher per-use fee, the customers become more sensitive about the further away neighbors’ non-purchasing probabilities. This shows that your purchasing behavior not only depends on your immediate neighbors’ decisions but also on their neighbors and their neighbors and so on.

Note how the explanation of the threshold vector with general $F$ is analogous to that of Bonacich centrality measure. Hereby, we can conclude that threshold value $T_i$ is determined as the centrality of customer-$i$, so that more central customers have higher threshold values. Since more central customers have higher thresholds, they are the ones that require stricter criterion to be satisfied in order to subscribe. In other words, the subscription behavior of customers acts in an unfavorable direction for the firm: more influential customers are more difficult to subscribe. Thus, if the firm can subsidize and attract a central customer as a subscriber, it may increase the firm’s revenue by not only converting that customer but also the ones that are influenced by her. This is a conflicting result with [9] since their result shows the optimal price the firm should charge to each customer is in proportion with the customer’s Bonacich centrality measure. Thus, the firm in their setting is better off charging higher price to more central customers, while in our setting the firm may be better off giving discounts to more influential ones. This is because the firm selling subscription earns disproportionately large amount of money from the fixed membership fee. Hence, it may be better off for them to attract more people as members, and this is made possible by charging low price to the more influential customers.

IV. CELL PHONE NETWORK APPLICATION

We now apply our analytical result to a real cell phone usage network culled from the MIT Media Lab Reality Mining data set [12] to visualize the equilibrium computed for the setting driven by real data. The data includes voice call information for a large group of individuals, including who they are calling and how long these calls last. We generate network $G$ by adding an edge between two people if at least one call is placed between them. The weight of each edge is determined as the total time in seconds the two people is connecting spent on the phone divided by the time in days over which the data was collected.

In order to apply our results we first need to model the belief function of customers about their neighbor’s usage. The belief distribution can also be thought of as the degree distribution of the customers’ service network when the beliefs are consistent with the real usage data. However, note that the belief of customers is not necessarily consistent with the real usage distribution, and what matters in the equilibrium of our game is the customers’ perceived distribution rather than the true usage distribution itself.

We consider three different distribution types, uniform, scale-free and Gamma distribution. Uniform distribution is e.g. [13] and [14]. The distribution is defined as

$$F(t) = 1 - \left( \frac{t}{t_{\min}} \right)^{-\beta+1},$$

where $t_{\min} = \min_i \sum_{j \in N(i)} \omega_{ij}$ and the parameter $\beta = 1.1740$. The specific value for parameter $\beta$ was computed using maximum likelihood estimation by fitting the distribution to the real usage data. From Figure 1 we see that the degree distribution for this data is not particularly well-fitted by the scale-free distribution although it is one of the most commonly assumed degree distributions in random graphs. Thus, we also consider Gamma distribution, which is defined as

$$F(t) = \frac{\int_{t}^{\infty} y^{k-1} e^{-y} dy}{\int_{0}^{\infty} y^{k-1} e^{-y} dy},$$

where shape parameter $\kappa = 1.7087$ and scale parameter $\theta = 756.4244$ were computed using maximum likelihood estimation. We incorporated Gamma distribution in our analysis in order to study a belief distribution that is consistent with the real usage data. From Figure 1, it is clear that Gamma distribution fits our data very well. For the three, uniform, scale-free and Gamma degree distributions, we
We showed that the equilibrium threshold value for each customer is proportional to her centrality on the network. This relationship between the threshold value and centrality measure implies that subsidizing more central customers may make the firm better off.

We developed an iterative algorithm that uses only the local degree information and the immediate neighbors’ threshold values. As the algorithm represents, customers could compute the equilibrium correctly by locally communicating and adjusting their threshold values before making their final decision. Furthermore, each iteration in our computation algorithm could be justified as each round of communication and adjustment among customers in the real setting.

We used the voice calling data culled from MIT Media Lab Reality Mining data set to fit the degree distribution and to discuss the difference in customers’ decisions depending on different distribution of beliefs. We also addressed the convergence of our iterative computation algorithm run based on the real data. The algorithm typically converges in few iterations which then means that only few rounds of communication among customers are necessary for them to correctly compute their threshold values in reality.

So far, we have focused solely on the customers’ side of the service subscription market. The natural step forward will be to include the firm into the picture and let the firm optimally decide its pricing scheme. Also, if the firm is allowed to perform price discrimination among customers as in [9], figuring out whom in the network it should subsidize in order to maximize the enrollment will be an interesting direction to pursue.

V. DISCUSSION AND CONCLUSION

In this paper we analyzed the behavior of customers who are facing a decision of purchasing a service subscription with positive network externality. We assumed a network of customers with the edges representing service sharing relationship and edge-weights capturing the amount of service shared by the two people they are connecting.

A one-shot simultaneous incomplete information game was used to model the interaction of customers, assuming that customers only know their own usage amount (i.e. degree) but not about their friends’ usage. Customers have beliefs in the form of probability distribution function about their immediate neighbors’ usage amount.

A threshold type equilibrium was shown to be the unique equilibrium of the game. Banach Fixed Point Theorem enabled us to both guarantee the existence of the equilibrium and to set up an iterative algorithm that computes the equilibrium threshold values.

Fig. 3. Examining the iterative method by which we compute the threshold, we find that after only 1 or 2 iterations the thresholds are accurate enough that continued updates no longer effect the membership fraction.