Output Regulation for Linear Systems with Unknown Exosystem Order

Riccardo Marino and Patrizio Tomei

Abstract—The regulator problem is considered for linear systems when the linear exosystem which generates the disturbances to be rejected and/or the reference signals to be tracked is unknown, that is both its order $2r+1$ and its parameters $\{\omega_1, \ldots, \omega_r\}$ are unknown. The proposed regulator makes use of two observers and contains an adaptive internal model which adapts $m$ internal parameters on the basis of the output regulation error. It is shown that: if $m \geq r$, that is the adaptive internal model can reproduce the required input, the regulation error tends exponentially to zero; if $m < r$, that is the adaptive internal model can only approximate the required input within an approximation error $\epsilon_M$, the regulation error tends exponentially into a closed ball whose radius is proportional to $\epsilon_M$, provided that $\epsilon_M$ is sufficiently small so that singularities are avoided.

I. INTRODUCTION

The main goal of feedback control design is to reject, cancel or attenuate the effect of external disturbances on the regulation error, on the basis of its measurements only: the reference signals are viewed as external disturbances in this context. When disturbances are constant, the integral control action can drive the regulation error to zero. When disturbances are sinusoidal, or periodic or sum of periodic signals, the disturbance cancellation problem becomes immediately very complex and is still unsolved in many instances also for linear systems [1]. The internal model principle [2], [3] provides the necessary and sufficient structural conditions as far as linear systems are concerned: the system should be stabilizable and detectable so that it may be stabilized by an error feedback compensator; the disturbances should be generated by a linear exosystem whose eigenvalues do not coincide with the zeros of the system, so that they are observable from the regulation error and can be generated by the controller. A linear exosystem can only produce disturbances which are made of finite sums of sinusoidal terms: it is the task of the controller to reproduce such disturbances on the basis of the regulation error only, without knowing the exact number of sinusoidal terms and their frequencies. Hence, the design problem can be recast as a robust adaptive observer design [4]: the adaptive observer may be interpreted as an adaptive internal model. The design of an adaptive internal model was first addressed for minimum phase continuous-time systems using adaptive controls by Feng and Palaniswami [5]. However, since both the system and the exosystem parameters are estimated by the adaptive control, sufficient persistency of excitation is required to guarantee robustness. Assuming that the system is known, the problem of rejecting unknown sinusoidal disturbances was studied in [6], [7], [8], [9]. An adaptive observer approach is followed in [10] assuming that the number of frequencies contained in the disturbance signal is known (see [11] for multivariable systems), while only an upper bound is needed in [12]. This crucial assumption has been recently removed in [13] as far as uncertain minimum phase systems are concerned, allowing for totally unknown exosystems. However, since the minimum phase assumption is not necessary for the regulator problem, we address the design of an adaptive regulator with an unknown exosystem (i.e. both its order and its parameters are unknown) assuming only the well known necessary conditions for the solution of the regulator problem.

II. PRELIMINARY RESULTS

Consider the linear single-input single-output system

\[
\begin{aligned}
\dot{x}(t) &= Ax(t) + bu(t) + Pw(t), & x(0) = x_0 \\
w(t) &= Rw(t), & w(0) = w_0 \\
e(t) &= cx(t) + qw(t)
\end{aligned}
\]  

(1)

in which $x \in \mathbb{R}^n$ is the state vector, $u \in \mathbb{R}$ is the scalar input, $e \in \mathbb{R}$ is the scalar output to be regulated to zero which is the only measured signal. Both the disturbances to be rejected $Pw$ and the signal to be tracked $-qw$ are generated by the linear exosystem

\[
w = Rw, & \quad w(0) = w_0, & \quad w \in \mathbb{R}^{2r+1}.
\]  

(2)

Necessary and sufficient conditions for the solution of the regulator problem are well known (see [2], [3]): (H1) the pair $(A,b)$ is stabilizable (i.e. rank $(A - \lambda I, b) < n$ implies $\text{Re}\lambda < 0$); (H2) the pair $(A,c)$ is detectable (i.e. rank $\begin{bmatrix} A - \lambda I & b \\ c & 0 \end{bmatrix} < n$ implies $\text{Re}\lambda < 0$); (H3) rank $\begin{bmatrix} A - \lambda I & b \\ c & 0 \end{bmatrix} = n + 1$, for any eigenvalue $\lambda$ of the matrix $R$. It is remarkable that the system $(A,b,c)$ is not restricted to be minimum phase: the zeros of the system are only required not to coincide with the eigenvalues of the exosystem $R$ by (H3). The sufficiency part of the proof of this theorem is constructive and leads to the design of a regulator which incorporates the exosystem itself (internal model principle), if the exosystem matrix $R$, the triple $(A,b,c)$ and the matrices $P$, $q$ are known. By virtue of (H3) there exists two matrices $\Gamma$ and $\gamma$ which are the unique
solutions to the regulator equations
\[ \Gamma R = A\Gamma + b\gamma + P \]
\[ e\Gamma + q = 0. \] (3)
Defining \( x_r = \Gamma w \) and \( u_r = \gamma w \), by virtue of (3) and (1), they satisfy
\[ \dot{x}_r = Ax_r + bu_r + Pw \]
\[ cx_r + qw = 0. \] (4)
in which \( x_r \) and \( u_r \) are the reference signals for \( x \) and \( u \), respectively, and in particular \( u_r \) is the open loop solution to the regulator problem. Typically the matrix \( R \) contains \( r \) frequencies \( \{\omega_1, \ldots, \omega_r\} \) which characterize the disturbances \( Pw \) to be rejected and/or the reference \( qw \) to be followed. If \( r \) is known, while the frequencies \( \omega_1, \ldots, \omega_r \), are distinct and unknown, it is shown in [10] that the regulator problem can still be solved by introducing an adaptive internal model: an adaptive observer is designed which at the same time generates exponentially converging estimates of the open loop solution \( \gamma w \) and of the \( r \) unknown frequencies, provided that all \( r \) frequencies which are actually contained in the open loop solution \( \gamma w \) are excited. In the cases in which only an upper bound \( r \) is known for the frequencies, an on-line detection of the actual number of frequencies is proposed in [12], so that the controller chooses among \( r \) adaptive internal models the correct one. We address the realistic case in which the matrix \( R \) is totally unknown, that is it contains an unknown number \( r \) of unknown distinct frequencies \( \{\omega_1, \ldots, \omega_r\} \), with the aim of re-obtaining the results of the classical regulator. We solve the regulator problem assuming that only the triple \((A, b, c)\) and a lower bound on the frequencies of the exosystem are known, i.e. we add the following technical assumption (H4) to (H1)-(H3): (H4) The spectrum of the exosystem matrix \( R \) in (2) is \( \{0, \pm \omega_i, 1 \leq i \leq r\} \) with \( \omega_i \geq \omega_{\min}, 1 \leq i \leq r, \omega_{\min} \) a known lower bound, \( \omega_1, \ldots, \omega_r \), positive distinct unknown parameters, \( r \geq 0 \) unknown.

III. OUTPUT REGULATOR DESIGN

Define the state regulation error \( \tilde{x} = x - x_r \) whose dynamics are obtained subtracting (4) from (1)
\[ \dot{\tilde{x}} = A\tilde{x} + b(u - u_r) \]
\[ e = c\tilde{x}. \] (5)
Since by virtue of (H2) the pair \((A, c)\) is detectable we operate a Kalman decomposition with respect to the unobservable part so that the known triple \((A, b, c)\) is transformed by the linear change of coordinates \((\tilde{x}_u, \tilde{x}_o) \in \mathbb{R}^{n-\nu}, \tilde{x}_o \in \mathbb{R}^\nu\)
\[ \begin{bmatrix} \tilde{x}_u \\ \tilde{x}_o \end{bmatrix} = T_1 \tilde{x} \] (6)
into
\[ \begin{bmatrix} A_u & A_{uo} \\ 0 & A_o \end{bmatrix}, \begin{bmatrix} b_u \\ b_o \end{bmatrix}, \begin{bmatrix} 0 & c_o \end{bmatrix} \] (7)
with \( A_u \) a Hurwitz matrix and \((A_o, b_o, c_o)\) a triple in observer canonical form, i.e.
\[ A_o = \begin{bmatrix} -a_{\nu-1} & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ -a_0 & 0 & \cdots & 0 \end{bmatrix}, b_o = \begin{bmatrix} b_{\nu-1} \\ \vdots \\ b_0 \end{bmatrix} \\
\]
\[ c_o = \begin{bmatrix} 1 & 0 & \cdots & 0 \end{bmatrix}. \] (8)
We may assume, without loss of generality, that the pair \((R, \gamma)\) is observable: in fact, if the pair \((R, \gamma)\) is not observable we may perform a Kalman decomposition and factor out the unobservable part which has no influence on \( \tilde{x} \) and on the output \( e \). Instead of considering the observable pair \((R, \gamma)\) to generate the open loop control \( u_r \), we may equivalently consider its observer canonical form
\[ \begin{bmatrix} \tilde{\eta} \\ \tilde{x}_u \\ \tilde{x}_o \end{bmatrix} = A_o \begin{bmatrix} \tilde{\eta} \\ \tilde{x}_u \\ \tilde{x}_o \end{bmatrix} + \begin{bmatrix} b_u \\ b_o \end{bmatrix} u \\
\]
\[ e = \begin{bmatrix} 0 & c_o & 0 \end{bmatrix} \begin{bmatrix} \tilde{x}_u \\ \tilde{x}_o \\ \tilde{\eta} \end{bmatrix} \] (9)
in which
\[ A_c = \begin{bmatrix} 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix}, c_c = \begin{bmatrix} 1 & 0 & \cdots & 0 \end{bmatrix}, \]
\( E_{2i} \) is the \( 2i \)-th column of the \((2r + 1) \times (2r + 1)\) identity matrix and \( \tilde{\theta} = [\theta_1, \ldots, \theta_r]^T \) is the unknown vector of coefficients of the exosystem characteristic polynomial
\[ s(2r + 1)^2 + \tilde{\theta}_1 s^2(r-1) + \cdots + \tilde{\theta}_r \triangleq s \prod_{i=1}^r (s + \omega_i^2). \] (10)
System (5), (9) becomes
\[ \begin{bmatrix} \dot{x}_u \\ \dot{x}_o \\ \dot{\eta} \end{bmatrix} = \begin{bmatrix} A_u & A_{uo} & -b_u c_c \\ 0 & A_o & -b_o c_c \\ 0 & 0 & R_c \end{bmatrix} \begin{bmatrix} x_u \\ x_o \\ \eta \end{bmatrix} + \begin{bmatrix} b_u \\ b_o \end{bmatrix} u \\
\]
\[ e = \begin{bmatrix} 0 & c_o & 0 \end{bmatrix} \begin{bmatrix} x_u \\ x_o \\ \eta \end{bmatrix} \] (11)
where, by virtue of (H3), the subsystem
\[ \begin{bmatrix} \dot{x}_o \\ \dot{\eta} \end{bmatrix} = \begin{bmatrix} A_o & -b_o c_c \\ 0 & R_c \end{bmatrix} \begin{bmatrix} x_o \\ \eta \end{bmatrix} + \begin{bmatrix} b_o \end{bmatrix} u \\
\]
\[ e = \begin{bmatrix} c_o & 0 \end{bmatrix} \begin{bmatrix} x_o \\ \eta \end{bmatrix} \] (12)
is observable. Since by (H1) the pair \((A, b)\) is stabilizable, a row vector \( k_c \) exists such that all eigenvalues of the matrix \((A_o + b_o k_c)\) have negative real part. The control input \( u \) is defined as
\[ u = k_c \dot{x}_o + v \] (13)
with \( \hat{x}_o \) given by the observer for the observable subsystem in (7)

\[
\dot{\hat{x}}_o = A_o \hat{x}_o + b_o k_o (e - c_o \hat{x}_o)
\]  

(14)

where, since by (H1) the pair \((A, c)\) is detectable, the vector \(k_o\) may be chosen such that the matrix \((A_o + k_o c_o)\) is Hurwitz with arbitrary eigenvalues, by virtue of the Kalman decomposition (7). The observer error dynamics are \((\hat{e}_o = \hat{x}_o - \hat{x}_o)\)

\[
\dot{\hat{e}}_o = (A_o + k_o c_o) \hat{e}_o - b_o u_r + b_o \nu
\]  

(15)

so that \(\hat{e}_o(t)\) is bounded, provided that \(v(t)\) is bounded. From (12), (13) and (14), we obtain the closed loop observable error dynamics

\[
\dot{\hat{x}}_o = (A + b_o k_o) \hat{x}_o + b_o (v - u_r) - b_o k_o \hat{e}_o
\]  

(16)

so that \(\hat{x}_o(t)\) is bounded, provided that \(v(t)\) and, consequently, \(\hat{e}_o(t)\), are bounded. From (11), since \(A_o\) is Hurwitz, it follows that if \(v(t)\) is bounded also \(\hat{x}_o(t)\) is bounded. Resuming, \(v(t)\) bounded implies \(\hat{x}(t)\) and \(z(t)\) bounded. Now, let \(u_r(t)\) be approximated by the signal \(u_{rE}(t)\) generated by the exosystem

\[
\eta_{E} = \frac{a}{c^T E}\]

\[
\eta_{E} = \frac{a}{c^T E} - \sum_{i=1}^{m} \theta_i E_{2i} \eta_{E1}
\]  

(17)

in which \(\theta = [\theta_1, \ldots, \theta_m]^T\) and \(\eta_{E0}\) are chosen as follows. If \(m < r\), let \(S \subset \mathbb{R}^m\) be the set of all possible vectors \(\theta_E = [\theta_{E1}, \ldots, \theta_{Em}]^T\) for which the roots of the polynomial \(s^{2m+1} + \theta_{E1}s^{2m} + \cdots + \theta_{Em}\) are a subset of \(\{\pm j\omega_1, \ldots, \pm j\omega_r\}\). Let

\[
\epsilon_M = \min_{\theta_E \in S} \left\{ \sup_{t \geq 0} |u_{rE}(t) - u_{rE}(t, \theta_E, \eta_{E0})| \right\}
\]  

(18)

where \(u_{rE}(t, \theta_E, \eta_{E0})\) is obtained from (17) with \(\theta = \theta_E\) and initial conditions \(\eta_{E0}\). Let \(\theta \in S\) and \(\eta_{E0} \in \mathbb{R}^{2m+1}\) be the vectors corresponding to such a minimum. If \(m \geq r\), let \(\theta \in \mathbb{R}^m\) be defined as

\[
\theta = [\theta_1, \ldots, \theta_r, 0, \ldots, 0]^T
\]  

and \(\eta_{E0} \in \mathbb{R}^{2m+1}\) be defined as

\[
\eta_{E0} = [\eta_0, 0, \ldots, 0]^T
\]  

(19)

It follows that for \(m \geq r, \epsilon_M = 0\). Define

\[
e(t) = u_{r}(t) - u_{rE}(t)
\]  

(20)

From (12), (17) and (20), we can write

\[
\left[ \begin{array}{c}
\dot{\hat{x}}_o \\
\eta_E
\end{array} \right] = \left[ \begin{array}{c}
A_o & -b_o c_o \\
0 & R_E(\theta)
\end{array} \right] \left[ \begin{array}{c}
\hat{x}_o \\
\eta_E
\end{array} \right] + \left[ \begin{array}{c}
b_o \\
0
\end{array} \right] (u - e) - e (b_0 + \sum_{i=1}^{m} \mu_i \xi_i)
\]  

(21)

Since (21) is observable by virtue of the assumption on \(\theta\), a parameter dependent change of coordinates \((T_2(\theta)\) is nonsingular)

\[
\zeta = T_2(\theta) \left[ \begin{array}{c}
\hat{x}_o \\
\eta_E
\end{array} \right], \quad \zeta \in \mathbb{R}^{n+2m+1}
\]  

(22)

exists such that in new coordinates, we obtain

\[
\dot{\zeta} = A_{\zeta} \zeta - a[0] e + b[0] (u - e) + \sum_{i=1}^{m} \xi_i \theta_i - e (b_0 + \sum_{i=1}^{m} \theta_i b[i]) (u - e)
\]  

\[
e = c_{\zeta} \zeta
\]  

(23)

where \((a_o = [a_{\nu-1}, \ldots, a_0]^T\) and \(b_o\) is as in (8))

\[
a[0] = [ \begin{array}{c}
a_0^T \\
0
\end{array} ]
\]  

\[
b[0] = [ \begin{array}{c}
b_0^T \\
0
\end{array} ]
\]  

\[
a[1] = [ \begin{array}{c}
a_1^T \\
0
\end{array} ]
\]  

\[
b[1] = [ \begin{array}{c}
b_1^T \\
0
\end{array} ]
\]  

\[
a[2] = [ \begin{array}{c}
a_2^T \\
0
\end{array} ]
\]  

\[
b[2] = [ \begin{array}{c}
b_2^T \\
0
\end{array} ]
\]  

\[
\vdots
\]  

\[
a[m] = [ \begin{array}{c}
a_m^T \\
0
\end{array} ]
\]  

\[
b[m] = [ \begin{array}{c}
b_m^T \\
0
\end{array} ]
\]  

(24)

with \(a[i] \in \mathbb{R}^{n+2m+1}, b[i] \in \mathbb{R}^{n+2m+1}, 0 \leq i \leq m\). Now, choose any vector \(d = [1, d_2, \ldots, d_{m+1}]^T \in \mathbb{R}^{n+2m+1}\) such that all the roots of the polynomial

\[
d(s) = s^{n+2m} + d_2 s^{n+2m-1} + \cdots + d_{m+1}
\]  

(25)

have negative real part and define the corresponding Hurwitz matrix

\[
D = \left[ \begin{array}{c}
-d_2 & 1 & 0 & \cdots & 0 \\
& \vdots & \ddots & \vdots & \vdots \\
& \vdots & & \ddots & \vdots \\
& \vdots & & & \ddots & \vdots \\
\end{array} \right]
\]  

(26)

The filtered transformation \((\xi_t \in \mathbb{R}^{n+2m}, \mu_t \in \mathbb{R}, 1 \leq i \leq m)\) (see [14], [15])

\[
\dot{\xi}_t = D \xi_t + \left[ \begin{array}{c}
0 & I_{n+2m} & \xi_0 & \xi_0 & \cdots & \xi_0_t \end{array} \right], \quad \xi_0 = \xi_0
\]  

\[
\mu_t = \left[ \begin{array}{c}
1 & 0 & \cdots & 0 \\
\xi_0 & \xi_0 & \cdots & \xi_0
\end{array} \right], \quad \xi_t, 1 \leq i \leq m
\]  

(27)

\[
z = \zeta - \left[ \sum_{i=1}^{m} \xi_i \theta_i \right], \quad z \in \mathbb{R}^{n+2m+1}
\]  

(28)

transforms system (23) into

\[
\dot{\zeta} = A_{\zeta} \zeta - a[0] e + b[0] (u + \sum_{i=1}^{m} \mu_i \theta_i)
\]  

\[
e = c_{\zeta} \zeta
\]  

(29)
An adaptive observer for (29) is provided by
\[
\dot{\hat{z}} = A_c \hat{z} - a[0]e + b[0]u + d \sum_{i=1}^{m} \mu_i \dot{\hat{\theta}}_i - \tilde{k}_o(e - \hat{c}_z \hat{z})
\]
(30)
in which \( \tilde{k}_o = -(A_c + \lambda I)d \), with \( \lambda > 0 \) a design parameter. This choice of \( \tilde{k}_o \) guarantees that the triple \((A_c + \tilde{k}_o c_c, d, c_v)\) is strictly positive real (SPR) [15], [16]. The adaptation dynamics for the estimates \( \dot{\hat{\theta}}_i(t), 1 \leq i \leq m \) are given by
\[
\dot{\hat{\theta}}_i = g_i \mu_i (e - \hat{c}_z \hat{z}), \quad 0 \leq t \leq T_M
\]
\[
\dot{\hat{\theta}}_i = g_i \mu_i (e - \hat{c}_z \hat{z}), \quad t > T_M,
\]
if \( \det[M_i(t)] \geq m_0 e^{-\lambda_0 t} > 0 \)
\[
\dot{\hat{\theta}}_i = g_i \mu_i (e - \hat{c}_z \hat{z}) - \alpha_i \hat{\theta}_i \quad \text{otherwise}
\]
(31)
where \( m_0 \) is a small threshold, \( \lambda_0 \) is a constant larger than all time constants in the matrix \( D \) in (26), \( \alpha_i > 0 \) and \( g_i > 0 \), \( 1 \leq i \leq m \) are design parameters, \( M_i(t), 1 \leq i \leq m \) become smaller than \( \alpha_i \) \( (i \times i) \) matrices given by
\[
M_i(t) = \int_{t-T_M}^{t} \mu_i[i] T[i][i] \mu_T[i][i] d\tau, \quad t \geq T_M
\]
(32)
with \( \mu[i] = [\mu_1, \ldots, \mu_i]^T \) and \( T_M = 2\pi/\omega_{\min} \), being the known upper bound on the exosystem periods, since by assumption \( \omega_{\min} \) is the lowest allowable frequency in the exosystem (2). Recalling (22) and (28), the control input \( v \) in (13) is chosen as
\[
v = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \end{bmatrix} \left( \det[T_2(\hat{\theta})] \right) \times \frac{\text{sat}(\det^2[T_2(\hat{\theta})])}{\text{sat}(\det^2[T_2(\hat{\theta})])} \left( \hat{z} + \left[ \sum_{i=1}^{m} \xi_i \dot{\hat{\theta}}_i \right] \right)
\]
(33)
in which
\[
\text{sat}(\det^2[T_2(\hat{\theta})]) = \begin{cases} \det^2[T_2(\hat{\theta})] & \text{if } |\det[T_2(\hat{\theta})]| \geq T_e \\ T_e & \text{otherwise} \end{cases}
\]
(34)
Note that the elements of \( \text{adj}[T_2(\hat{\theta})] \) (adjoint matrix of \( T_2(\hat{\theta}) \)) are polynomial functions of \( \hat{\theta} \) so that \( T_e \) avoids the problem of division by zero when \( \hat{\theta} \) is approaching an unobservable pair
\[
\begin{bmatrix} A_o & -b_o c_c \\ 0 & R_{c_E}(\hat{\theta}) \end{bmatrix}, \begin{bmatrix} c_o & 0 \end{bmatrix}
\]
(35)
This concludes the design of the controller (13), (33) which is based on two distinct observers: the non-adaptive one (14) which observes the observable part; the adaptive one (30), (31), (27) which reconstructs the approximation \( u_{r,E} \) of the reference input \( u_r \).

IV. STABILITY ANALYSIS

Theorem 4.1: Consider system (1). Under assumptions (H1)-(H4) the adaptive output error feedback controller (13), (14), (33), (31), (30), (27) guarantees that the errors \((\theta - \hat{\theta}, z - \hat{z}, x - x_r, e, \tilde{x}_o - \tilde{x}_o)\), in the closed loop system, are bounded for any \( t \geq 0 \) and for any initial condition.

Moreover, if the approximation error \( \epsilon_M \) is sufficiently small so that singularities are avoided in \( T_2(\hat{\theta}) \), then the errors \((\theta - \hat{\theta}, z - \hat{z}, x - x_r, e, \tilde{x}_o - \tilde{x}_o)\) converge exponentially into a closed ball whose radius is proportional to \( \epsilon_M \) as \( t \) tends to infinity. If \( m \geq r \) (i.e. the order of the modeled exosystem is larger than or equal to the order of the true exosystem), since \( \epsilon_M = 0 \), exponential convergence to zero is achieved. Proof. From (29) and (30), we obtain the observer error dynamics
\[
\dot{\hat{z}} = (A_c + \tilde{k}_o c_c) \hat{z} + d \mu_T \hat{\theta} - \epsilon(b[0] + \sum_{i=1}^{m} \theta_i b[i])
\]
(36)
in which \( \mu = [\mu_1, \ldots, \mu_m]^T, \hat{z} = z - \hat{z} \) and \( \hat{\theta} = \theta - \hat{\theta} \).

Define
\[
\chi_i = \xi_i - N_i x_o, \chi_i \in R^{\nu + 2m}, 1 \leq i \leq m
\]
\[
N_i = \begin{bmatrix} 0 & I_{\nu} \\ 0 & 0 \\ \vdots & \vdots \\ 0 & 0 \end{bmatrix}, \quad \ldots, \quad N_m = \begin{bmatrix} 0 & 0 \\ 0 & I_{\nu} \end{bmatrix}
\]
(37)
where \( I_{\nu} \) is the \((\nu \times \nu)\) identity matrix. The filter outputs \( \mu_i \) in (27) can be equivalently generated by
\[
\dot{\chi}_i = D \chi_i + \begin{bmatrix} 0 & I_{\nu + 2m} \end{bmatrix} b[i]\nu_r, \chi_i \in R^{\nu + 2m}
\]
\[
\mu_i = \begin{bmatrix} 1 & 0 & \cdots & 0 \end{bmatrix} \chi_i, 1 \leq i \leq m
\]
(38)
which implies that the signal \( \mu_i(t) \) are bounded. Hence by the arguments reported in [16]. Th. 2.7.2, the vector \( \mu[i^*](t) = [\mu_1(t), \ldots, \mu_r(t)]^T \in R^r, i^* \leq m \), is persistently exciting if the signal \( u_r(t) \) contains \( i^* \) distinct frequencies. Therefore, if \( m \leq r \), we obtain from (31)
\[
\dot{\hat{\theta}}_i = -g_i \mu_i(e - \hat{c}_z \hat{z}), \quad t \geq 0, \quad 1 \leq i \leq m
\]
(39)
while if \( m > r \), we have (recall (18))
\[
\dot{\hat{\theta}}_i = -g_i \mu_i(e - \hat{c}_z \hat{z}), 1 \leq i \leq r, \quad t \geq 0
\]
\[
\dot{\hat{\theta}}_i = -g_i \mu_i(e - \hat{c}_z \hat{z}), r < i \leq m, \quad t \in S_{1i}
\]
\[
\dot{\hat{\theta}}_i = -g_i \mu_i(e - \hat{c}_z \hat{z}) - \alpha_i \hat{\theta}_i, r < i \leq m, \quad t \in S_{2i}(40)
\]
in which \( S_{1i} \) is the union of \([0, T_M] \) with the time intervals in which \( \det[M_i(t)] \geq m_0 e^{-\lambda_0 t} \) and \( S_{2i} \) is the union of the time intervals in which \( \det[M_i(t)] < m_0 e^{-\lambda_0 t} \). Since \( i > r \), it follows that after the exponential decaying transient due to the initial conditions in (27), \( \det[M_i(t)] \) will definitely become smaller than \( m_0 e^{-\lambda_0 t} \) for \( r < i \leq m \), so that \( S_{2i} \) extends until \(+\infty\). Since the signal \( u_r(t) \) contains \( r \) distinct frequencies by virtue of assumption (H4), there exists a positive real \( k_T \) such that
\[
\int_{t}^{t+T_M} \mu[i^*](\tau) \mu^T[i^*](\tau) d\tau \geq k_T I > 0, \forall t \geq 0
\]
(41)
with \(i^* = m\), if \(m \leq r\), and \(i^* = r\) if \(m > r\). Now, consider the function
\[
V = \tilde{z}^T P \tilde{z} + \tilde{\theta}^T G \tilde{\theta} + \frac{1}{2\beta} \left( Q_2 \tilde{\theta}[i^*] - \frac{\mu[i^*]}{d^T z} \right)^T
\]
(42)
in which \(G = \text{diag}[g_1, \ldots, g_m]\), \(\beta\) is a positive real yet to be chosen, \(\tilde{\theta}[i^*] = \left[\hat{\theta}_1, \ldots, \hat{\theta}_r\right]^T\), \(\mu[i^*] = [\mu_1, \ldots, \mu_r]^T\), the matrix \(P\) is solution of
\[
(A_c + \hat{k}_o c_c)^T P + P (A_c + \hat{k}_o c_c) = -Q_1 < 0
\]
(43)
which exists and is unique since the triple \((A_c + \hat{k}_o c_c, d, c_c)\) is SPR, and \(Q_2\) is generated by the matrix differential equation
\[
\dot{Q}_2 = -Q_2 + \mu[i^*] \mu^T[i^*], \quad Q_2(0) = e^{-T_M} k_T I
\]
(44)
Since \(\mu[i^*](t)\) is bounded we can write
\[
\mu[i^*](t) \leq \mu_M, \quad 1 \leq i \leq i^*, \quad \forall t \geq 0.
\] (45)
From (41), (44) and (45), it follows that
\[
\mu_M k_T e^{-2T_M} I > 0, \quad \forall t \geq 0.
\] (46)
If \(m > r\), the time derivative of (42), according to (36) and (40), is such that for \(t \in \{S_{2r+1} \cap \cdots \cap S_{2m}\}\)
\[
V = \tilde{z}^T Q_1 \tilde{z} + 2\tilde{z}^T \mu \tilde{\theta} - 2\epsilon \tilde{z}^T P(b[0] + \sum_{i=1}^m \theta_i b[i])
\]
(47)
in which \(G[i^*] = \text{diag}[g_1, \ldots, g_r]\) and \(\epsilon = 0\) by virtue of (18) and (19). Since \(\mu_i\) and \(\hat{\mu}_i\) are bounded, by choosing \(\beta\) sufficiently small, from (42) and (47) we obtain that the error vector \([\tilde{z}^T, \tilde{\theta}^T]^T\) exponentially converges into the origin. From (33), it follows that \(\nu(t)\) is bounded so that \(\tilde{x}(t)\) and \(x(t)\) are also bounded. From (33), we can write
\[
\nu = v(\hat{\theta}, \tilde{z})
\] (48)
so that when \(t\) is sufficiently large to avoid problems in the inversion of \(T_2(\tilde{\theta})\), we obtain (see [17])
\[
\nu = v(\theta, z) + [v(\hat{\theta}, \tilde{z}) - v(\theta, z)]
\]
(49)
in which \(v_\theta\) and \(v_z\) are suitable continuous functions. Since \(z(t), \tilde{z}(t)\) and \(\hat{\theta}(t)\) are bounded, \(\epsilon = 0\), and \(\hat{\theta}(t), \tilde{z}(t)\) exponentially converge to zero, the stable closed loop systems (15) and (16) are perturbed by bounded disturbances exponentially converging to zero, so that \(\tilde{x}_o, \hat{x}_o\) and, through (5), (7), \(\hat{x}_u\) exponentially converge to zero. It is easy to see that all signals remain bounded in the time interval \(t \notin \{S_{2r+1} \cap \cdots \cap S_{2m}\}\) by using the function \(V = \tilde{z}^T P \tilde{z} + \theta^T G \theta\). If \(m \leq r\), from the previous analysis it is straightforward to see that the error vector \([\tilde{z}^T, \tilde{\theta}^T]^T\) exponentially converges into a residual ball centered at the origin of radius proportional to \(\epsilon_M\) (zero, if \(m = r\)) and, consequently, from (15), (16), (5) and (7), \(\tilde{x}_o, \hat{x}_o, \hat{x}_u\) exponentially converge into a residual ball centered at the origin of radius proportional to \(\epsilon_M\) (zero, if \(m = r\)).

\[\Box\]

V. EXAMPLE

As an illustrating example, we consider the following second order non-minimum phase system
\[
\begin{align*}
\dot{x}_1(t) &= x_2(t) + u(t) - w(t) \\
\dot{x}_2(t) &= -u(t) + w(t) \\
y(t) &= x_1(t)
\end{align*}
\]
with the disturbance: \(w(t) = \sin t + 0.1 \sin 2t\) for \(t \in [0, 50]\) s, \(w(t) = \sin t\) for \(t \in [50, 100]\) s, \(w(t) = 0\) for \(t > 100\) s and the trajectory to be followed \(y_r(t) = 0\). It may be verified that assumptions (H1)-(H4) in Theorem 4.1 are satisfied while the order of the unknown exosystem is 5 for the first period, 3 for the second period and 1 for the final period. The transformation \(T_1\) in (6) is in this case equal to the identity. For the adopted exosystem (17) we choose \(m = 1\) so that only one unknown parameter \(\theta\) is needed. Following the design procedure outlined in the previous section, we introduce the observer
\[
\begin{align*}
\dot{\hat{x}}_1 &= \hat{x}_2 - k_{o_1}(y - \hat{x}_1) + k_{c_1} \hat{x}_1 + k_{c_2} \hat{x}_2 \\
\dot{\hat{x}}_2 &= -k_{o_2}(y - \hat{x}_1) - k_{c_1} \hat{x}_1 - k_{c_2} \hat{x}_2
\end{align*}
\]
The filter (27) becomes
\[
\begin{align*}
\hat{\xi}_1 &= -d_2 \xi_1 + \xi_2 - y \\
\hat{\xi}_2 &= -d_3 \xi_1 + \xi_3 + u \\
\hat{\xi}_3 &= -d_4 \xi_1 + \xi_4 - u \\
\hat{\xi}_4 &= -d_5 \xi_1 \\
\mu &= \xi_1
\end{align*}
\]
while the adaptive observer (30), (31) becomes
\[
\begin{align*}
\dot{\hat{z}}_1 &= \hat{z}_2 + u + \mu \hat{\theta} - \bar{k}_o(y - \hat{z}_1) \\
\dot{\hat{z}}_2 &= \hat{z}_3 - u + d_2 \mu \hat{\theta} - \bar{k}_o(y - \hat{z}_1) \\
\dot{\hat{z}}_3 &= \hat{z}_4 + d_3 \mu \hat{\theta} - \bar{k}_o(y - \hat{z}_1) \\
\dot{\hat{z}}_4 &= \hat{z}_5 + d_4 \mu \hat{\theta} - \bar{k}_o(y - \hat{z}_1) \\
\dot{\hat{\theta}} &= g \mu(y - \hat{z}_1), \ t \in S_1 \\
\dot{\hat{\theta}} &= g \mu(y - \hat{z}_1) - \alpha \hat{\theta}, \ t \in S_2.
\end{align*}
\]

The matrix \( M(t) \) in (32) becomes the scalar
\[
M(t) = \int_{t-T_M}^{t} \mu^2(\tau) d\tau, \ t \geq T_M.
\]

Finally, the signal \( v \) in (33) obtained through the inverse transformation and the control \( u \) are given by
\[
v = \frac{1}{1 + \hat{\theta}} \left[ -\hat{\theta}(\hat{z}_1 + \hat{z}_2 + \hat{\theta}(\hat{z}_1) + \hat{z}_3 + \hat{z}_4 + \hat{z}_5 + \hat{\theta}(\hat{z}_2 + \hat{z}_3 + \hat{z}_4 + \hat{z}_5)) + \hat{z}
\right]
\]
\[
u = k_{c_1} \hat{z}_1 + k_{c_2} \hat{z}_2 + v.
\]

Some numerical simulations have been carried out with reference to the following values for parameters and initial conditions: \( d = [1, 6, 13, 12, 4]^T \), \( \lambda = 1 \), \( g = 50 \), \( \alpha = 1 \), \( [k_{c_1}, k_{c_2}] = [1, 3], [k_{o_1}, k_{o_2}] = [-4, -4] \) and all initial conditions set to zero. The vectors \( k_c \) and \( k_o \) are chosen to assign the eigenvalues of the closed loop system when disturbances are not present (see (13) and (14)). The vector \( d \) and the scalar \( \lambda \) may be chosen to attenuate sensor and actuator noises (see (27) and (30)), while \( \alpha \) and \( g \) are adaptation gain and forgetting factor, respectively. The simulation results are illustrated by Fig. 1 in which the regulation error \( y(t) \), the disturbance \( w(t) \), the control input \( u(t) \) and the frequency estimate \( \hat{\theta}(t) \) are reported.

VI. CONCLUSIONS

The problem of adaptive output regulation is addressed for linear systems with unknown exosystems assuming only the necessary conditions for its solution: the minimum phase assumption is not required. A fixed regulator is proposed which incorporates two observers and an adaptive internal model. It is shown that: if the required reference input can be generated by the controller internal model, the regulation error tends exponentially to zero; if the required reference input can only be approximated within an approximation error \( \epsilon \) by the internal model, then the regulation error is bounded and tends exponentially into a ball whose radius is proportional to \( \epsilon \), provided that singularities are avoided. Extensions to the case of time-varying frequencies require further investigations (see [18], [19] for contributions on this topic).

REFERENCES


