Extension of Zeheb-Walach Absolute Stability Criteria for Robot-Human Interactions

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Abstract—The notion of absolute stability has been used to analyze the stability of man-machine interfaces in which a human operator commands a robot to do a task in a real or simulated environment. In this paper, we present and prove an extended version of the Zeheb-Walach absolute stability theorem that can be applied to $n$-port networks with “simple imaginary poles”. Conditions on the Laurent expansion of the elements and the principal minors of the impedance matrix are added to the theorem and the proof is conducted using the zero set of multiparameter functions. The unique feature of the proposed method is the graphical representation of the loci of the roots of the system, which allows for studying the effect of various system and control parameters on stability margins. To demonstrate the applicability of the new theorem, the stability of a position-position controlled teleoperation system, modeled by a 2-port network, coupled with the passive impedances of the human operator’s hand, and the environment, is investigated.

I. INTRODUCTION

It frequently happens that new engineering applications require stability analysis of a system that interacts dynamically with its neighboring environments. Examples of such systems include a microwave amplifier circuit terminated by an unknown load [1], a robotic rehabilitation system interacting with a human operator [2] or a teleoperation system interacting with a human operator and an unknown environment [3]. While the environments with which a robot interacts can display different known or unknown dynamic behaviors, the majority of them can be considered as passive entities. Robots may also interact with human operators in rehabilitation or teleoperation applications. Human limb exhibits complex, time-varying and nonlinear behavior. However, it has been argued that human limb displays passive dynamics [4]. As a result, the coupled system can be viewed as a known network terminated by one or more passive 1-port networks that may include dynamic uncertainties, see Figure 1.

Coupled stability of a network is important in various applications to ensure performance and safe interaction. Passivity theory has been used to discuss coupled stability. It has been shown that coupling of a number of passive networks results in a stable system [5]. For example, if a 2-port passive network is terminated by any two passive 1-port networks, the resulting coupled system is stable. Passivity of a network is a sufficient condition for absolute stability and it results in a conservative design, because in addition to stability, it ensures that maximum energy extractable from the network ports is limited. Less conservative conditions to passivity are Llewellyn’s criteria for absolute stability which provides necessary and sufficient conditions for absolute stability of 2-port networks [6], [7]. However, emerging applications are modeled by networks with higher number of ports. For example, single-user teleoperation systems with actuator/sensor asymmetry [8] and multi-user teleoperation systems [9] are two to name.

Another method to analyze absolute stability of networks is the one that has been developed by Zeheb and Walach [10]. This method is based on checking the stability of the network while constrained to certain boundary conditions, such as short-circuit and open-circuit configurations of the network or when it is terminated by inductors and capacitors. The benefits of the method to Llewellyn’s is that it can extend to multi-port networks and also provides the loci of the roots of the characteristic equation of the closed-loop system, and hence, offers insight about system’s stability margin. The original conditions of the Zeheb-Walach (Z-W) theorem were stated for asymptotic stability of a network. This restriction limits the application of the conditions to the networks with no poles on the $j\omega$-axis. Such systems use cross and through variables, but not their integrals, for control. For robotic applications this translates to the use of velocity and force. In this paper, we will extend the original Z-W theorem to allow $j\omega$-axis poles in the system design. The proof that builds upon the proof of the original theorem will be provided. The second contribution of the paper is the analysis of coupled stability for a single-user position-position teleoperation control system and verifying the results with the Llewellyn’s criteria.

The rest of this paper is organized as follow. In Section II, we review important results on the zero set of multiparameter functions, which are used in Section III to prove the extended Z-W theorem. Section IV contains an illustrative example to show the effectiveness of the proposed extended method in analyzing the absolute stability of a 2-
II. ZERO SET OF MULTI-PARAMETER FUNCTIONS

In this section, we briefly review the definition of multi-parameter functions and a lemma that describes the relation between the domain of definition of the parameters and the zero set of the function. This lemma will be used later in Section III to prove the extended Z-W theorem. Let

\[ F = F(A, s) \]  

be any rational transfer functions in the complex variable \( s \), depending on an \( n \)-dimensional vector of parameters

\[ A = (A_1, ..., A_n) \in D^n \subseteq C^n \]  

where \( D^n \) is a closed subspace included in the closed complex space \( C^n \). We also assume that the boundaries \( \partial D_k \) of all the regions \( D_k \) (\( k = 1, ..., n \)) are piecewise Jordan arcs or curves, i.e.,

\[ A_k \in \partial D_k \iff A_k = f_k(t_k); \ a_k \leq t_k \leq b_k, t_k \in \mathbb{R} \]  

where \( f_k(l_1) \neq f_k(l_2), a_k < l_1 < l_2 < b_k \). Also, the derivates \( \frac{df_k}{dt_k} \) exist at every point on the boundaries \( \partial D_k \), except at a finite number of points \( t_{nk} \).

The objective is to form a relation between the parameters space \( D^n \) and the zero set \( V \) in the complex plane at which \( F \) vanishes. The zero set of \( F \) is defined as

\[ V = \{ s_i : \exists A \in D^n \ni F(A, s_i) = 0 \} \]  

The zeros \( s_i \), with its parameters satisfying (2) and (3), form a closed region \( V \) in the complex plane \([11], i.e.,\)

\[ \partial V \subset V \]  

where \( \partial V \) is the boundary of \( V \). The following lemma demonstrates the relation between the boundary of the zero set, i.e., \( \partial V \), and the boundary of the parameter set, i.e., \( \partial D \).

**Lemma 1** \([11]\): let \( s_0 \) be any point on \( \partial V \), that is \( s_0 \in \partial V \)

then,

\[ \exists A^0 \in \partial D \ni F(A^0, s_0) = 0 \]

The above lemma states that the parameter vectors which correspond to the boundary points \( \partial V \) are points on \( \partial D \).

III. EXTENDED ZEHEB-WALACH THEOREM

Let an \( n \)-port network be characterized by the impedance matrix \( Z = \{ Z_{ij} \}, i, j = 1, ..., n \), such that

\[ F = ZV \]  

where \( F \) and \( V \) are vectors of forces and velocities, respectively, as shown in Figure 1. Matrix \( Z \) is an \( n \times n \) matrix of rational transfer functions \( Z_{ij}(s) \) in the complex variable \( s = \sigma + j\omega \). We are interested in a set of necessary and sufficient conditions for \( Z \) such that the \( n \)-port network remains stable while coupled with any set of passive terminations

\[ F = -zV \]  

where

\[ z = \begin{bmatrix} z_1 & 0 \\ z_2 & \ddots \\ 0 & \ddots & \ddots \\ \end{bmatrix} \]

\[ Re(z_i) \geq 0 \text{ for } Re(s) \geq 0, \ i = 1, ..., n. \]

The characteristic equation of the closed-loop system is given by

\[ \text{det}(Z(s) + z(s)) = 0. \]  

Absolute stability of the network \( Z \) requires that all the roots of (9) remain in the closed left-half-plane (LHP) for all possible terminations \( z_i \) while imaginary poles are simple. Moreover, since \( z_i \)s have infinite values for open-loop terminations, absolute stability also requires that the poles of (9) remain in the closed LHP.

**Theorem 1** * (Extended Z-W theorem)*: An \( n \)-port system defined by (6)-(7) is absolutely stable if, and only if, the following four conditions are satisfied:

i) \( \text{det}(Z(s)) \neq 0 \) in the RHP and all principal minors of \( Z(s) \) (of order \( i = 1, ..., n \)) are analytic in the RHP with simple imaginary poles.

ii) All principal minors of \( Z(s) \) of order \( n \) and \( n-1 \) are positive for \( s = 0 \) if they are analytic at \( s = 0 \); otherwise, the corresponding coefficient of the first term in the Laurent expansion of the principal minor, i.e., the coefficient of \( s^{-k} \) for the pole of order \( k \) at zero, is positive.

iii) The boundary of the zero set of the following function lies in the closed LHP

\[ F(s, jx_1, ..., jx_n) = \text{det} \left( Z(s) + \begin{bmatrix} jx_1 & 0 \\ \vdots & \ddots \\ 0 & jx_n \end{bmatrix} \right); x_1, ..., x_n \in \mathbb{R} \]  

iv) The boundary of the zero set for each of the \( n - 1 \) functions

\[ F_k(s) = \text{det} \left( Z(s) + \begin{bmatrix} 0 & 0 \\ s & \epsilon \end{bmatrix} \right) \]  

\[ k = 1, ..., n - 1, x_1, ..., x_n \in \mathbb{R} \]

is in the closed LHP, where \( s_i = 0 \) for analytic systems at \( s = 0 \) and \( s_i = \epsilon j \) for systems with poles at \( s = 0 \), where \( \epsilon \) is an arbitrarily small number.

Similar to the original Z-W theorem, the extended theorem applies to admittance (Y) and hybrid (H) representations.
of the n-port network. For systems with simple poles on the imaginary axis other than \( s = 0 \), condition (ii) has to be checked for those poles as well. Condition (iii) has remained the same. In condition (iv), point \( s_c \) could be replaced by any point on the imaginary axis arbitrarily close to the open-loop imaginary pole \( s_i \) of \( Z(s) \), that is \( s_c = s_i \pm \epsilon j \).

Proof:

**Sufficiency Proof of Theorem 1:**

Condition (i) related to poles of principal minors ensures stability in open termination conditions. We prove sufficiency of condition (i) – (iv) by contradiction. Suppose an n-port system characterized by matrix \( Z \), complies with conditions (i) – (iv) of Theorem (1) and yet is not absolutely stable. Then, there exist passive terminations \( z_1^0, \ldots, z_n^0 \) such that the solution \( s^0 \) of the closed-loop system characteristic equation

\[
F_0(s^0, z_1^0, \ldots, z_n^0) = \det \left( Z(s^0) + \begin{bmatrix}
    z_1^0 & 0 \\
    \vdots & \ddots \\
    0 & \cdots & z_n^0
\end{bmatrix} \right) = 0
\]

is in the RHP. In the following, we will use conditions (i) to (iii) to show that the RHP zero \( s^0 \) contradicts condition (iv). From the passivity of the terminations, we state that \( z_i^0 \) are positive-real, that is

\[
\Re(z_i^0) \geq 0 \quad \text{for} \quad \Re(s^0) \geq 0, \quad i = 1, \ldots, n
\]

By continuously varying \( z_i^0 \) towards the short-circuit condition, that is \( z_i \to 0 \), the solution \( s^0 \) of (12) will move along a continuum of loci denoted by \( L \). By condition (i), for \( z_i = 0 \) \( \forall i \), the roots must lie in the closed LHP, whereas \( s^0 \) originally lied in the RHP. Hence, \( L \) intersects the imaginary axis of the complex s-plane., that is

\[
\exists(s = j\omega) \in L
\]

Now consider the zero set \( V \) of

\[
F(s, z_1, \ldots, z_n) = \det \left( Z(s) + \begin{bmatrix}
    z_1 & 0 \\
    \vdots & \ddots \\
    0 & \cdots & z_n
\end{bmatrix} \right)
\]

where \( \Re(z_i) \geq 0 \forall i \). We denote the boundary of \( V \) by \( \partial V \). We now apply Lemma 2 to find \( \partial V \) by considering \( A_i = z_i, D_i \) the RHP, and the boundary of \( D_i \) the imaginary axis, that is \( \partial D_i = jx_i \). By Lemma 1, \( \partial V \) is the boundary of the zero set of \( F \) in (10) in condition (iii). By condition (iii), \( \partial V \) lies in the closed LHP. Considering from (14) that a point on the imaginary axis is in the zero set \( V \), we conclude that the entire imaginary axis belongs to \( V \). If we pick a point \( s_c \) on the imaginary axis, it belongs to \( V \). Therefore, there exist \( n \) positive-real numbers \( z_1', \ldots, z_n' \) that satisfy

\[
F(s, z_1', \ldots, z_n') = 0
\]

Now consider the zero set \( V_1 \) for the following function \( F_1 \)

\[
F_1(s_c, s, z_2, \ldots, z_n) = \det \left( Z(s_c) + \begin{bmatrix}
    s \\
    2 & 0 \\
    \vdots & \ddots \\
    0 & \cdots & z_n
\end{bmatrix} \right)
\]

where \( z_i, i = 2, \ldots, n \) are positive-real numbers, and \( s_c = 0 \) if \( Z(s) \) is analytic at \( s = 0 \); otherwise \( s_c = \epsilon j \). Since (17) is a first-order equation in the independent variable \( s \), for every set of \( z_2, \ldots, z_n \), there is only one point \( s \) vanishing (17). From (16) and (17), one concludes that \( z_1' \in V_1 \). In other words, a positive-real solution for (17) is found. On the other hand, for \( z_i = 0, i = 2, \ldots, n \), we have

\[
F_1(s_c, s, 0, \ldots, 0) = \det(Z(s_c)) + s\Delta_{11}(s_c)
\]

where \( \Delta_{11} \) is the first principal minor of order \( n - 1 \) of the matrix \( Z \). For the case that \( \det(Z(s_c)) \) and \( \Delta_{11}(s_c) \) are analytic at \( s_c = 0 \), \( \det(Z(0)) \) and \( \Delta_{11}(s_c) \) are positive by condition (ii). Thus, \( s \), the zero of (18), which also belongs to \( V_1 \), must be real and negative. For the case that \( \det(Z(s_c)) \) and \( \Delta_{11}(s_c) \) are not analytic at \( s_c = 0 \), the first term of the Laurent series for the function \( F_1 \) about the point \( s_c = 0 \) is

\[
F_1(s, s, 0, \ldots, 0) = \frac{a_{m_1}}{(\epsilon j)^{m_1}} + \frac{a_{m_2}}{(\epsilon j)^{m_2}}
\]

where it is assumed that \( a_{m_1} = \lim_{s \to 0} s^{m_1} \det(Z(s)) \) and \( a_{m_2} = \lim_{s \to 0} s^{m_2} \Delta_{11}(s) \). Following the same argument as before, since \( a_{m_1} \) and \( a_{m_2} \) are both positive and \( -1 \leq m_2 - m_1 \leq 0 \), the variable \( s \) should be in the closed LHP. Therefore, we have shown that the zero of (18), for the two cases discussed above, is in the closed LHP. Thus, while varying \( z_i \) toward zero, the locus \( L_1 \) of \( s \) moves from \( s = z_1' \) to the closed LHP. This implies that at least one point on the imaginary axis belongs to \( V_1 \), that is

\[
\exists(s = j\omega) \in L_1
\]

We denote the boundary of \( V_1 \) by \( \partial V_1 \). We now apply Lemma 2 to find \( \partial V_1 \) by considering \( A_i = z_i, D_i \) the RHP, and the boundary of \( D_i \) the imaginary axis, that is \( \partial D_i = jx_i \), where \( i = 2, \ldots, n \). By Lemma 1, \( \partial V \) is the boundary of the zero set of \( F_1 \) in (11) in condition (iv). By condition (iv), \( \partial V \) lies in the closed LHP. Considering a point on the imaginary axis is in the zero set \( V_1 \), as proved in above, we conclude that the entire imaginary axis belongs to \( V_1 \). In particular, \( s = 0 \in V_1 \). Repeating this line of thought for \( F_2 \) towards \( F_{n-1} \), we finally conclude that there exist a positive-real \( z_n \) such that

\[
F_{n-1}(s_c, s = 0, z_n) = \det \left( Z(s_c) + \begin{bmatrix}
    0 & 0 \\
    \vdots & \ddots \\
    0 & \cdots & s = 0
\end{bmatrix} \right)
\]

However, (21) contradicts condition (ii) by the same reasoning leading to the existence of a root in (18) belonging to the closed LHP. Thus, any n-port that complies with the
conditions (i) – (iv) of Theorem 1 cannot comply with (12) and (13), and hence is absolutely stable.

**Necessity Proof of Theorem 1:**

Here we investigate conditions (i)-(iv), assuming absolute stability of the n-port network.

**Condition (i):** The extreme operating conditions for an absolutely stable network are short and open circuits. Stability in these conditions requires condition (i).

**Condition (ii):** To show the necessity of condition (ii), we start with diagonal elements of the matrix $Z$, that is, $Z_{ii}$. Terminating port $i$ by a passive load $z_i$ and leaving all other ports open-circuit results in the following closed-loop characteristic equation

$$Z_{ii} + z_i = 0, \forall i = 1, \ldots, n \tag{22}$$

Since $Z(s)$ is an absolutely stable network, a necessary and sufficient condition for the above equation to be Hurwitz is that $Z_{ii}$ be a passive impedance or positive-real, which translate to the following conditions at $s = 0$

$$\begin{cases} Z_{ii}(0) \geq 0, & \text{if } Z_{ii} \text{ is analytic at } s = 0, \\ \text{Res}(Z_{ii}, 0) > 0, & \text{otherwise} \end{cases} \tag{23}$$

for open circuit and

$$\begin{cases} Y_{ii}(0) \geq 0, & \text{if } Y_{ii} \text{ is analytic at } s = 0, \\ \text{Res}(Y_{ii}, 0) > 0, & \text{otherwise} \end{cases} \tag{24}$$

for short circuit. Now terminating ports $i$ and $j$ with arbitrary passive impedances $z_i$ and $z_j$ and leaving all of the other ports open yields

$$Y_{ij} = \frac{Z_{ij}}{\Delta} \tag{25}$$

where

$$\Delta := Z_{ii}Z_{jj} - Z_{ij}Z_{ji} \tag{26}$$

Equations (23), (24), and (25) imply $\Delta(s) > 0$ or $s_m > 0, \forall i, j$, where $s_m = \lim_{s \to 0} s^m \Delta(s)$. Repeating this reasoning with adding another port terminated by a passive impedance, and working with a higher order principal minor, leads to condition (ii).

**Conditions (iii) and (iv):** Assume that condition (iii) is not met, meaning that the boundary of the zero set of (10) is in the RHP. In this case, $s = \sigma_0 \pm j\omega_0$, with arbitrary small $\sigma_0$, belongs to the zero set of $F$ in (10). This means that there exist real numbers $x_1, \ldots, x_n$ such that

$$F(\sigma_0 \pm j\omega_0, jx_1, \ldots, jx_n) = 0 \tag{27}$$

Now, notice that the boundary of the zero set of

$$F(s, z_1, \ldots, z_n) \tag{28}$$

belongs to the boundary of the zero set of $F$ in (27). Since the parameter space is a closed subspace in the complex plane, (i.e., $z_i$ varies in the closed RHP), the zero set of $F$ is a closed subspace [11], and as a result the boundary of the zero set of $F$ also belongs to the zero set of $F$. Thus, $s = \sigma_0 \pm j\omega_0$ is also a zero of (28). Varying the parameters $z_i$ toward their domain of definition in the RHP moves $\sigma_0 \pm j\omega_0$ toward inside of the boundary.

$$F(\sigma_0 + j\omega_1, jx_1 + \sigma_1, \ldots, jx_n + \sigma_n) = 0 \tag{29}$$

Finding a passive load $z_i$ in (28) such that $z_i(\sigma_0 + j\omega_1) = \sigma_i + jx_i$ completes the necessity proof of condition (iii).

One choice for the case of $x_i > 0$ is

$$z_i(s) = \frac{K}{s} + B, \quad K = \frac{\omega_1 \sqrt{\sigma_0^2 + \omega_1^2}}{-x_i}, \quad B = (\sigma_i - \frac{\omega_1}{\omega_1^2}) \tag{30}$$

since $\sigma_0$ can be chosen arbitrarily close to the origin, $B$ is positive and, hence, $z_i$ is passive. A choice for the case of $x_i < 0$ is

$$z_i(s) = \frac{K}{s} + B, \quad K = \frac{\omega_1 \sqrt{\sigma_0^2 + \omega_1^2}}{-x_i}, \quad B = (\sigma_i + \frac{\omega_1}{\omega_1^2}) \tag{31}$$

Again, with arbitrary small $\sigma_0, B$ is positive. The necessity proof for condition (iv) is similar to that of condition (iii).

It should be noted that simple imaginary zeros of the characteristic equation can occur just for open termination configurations (condition (i)). It can be easily shown that the existence of imaginary poles for a set of finite passive terminations leads to the existence of RHP zeros for another finite set of terminations in the neighboring of the original set. Thus, checking conditions of the extended Z-W theorem will ensure that no imaginary zeros of any order exist for finite passive terminations.

Calculation of the zero set of the function $F$ in the condition (iii) can be facilitated by the methods described in [11]. The following Lemma provides a simple way for the calculation of the zero set of a 2-parameter function.

**Lemma 2** [10]: Let the zero set of

$$F(s, x_1, x_2) = G_{10}(s) + x_1 G_{11}(s) + x_2 G_{01}(s) + x_1 x_2 G_{11}(s) \tag{32}$$

$x_1, x_2 \in \mathbb{R}^2$

be denoted by $V$ and its boundary by $\partial V$. Then, $s_0 \in \partial V$ implies

$$f_1(s_0) = (G_{00} \times G_{11})^2 + 2(G_{11} \times G_{10})(G_{00} \times G_{01}) \tag{33}$$

$$(G_{10} \times G_{01})^2 + 2(G_{11} \times G_{10})(G_{00} \times G_{10})|_{s=s_0} = 0$$
where the cross operator $\times$ defines a vector product of two complex numbers as follows

$$[A \times B] = Re(A)Im(B) - Im(A)Re(B)$$

(34)

Furthermore, any point for which $f_1(s_0)$ is non-negative, belongs to the zero set $V$, that is

$$f_1(s_0) \geq 0 \Rightarrow s_0 \in V$$

(35)

This Lemma will be used in next section.

IV. ILLUSTRATIVE EXAMPLE

A. Position-Position Teleoperation System

In this section, we apply the extended Z-W theorem to analyze the coupled stability of a master-slave teleoperation system controlled by a position-position control architecture [12], [13], [14]. Figure 3 shows the block diagram of the network model of the teleoperation system, consisting of the operator and the environment one port networks and the master and slave robot dynamics lumped into the master-slave two-port network. Here, $V_h$ and $V_e$ denote master and slave velocities, $F_h$ and $F_e$ represent the forces exerted by the operator to the master and by the slave to the environment, and $F_{cm}$ and $F_{cs}$ denote the master and slave control commands. The master and slave dynamics are assumed to be modeled by linear-time-invariant mass-damper systems, that is

$$Z_{cm}V_h = F_h - F_{cm}, \quad Z_{cs}V_e = -F_e + F_{cs}$$

(36)

where $Z_{cm} = m_ms + b_m$ and $Z_{cs} = m_s\sigma + b_s$. The master and slave are controlled by a symmetric position-position control architecture. The control commands are symmetric coordinating forces, creating a virtual elastic band between the master and slave for position correspondence and are given by

$$F_{cm} = \frac{Kp}{s}(V_e - V_h), \quad F_{cs} = \frac{Kp}{s}(V_h - V_e)$$

(37)

The master-slave network (36) with the feedback controller (37) can be represented by the impedance matrix

$$Z = \begin{bmatrix} Z_{cm} + \frac{Kp}{s} & \frac{Kp}{s} \\ \frac{Kp}{s} & Z_{cs} + \frac{Kp}{s} \end{bmatrix}$$

(38)

It can be seen that position feedback introduces a pole at $s = 0$ and the original Z-W theorem cannot be applied to analyze such problem. In the following, we will study the effect of various controller parameters on the stability margins of the system. To this purpose, we check for the four conditions of the extended Z-W theorem for the impedance matrix given in (38).

**Condition (i):** The determinant of the $Z$-matrix is given by

$$\det(Z) = s^2 + (b_m + b_s)s + (b_m b_s + 2Kp)$$

(39)

The zeros of $\det(Z(s))$ are in the LHP for $b_m > 0$ and $b_s > 0$. Also, the principal minors of order 1, i.e., $Z_{11}$ and $Z_{22}$, are analytic in the RHP (no RHP poles).

**Condition (ii):** The coefficients of the term $\frac{1}{2}$ in $Z_{11}, Z_{22}$, and $\det(Z(s))$ are $Kp, Kp$ and $Kp(b_m + b_s)$, respectively. These coefficients are all positive with assuming $b_m > 0$ and $b_s > 0$ and $Kp > 0$.

**Condition (iii):** The zero set of the 2-parameter function

$$F(s, jx_1, jx_2) = \det \left\{ Z(s) + j \begin{bmatrix} x_1 & 0 \\ 0 & x_2 \end{bmatrix} \right\}; \quad x_1, x_2 \in \mathbb{R}^2$$

should be in the closed LHP. The calculation of the zero set is carried out using Lemma 2. After substituting for $Z(s)$ from (38) into the above function $F$, the coefficients $G_{ii}, i = 0, 1$ defined by (32) are found as

$$G_{00} = 2Kp + b_m b_s + (b_m + b_s)\sigma + \sigma^2 - \sigma^2 + \frac{Kp(b_m + b_s)}{\sigma^2 + \omega^2}$$

(40)

$$G_{10} = -\sigma + \frac{Kp\sigma}{\sigma^2 + \omega^2} + \left( \frac{\sigma + b_s + Kp\sigma}{\sigma^2 + \omega^2} \right)j$$

$$G_{01} = -\sigma + \frac{Kp\sigma}{\sigma^2 + \omega^2} + \left( \frac{\sigma + b_m + Kp\sigma}{\sigma^2 + \omega^2} \right)j$$

$$G_{11} = -1$$

Using $G_{ii}, i = 0, 1$, the polynomial $f_1$ in (33) that shows the equation of boundary is given by

$$f_1(\sigma, \omega) = \frac{\begin{array}{c} a_3(\sigma)\omega^4 + a_2(\sigma)\omega^2 + a_1(\sigma) \\ \sigma^4 + 2\sigma^2 + \omega^2 + \omega \end{array}}{\begin{array}{c} \sigma^4 + 2\sigma^2 + \omega^2 + \omega \end{array}}$$

(41)

The typical boundaries of the zero set of $f_1$ are shown in Figure 4 for two sets of values (a) $b_m = b_s = K_p = 1$, and (b) $b_m = 0.5, b_s = K_p = 1$. The behavior of the asymptotes are given by

$$a_3(\sigma) = -4(\sigma + b_s)^2(\sigma + b_s)^2 = 0$$

(42)
which is the coefficient of the highest-order term in $\omega$. The solutions are given by $\sigma_{1,2} = -b_m$ and $\sigma_{3,4} = -b_s$. To ensure stability, based on the conditions (i) and (ii), both $b_m$ and $b_s$ should be positive. This condition would place the asymptotes in the LHP and prevent $j\omega$ crossing.

**Condition (iv):** The zero set of the function

$$F(s, jx_2) = \det \left\{ Z(e^{j\omega}) + j \begin{pmatrix} s & 0 \\ 0 & x_2 \end{pmatrix} \right\}; \quad x_2 \in R$$

when $\epsilon \to 0$ is $s = -(b_m + b_s) - x_2j$, which is always in the LHP for $b_m, b_s > 0$.

**B. Verification with Llewellyn’s Criteria**

Here, we will verify the results obtained using the extended Z-W theorem with Llewellyn’s absolute stability criteria, which states that a 2-port network modeled by the impedance matrix $Z$ is absolutely stable if, and only if, the two conditions

1. $Z_{11}$ and $Z_{22}$ are positive real
2. $\eta(\omega) = \frac{\text{Re}\{Z_{21}\}}{\text{Re}\{Z_{22}\}} + \frac{\text{Re}\{Z_{11}\}}{\text{Re}\{Z_{22}\}} \geq 1$,

hold for all $\omega$, where $\eta(\omega)$ is called the network stability parameter [7]. For our impedance matrix, $Z_{11}$ and $Z_{22}$ are positive real for $b_m, b_s > 0$, and the condition

$$\eta(\omega) = \frac{2b_m b_s \omega^2}{K_p} + 1 \geq 1$$

is satisfied if $b_m b_s > 0$. These results confirm the results obtained using Zeheb-Walach theorem. The Zeheb-Walach theorem has two advantages over the Llewellyn’s criteria. One is that it considers the roots of the closed-loop characteristic equation, and hence can show the stability margins. For the example discussed, the stability margin becomes smaller as the damping gains decrease, as seen from Figure 4. The second advantage is that Z-W method can be extended to multi-port networks.

**V. CONCLUSIONS AND FUTURE WORKS**

In this paper, an extension of Zeheb-Walach(Z-W) absolute stability criteria was presented and proven. The extended theorem is applicable to systems with $j\omega$-axis poles. Comparing to the original theorem, we added new conditions on the Laurent expansion of the elements and principal minors of the impedance matrix. The proof follows a similar approach used in [10], which applies results from zero set of multi-parameter functions. We used the extended criteria to analyze the coupled stability of a benchmark teleoperation control system and verified the results by Llewellyn’s absolute stability criteria. In comparison with Llewellyn’s method, the Z-W method can be applied to multi-port networks and can also be used to study stability margins in terms of the loci of the roots of the closed loop systems.

Future work will include the application of the extended Z-W method to networks with time delays. This may require approximation of the delay with rational transfer functions. In addition, routines will be developed to numerically compute the zero set of functions with three or more parameters needed to analyze the absolute stability of networks with a higher number of ports.

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**REFERENCES**