A Convex Formulation of Controller Synthesis for Piecewise-Affine Slab Systems Based on Invariant Sets

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Abstract—This paper presents a controller synthesis method to stabilize piecewise-affine (PWA) slab systems based on invariant sets. Inspired by the theory of sliding modes, sufficient stabilization conditions are cast as a set of Linear Matrix Inequalities (LMIs) by proper choice of an invariant set which is a target sliding surface. The method has two steps: the design of the attractive sliding surface and the design of the controller parameters. While previous approaches to PWA controller synthesis are cast as Bilinear Matrix Inequalities (BMIs) that can, in some cases, be relaxed to LMIs at the cost of adding conservatism, our method leads naturally to a convex formulation. Furthermore, the LMIs obtained in this paper have lower dimension when compared to other methods because the dimension of the closed-loop state space is reduced. A numerical example on flutter suppression is included to demonstrate the effectiveness of the approach.

I. INTRODUCTION

Piecewise-affine (PWA) systems are a special class of hybrid systems. PWA systems can also be used to approximate a wide variety of nonlinear systems. Several promising methods have emerged for analysis and synthesis of PWA control systems such as those proposed in [1]–[9] and references therein. Unfortunately, synthesis of PWA controllers naturally leads to non-convex problems. Solving these problems is therefore a non-trivial task. PWA slab systems [6] are a subclass of PWA systems where the regions partitioning the domain are slabs. Reference [2] shows that sufficient conditions for quadratic stabilization using piecewise-linear state feedback for PWA slab systems can be cast as a convex optimization problem. Unfortunately, if affine terms are included in the controller, the convex structure is apparently destroyed, making it hard to solve the problem globally. However, under certain additional assumptions, reference [6] shows that one can develop algorithms to solve these non-convex problems with optimality guarantees. References [10], [11] present algorithms for state-feedback design of PWA systems based on LMIs which can be efficiently solved using software packages such as SeDuMi [12] and YALMIP [13]. In fact, to the best of our knowledge, the methods proposed in [10], [11] are the only ones that can formulate PWA state feedback as a set of LMIs. The method in [10] shows that one can avoid solving the Bilinear Matrix Inequalities (BMIs) proposed in [6] by using a convex relaxation which leads to a set of LMIs. Unfortunately, using more conservative conditions may lead to infeasibility. In [11] a backstepping approach is developed for PWA systems in strict feedback form. Controller synthesis was formulated as a convex problem but one cannot control the way in which the trajectories converge to the origin. This limitation motivates the work that will be presented here. It will be shown that the synthesis procedure proposed in this paper leads to a convex problem in a reduced state space and the closed-loop trajectories converge to the origin along a desired direction.

The contribution of this paper is to use invariant set ideas to formulate the PWA synthesis problem as a set of LMIs. Inspired by the theory of sliding modes, sufficient stabilization conditions are cast directly as a set of LMIs by proper choice of an invariant set which is a target sliding surface. It is shown that the dimension of the LMIs obtained in this paper is lower than in the other convex methods in the literature because the dimension of the state space is reduced, which further simplifies the synthesis problem. We have considered active flutter suppression (AFS), an interesting and hard control problem in aerospace systems, as an application of our method. Flutter is inherently a nonlinear phenomenon. However, one can approximate the nonlinearity by a PWA function using for example the method detailed in [4]. Simulation results will show how the method proposed in this paper can actively suppress flutter in a wing section.

This paper is divided into five parts. The problem formulation is presented first. The proposed controller synthesis for PWA systems is explained in section III. The controller synthesis method is then applied to AFS in section IV. Finally, after showing simulation results, conclusions will be drawn.

II. PRELIMINARIES

The dynamics of a PWA system can be written as

$$x(t) = Ax(t) + a_i + Bu(t), \quad x(t) \in R_i \quad (1)$$

where $x(t) \in \mathbb{R}^n$ is the state, $u(t) \in \mathbb{R}^p$ the control input and a forward invariant set $\mathcal{X} \subset \mathbb{R}^n$ is partitioned into $M$ polytopic cells $R_i, i \in I = \{1, \ldots, M\}$ such that $\cup_{i=1}^{M} \bar{R}_i = \mathcal{X}, \bar{R}_i \cap \bar{R}_j = \emptyset$ where $\bar{R}_i$ denotes the closure of $R_i$ (see [4] for generating such partition).

A slab region is defined as

$$R_i = \{x \mid \beta_i < \lambda^T x < \beta_{i+1}\} \quad (2)$$

where $\lambda \in \mathbb{R}^n, \lambda \neq 0$ and $\beta_i, \beta_{i+1} \in \mathbb{R}, i = 1, \ldots, M$. The slab region $R_i$ can also be cast as a degenerate ellipsoid

$$R_i = \{x \mid \|Li + li\| < 1\} \quad (3)$$

where

$$Li = 2\lambda^T / (\beta_{i+1} - \beta_i), \quad li = - (\beta_{i+1} + \beta_i) / (\beta_{i+1} - \beta_i). \quad (4)$$
A PWA system whose regions are slabs is called a PWA slab system [6]. Using a PWA control input of the form
\[ u = K_i x(t) + k_i, \quad x(t) \in \mathcal{R}_i \]
into system (1) yields the closed-loop dynamics
\[ \dot{x}(t) = A_i x(t) + a_i, \quad x(t) \in \mathcal{R}_i, \]
where \( A_i = A_i + B_i k_i \), \( a_i = a_i + B_i k_i \)
(8)

The following lemma gives sufficient conditions for asymptotic stability of closed-loop PWA slab systems.

Lemma 1: [6] Consider the PWA slab system (1). If there exist \( Q = Q^T > 0 \), \( \alpha > 0 \), \( \mu_i > 0 \) satisfying
\[ \Omega_{i0} + \alpha Q < 0 \quad \text{if} \quad 0 \in \mathcal{R}_i, \]
\[ \Omega_i = \begin{bmatrix} \Omega_{i1} & \Omega_{i2} \\ \Omega_{i3} & \Omega_{i4} \end{bmatrix} < 0 \quad \text{otherwise} \]
(9)\( (10) \)

where
\[ \begin{align*}
\Omega_{i0} &= A_i Q + Q A_i^T \\
\Omega_{i1} &= A_i Q + Q A_i^T + \alpha Q - \mu_i \pi_i \pi_i^T \\
\Omega_{i2} &= -\mu_i \pi_i \pi_i^T + Q L_i^T \\
\Omega_{i3} &= -\mu_i \pi_i \pi_i^T + L_i Q \\
\Omega_{i4} &= \mu_i (1 - l_i T_i)
\end{align*} \]
with \( L_i \) and \( l_i \) defined in (4) and (5), then the origin is an exponentially stable equilibrium point.

Proof: See reference [6].

Note that the constraint (10) is nonconvex. The nonconvexity of BMIs (10) is due to the existence of the term
\[ -\mu_i \pi_i \pi_i^T \]
(11)
which includes a product of unknown gains \( k_i \). Therefore controller synthesis for PWA slab systems is a non-convex problem. The method proposed in [10], shows that one can avoid solving the BMIs (10) by ignoring the negative definite term (11), which is a convex relaxation. Note that the conditions of Lemma 1 are sufficient conditions and therefore conservatism has been already introduced to the problem. Reference [10] adds more conservativeness by ignoring the negative definite term (11). Unfortunately, the resulting conditions may lead to infeasibility. Motivated by the drawbacks of existing methods, the next section presents a convex formulation of the synthesis problem using an invariant set approach.

III. CONTROLLER SYNTHESIS

Consider the following class of PWA slab systems
\[ \dot{x}(t) = A_i x(t) + a_i + \begin{bmatrix} 0 \\
B_2 \end{bmatrix} u(t), \quad x(t) \in \mathcal{R}_i \]
(12)

where \( u \in \mathbb{R}^p, B_2 \in \mathbb{R}^{m \times p} \) and \( m \in \mathcal{M} = \{1, \ldots, n - 1\}, m \geq p \).

Remark 1: Note that the equations of motion of several physical systems of interest come naturally in this form, in particular if one write the equations of motion of mechanical systems divided into the kinematics (without input forcing terms) and the dynamics (with input forcing terms).

We can rewrite equations (12) in the following form
\[ \begin{bmatrix} \dot{x}_1 \\
\dot{x}_2 \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\
A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} x_1 \\
x_2 \end{bmatrix} + \begin{bmatrix} a_1 \\
a_2 \end{bmatrix} + \begin{bmatrix} 0 \\
B_{22} \end{bmatrix} u, \quad x(t) \in \mathcal{R}_i \]
(13)
where \( x_1 \in \mathbb{R}^{n-m}, x_2 \in \mathbb{R}^m \). Assume further that in this class of PWA systems, the slab regions are only functions of \( x_1 \). Therefore, the definition of slab regions (3) can be rewritten as
\[ \mathcal{R}_i = \{ x \mid \| L_i x + l_i \| = \| L_{1i} x + l_i \| \leq 1 \} \]
(14)

where \( L_{1i}^T \in \mathbb{R}^{n-m} \). This paper proposes a new method to formulate PWA controller synthesis for system (13) as a convex feasibility problem. The main result is presented in the next theorem.

Theorem 1: Assuming that either \( B_{2i} = 0 \) or \( B_{2i} = B_2 \) is constant and full rank for all regions, the PWA controller
\[ u = -(S_2 B_2)^{-1} [S_1 (A_{11} x_1 + A_{12} x_2 + a_1) + S_2 (A_{21} x_1 + A_{22} x_2 + a_2) + \gamma \| S_1 x_1 + S_2 x_2 \|_2] \]
(15)
for \( x \in \mathcal{R}_i, i = 1, \ldots, M \), exponentially stabilizes system (13) defined in a forward invariant set \( \mathcal{X} \) if given \( \gamma > 0 \) and \( \alpha > 0 \), there exist \( Q = Q^T > 0 \), \( \mu_i > 0 \), and \( Y = S_1 Q \), satisfying the following LMIs

where
\[ \omega_{i0} + \alpha Q < 0 \quad \text{if} \quad 0 \in \mathcal{R}_i, \]
\[ \omega_i = \begin{bmatrix} \omega_{i1} & \omega_{i2} \\ \omega_{i3} & \omega_{i4} \end{bmatrix} < 0 \quad \text{otherwise} \]
(16)\( (17) \)
\[ \omega_{i0} = A_{11} Q + Q A_{11}^T - A_{12} S_2^T Y - Y^T (S_2^T)^T A_{12}^T, \]
\[ \omega_{i1} = A_{11} Q + Q A_{11}^T - A_{12} S_2^T Y - Y^T (S_2^T)^T A_{12}^T + \alpha Q - \mu_i a_{1i} a_{1i}^T, \]
\[ \omega_{i2} = -\mu_i a_{1i} a_{1i}^T + Q L_{1i}^T, \]
\[ \omega_{i3} = -\mu_i a_{1i} a_{1i}^T + L_{1i} Q, \]
\[ \omega_{i4} = \mu_i (1 - l_{1i} T_{1i}) \]
with
\[ S_i^T = S_i^T (S_i S_i^T)^{-1} \]
(18)

Proof: Consider a surface of the form
\[ \sigma(x) = S x = 0 \]
(19)
where
\[ S = \begin{bmatrix} S_1 & S_2 \end{bmatrix} \]
(20)
with \( S_1 \in \mathbb{R}^{p \times (n-m)} \) and \( S_2 \in \mathbb{R}^{p \times m} \). In order to make \( \sigma(x) = 0 \) an attractive invariant set, we define a candidate Lyapunov function of the form
\[ V(\sigma(x)) = \frac{1}{2} \sigma^T(x) \sigma(x). \]
(21)
Note that, although $V(\sigma(x))$ is implicitly based on $x(t)$, it is not a Lyapunov function for $x$, but it is rather a Lyapunov function for $\sigma(x)$. As a function of $\sigma(x)$, $V(\sigma(x))$ is obviously positive definite because it is a norm. In order to have finite-time convergence to $\sigma(x) = 0$, according to [14] and [15] one needs to ensure

$$\dot{V}(\sigma(x)) \leq -\mu \|\sigma(x)\|$$  

(22)

where $\mu > 0$. Note that the Lie derivative of the Lyapunov function in (21) is

$$\dot{V}(\sigma(x)) = \frac{\partial V(\sigma(x))}{\partial \sigma(x)} \sigma(x) = \sigma^T(x) \dot{\sigma}(x).$$

(23)

Designing $\sigma(x)$ such that

$$\dot{\sigma}(x) = -\gamma \left( \frac{\sigma(x)}{\|\sigma(x)\|} \right)$$

(24)

with $\gamma \geq \mu > 0$, the time rate of change of the Lyapunov function in (21) will be

$$\dot{V}(\sigma(x)) = -\gamma \sigma^T(x) \left( \frac{\sigma(x)}{\|\sigma(x)\|} \right)$$

$$= -\gamma \|\sigma(x)\| \leq -\mu \|\sigma(x)\|,$$

(25)

which verifies (22). Using (13), (19) and (20), for $t \geq t_c$, we can write

$$\dot{x} = S \dot{x} = S_1 (A_{11}x_1 + A_{12}x_2 + a_1)$$

$$+ S_2 (A_{21}x_1 + A_{22}x_2 + a_2) + (S_2 B_2) u$$

(26)

where $S_2$ is invertible or constant for all $i \in I$ and full rank, $S_2 B_2$ is invertible (for example with the choice $S_2 = B_2^T$ when $B_2 = B_2$), and replacing the control law (15) into (26) ensures that (25) is verified. Therefore makes the target surface $\sigma(x) = 0$ an attractive invariant set. We now show that the trajectories converge to this target surface in finite time. Observe that (25) is equivalent to

$$\dot{V}(\sigma(x)) = -\gamma \sqrt{2} V^{\frac{1}{2}}(\sigma(x))$$

(27)

for the Lyapunov function defined in (21). This is a differential equation. Assuming $V(\sigma(x(t_0)))$ as the initial condition, the solution to (27) can be found as

$$V^{\frac{1}{2}}(\sigma(x(t))) = V^{\frac{1}{2}}(\sigma(x(t_0))) - \frac{\sqrt{2} \gamma}{2} (t - t_0).$$

(28)

One now can see that

$$\exists t_c \in \mathbb{R}, \text{ such that } V(\sigma(x(t_c))) = 0$$

(29)

where $t_c \geq t_0$ is the finite time of convergence to the surface. In fact, replacing $V(\sigma(x(t_0))) = 0$ in (28) yields

$$t_c = \sqrt{2} \gamma^{-1} V^{\frac{1}{2}}(\sigma(x(t_0))) + t_0.$$

(30)

Furthermore, the differential equation (27) implies that

$$\dot{V}(\sigma(x(t_c))) = -\gamma \sqrt{2} V^{\frac{1}{2}}(\sigma(x(t_c))) = 0,$$

(31)

which yields

$$V^{\frac{1}{2}}(\sigma(x(t))) = 0, \quad \forall t > t_c$$

(32)

and therefore

$$V(\sigma(x(t))) = 0, \quad \forall t \geq t_c.$$

(33)

Since the trajectories converge in finite time to the surface $\sigma(x) = 0$ and remain on that surface for all future times, using (19) and (20), for $t \geq t_c$ we can write

$$S_1 x_1 + S_2 x_2 = 0.$$  

(34)

Assuming

$$x_2 = S_2 T Z$$

(35)

where $Z \in \mathbb{R}^p$, we can rewrite (34) as

$$Z = -(S_2 S_2^T)^{-1} S_1 x_1.$$  

(36)

Hence

$$x_2 = -S_2^T S_1 x_1$$

(37)

is the pseudo-inverse of the matrix $S_2$. Therefore, using (34) and (37) we can rewrite the dynamics of the PWA system (13) for $t \geq t_c$ as

$$x_2 = -S_2^T S_1 x_1$$

(39)

$$\dot{x} = (A_{11} - A_{12} S_2^T S_1) x_1 + a_1, \quad x \in R_i.$$  

(40)

Due to (39), if $x_1(t)$ exponentially converges to the origin, then $x_2(t)$ will also exponentially converge to the origin. Therefore, exponential stability of the reduced order system (40) ensures that the PWA slab system (13) is exponentially stable under the control law (15). However, exponential stability of the reduced order system (40) is guaranteed if the LMIs (16)–(17) hold, based on Lemma 1 using

$$\overline{A}_i := (A_{11} - A_{12} S_2^T S_1)$$

(41)

$$a_i := a_1.$$  

(42)

This finishes the proof.

Remark 2: As one can see, Theorem 1 results in a set of LMIs. Moreover no relaxation is used in the proof. In fact since (42) is always a constant vector (in each region), the term (11) is known, which makes the problem convex.

Remark 3: Theorem 1 reduces the complexity of the LMIs that must be solved because transforming the closed-loop stability problem for system (13) into a stability problem for system (40) makes the dimension of the closed-loop state space smaller than the dimension of the open-loop state-space. The control methods in [10], [11] do not perform this transformation and therefore are more complex because of two reasons: i) they lead to BMIs and ii) the dimension of the state space is larger.
IV. APPLICATION TO ACTIVE FLUTTER SUPPRESSION

In this section, we consider the problem of Active Flutter Suppression (AFS) to demonstrate how the proposed method works. The flutter model is taken from [16] and is given by

\[
\dot{x} = \begin{bmatrix} 0 \\ -M^{-1}(K_0 + K) \end{bmatrix} x + \begin{bmatrix} 0 \\ -M^{-1}(C_0 + C) \end{bmatrix} \begin{bmatrix} I \\ 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ -2 x_2 k_\alpha(\alpha) \end{bmatrix} + \begin{bmatrix} 0 \\ \mu M^{-1} \end{bmatrix} Bu \tag{43}
\]

where

\[
C_\mu = \begin{bmatrix} \rho Ub C_{La} & \rho Ub^2 C_{La} (\frac{1}{2} - a) \\ \rho Ub^2 C_{ma} & -\rho Ub^2 C_{ma} (\frac{1}{2} - a) \end{bmatrix},
\]

\[
C_\sigma = \begin{bmatrix} C_b & 0 \\ 0 & C_\sigma \end{bmatrix},
\]

\[
M = \begin{bmatrix} m & mx_\alpha b \\ mx_\alpha b & I_\alpha \end{bmatrix},
\]

\[
K_0 = \begin{bmatrix} k_h & 0 \\ 0 & 0 \end{bmatrix},
\]

\[
K_\mu = \begin{bmatrix} 0 & \rho U^2 b C_{La} \\ 0 & -\rho U^2 b^2 C_{ma} \end{bmatrix},
\]

\[
B = \begin{bmatrix} -p b C_{L\beta_1} & -p b C_{L\beta_2} \\ -p b^2 C_{m\beta_1} & -p b^2 C_{m\beta_2} \end{bmatrix} U^2.
\]

\(U\) is the airspeed, \(C_{La}\) and \(C_{ma}\) are aerodynamic lift and moment coefficients, \(\rho\) is the air density, \(C_{L\beta}\) and \(C_{m\beta}\) are lift and moment coefficients per control surface deflection, respectively, \(m\) is the mass of the airfoil, \(I_\alpha\) is the mass moment of inertia about the elastic axis, \(C_h\) and \(C_\alpha\) are plunge and pitch structural damping coefficients, respectively, and \(L\) and \(M\) are the aerodynamic lift and moment about the elastic axis. Structural stiffness is represented by \(k_h\) and \(k_\alpha\) for plunge and pitch motions, respectively. The source of nonlinearity is the torsional stiffness, which is

\[
k_\alpha(\alpha) = k_{\alpha_0} + k_{\alpha_1}(\alpha) + k_{\alpha_2}(\alpha^2) + k_{\alpha_3}(\alpha^3) + k_{\alpha_4}(\alpha^4).	ag{44}
\]

All the model parameters are taken from [16] and are available in the appendix. A controller is now designed for a PWA approximation of the system (43). Therefore, first we approximate the AFS nonlinear system by a PWA model using the method explained in [4]. The slab regions used for the approximation are defined by

\[
R_1 = \{x \in \mathbb{R}^4 \mid 0.6 < x_2 = \alpha < 1\},
\]

\[
R_2 = \{x \in \mathbb{R}^4 \mid -1 < x_2 = \alpha < -0.6\},
\]

\[
R_3 = \{x \in \mathbb{R}^4 \mid -0.6 < x_2 = \alpha < -0.2\},
\]

\[
R_4 = \{x \in \mathbb{R}^4 \mid -0.2 < x_2 = \alpha < 0.2\},
\]

\[
R_5 = \{x \in \mathbb{R}^4 \mid 0.2 < x_2 = \alpha < 0.6\}.
\]

The dynamic equations in all regions will not be presented here for lack of space but, for example, the equations of the PWA model for AFS in \(R_5\) are

\[
\dot{x} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ -293.27 & 2272.13 & -5.90 & -0.40 \\ 1885.94 & -69573.59 & 34.72 & 2.47 \\ 13972.34 & 14250.44 & 9021.92 \end{bmatrix} x + \begin{bmatrix} 0 \\ 0 \\ -471.47 \\ -7606.78 \end{bmatrix} u + \begin{bmatrix} -7642.55 \end{bmatrix} u. \tag{45}
\]

This approximation belongs to the class of PWA systems defined in (12). To design a controller, we first define the parameters

\[
\gamma = 2 \quad \alpha = 0.01 \tag{46}
\]

and then we assign

\[
S_2 = B_2^{-1} = \begin{bmatrix} 0.0002 & 0.0002 \\ -0.0004 & -0.0002 \end{bmatrix}.	ag{47}
\]

Using these parameters and solving LMIs (16) and (17), the matrix \(S_1\) is obtained as

\[
S_1 = \begin{bmatrix} 0.00017 & -0.0198 \\ -0.0017 & 0.0320 \end{bmatrix}.	ag{48}
\]

Therefore the sliding surface defined in (19) for the AFS problem is

\[
\sigma(x) = \begin{bmatrix} 0.0017 & -0.0198 & 0.0002 & 0.0002 \\ -0.0017 & -0.0198 & 0.0002 & 0.0002 \end{bmatrix} x.	ag{49}
\]

After computing \(\sigma(x)\) and using (15), we are able to derive control laws for all five regions. For example the control input for the fifth region is

\[
\begin{bmatrix} -0.292 & 12.691 & -0.007 & 0.019 \\ 0.252 & -12.334 & 0.006 & -0.031 \\ 0.0017 & -0.0198 & 0.0002 & 0.0002 \\ -0.0017 & -0.0004 & -0.0004 & -0.0002 \end{bmatrix} x + \begin{bmatrix} 0.0002 \\ -0.0004 \\ -0.0004 \\ -0.0002 \end{bmatrix} x + \begin{bmatrix} -2.545 \\ -2.471 \end{bmatrix}.	ag{50}
\]

Fig. 1 shows simulation results for \(x_0 = [0.15 \ 0.1 \ 0.5 \ -0.2]^T\) as an initial condition. It can be clearly seen in the figure that flutter was effectively suppressed as desired. Fig. 2 shows the simulation results for the same AFS model controlled by the method proposed in [10]. The simulation results in Fig. 2 show that there is a high frequency oscillatory behavior of the state variables using the approach suggested in [10].
V. Conclusion

This paper proposes a novel controller design method for PWA systems. The method has two steps: the design of an attractive invariant set and the design of the controller parameters. To the best of our knowledge, all previous approaches to continuous-time PWA controller synthesis are cast as non-convex BMIs that can, in some cases, be relaxed to LMIs at the cost of adding conservatism. By contrast, the method proposed in the paper leads naturally to a convex cast as non-convex BMIs that can, in some cases, be relaxed to LMIs at the cost of adding conservatism. By contrast, the method proposed in the paper leads naturally to a convex approach,
\[k_h \quad 2844.4\]

\[K_{\mu} \quad \begin{bmatrix} 0 & 935.1 \\ 0 & -6.3 \end{bmatrix} \text{ kg/s}^2\]

\[C_o \quad \text{diag}(27.43, 0.036)\]

\[C_h \quad 27.43\]

\[C_\alpha \quad 0.036 \text{ kgm}^2/\text{s}\]

\[C_{\mu} \quad \begin{bmatrix} 31.17 & 3.99 \\ 0.21 & -0.027 \end{bmatrix}\]

\[K_{\alpha}(\alpha) \quad q\alpha\]

\[q \quad \begin{bmatrix} q_1 & q_2 & q_3 & q_4 & q_5 \end{bmatrix}^T\]

\[q_1 \quad 2.82\]

\[q_2 \quad -62.322\]

\[q_3 \quad 3709.71\]

\[q_4 \quad -24195.6\]

\[q_5 \quad 48756.954\]

\[\mu \quad 176.609\]

\[C_{L\beta_1} \quad 3.358\]

\[C_{L\beta_2} \quad 3.458\]

\[C_{m\beta_1} \quad -0.635\]

\[C_{m\beta_2} \quad -0.735\]

\[\rho \quad 1.225 \text{ kg/m}^3\]

\[C_{m\alpha} \quad (0.5 + a)C_{L\alpha}\]

\[C_{L\alpha} \quad 6.28\]