Stochastic Controllability and its role in Network Congestion Control

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Abstract—Controllability and reachability concepts are developed for nonlinear stochastic systems motivated by a problem in network congestion control described by a Hidden Markov Model. These definitions are posed in terms of the capability to steer the state distribution towards certain target distributions, as measured by relative entropy. It is shown that this definition extends earlier approaches and, in the linear case, concurs with the usual rank and range conditions from deterministic analysis. For the network congestion control problem, these ideas are analyzed from two viewpoints: the capacity to steer the state distribution towards a specific target, reachability; the capacity to yield bottleneck node state distributions which maintain stochastic observability at the data source from acknowledgement packet and input rate data, controllability.

I. INTRODUCTION

We revisit the network congestion control problem discussed in the authors’ recent paper [1]. The system features a computer which sends data into a network through a bottleneck router to a destination computer. The destination responds with acknowledgment packets when data is successfully delivered. From the perspective of a user connected to such a network, there are two objectives: the first is to increase the throughput of data being sent through the network, and the second is to avoid congesting the network, requiring that a significant portion of the throughput be dedicated to the re-transmission of lost data due to congestion. These objectives inform the design of throughput-control algorithms for source computers including the widely implemented Additive-Increase-Multiplicative-Decrease algorithm (AIMD) [2] and variations which roughly determine the level of congestion in the network based on the proportion of successfully delivered data indicated by the acknowledgment sequence. Underlying the algorithm design is the assumption that the acknowledgment sequence contains enough information about the network for meaningful control action; [1] addresses the validity of this assumption.

In [3] the authors demonstrated connections between the selection of control algorithm and the information content of the sequence of acknowledgments. That the control can influence the system behavior appears to be a question of stochastic controllability, but existing concepts of controllability do not address the role of and impact on estimation, and so are insufficient for describing our observations even in the simple network model used in [3]. Our goal in the present work is to demonstrate that a meaningful concept of stochastic controllability can be defined, which in our network model distinguishes control algorithms based on their ability to produce informative outputs, and which conforms to the well-known classical definitions of controllability for linear, deterministic systems [4], [5]. We develop in parallel the closely related notion of stochastic reachability.

The paper begins in Section II with the Hidden Markov Model (HMM) formulation of the network congestion control problem. Both the structure of the HMM and the implementation of network congestion control algorithms provide context to the theoretical development that follows. Section III establishes some mathematical preliminaries. Section IV introduces our definitions of stochastic controllability and reachability in the framework of linear Gaussian systems; this simultaneously demonstrates compatibility with the familiar deterministic definitions and provides intuitions for the nonlinear extensions. Section V offers the nonlinear versions of the definitions of Section IV; a key result establishes the connection to the definitions of stochastic observability given in [1]. The result provides insight into the network congestion control problem and currently implemented algorithms. We conclude V with a brief return to the HMM. The structure of the HMM allows additional links to be drawn to the ideas of model reducibility, a subject closely related to deterministic controllability and reachability, but rarely discussed in studies on stochastic controllability and reachability.

II. NETWORK MODEL

We reprise a Hidden Markov Model from [1] for congestion control in Transmission Control Protocol (TCP) networks [6]. This provides the concrete motivation for our analysis. Figure 1 depicts the behavior of a single bottleneck node [7] in such a network. The source computer sends packets into the network at rate \( r_k \). The router possesses a buffer of length \( b_{\text{max}} \) into which it may place these packets; its occupancy is denoted \( b_k \). Traffic from other sources arrives into other buffers and acts to compete for available downlink capacity. The capacity available for onward transmission of the \( b_k \) packets is denoted \( c_k \) and may be modeled as a Markov process, \( c_k \sim \text{Markov}(\Pi_c, P_c) \) [8], where \( \Pi_c \) is the probability distribution of \( c_0 \) and \( P_c \) is the associated transition matrix. This model (fixed buffer segment and variable capacity) is used to capture the dynamics of a shared buffer and varying arrivals. Because each component, \( b_k \) and \( c_k \), has finite cardinality, we can lexicographically assign pairs of \( (b_k, c_k) \) values to single values of a state

\[ x_k = (b_k, c_k). \]
The router runs a simplified form of the Random Early Detection algorithm (RED) [9] where $b_k$ is the instantaneous, rather than averaged, queue length, and the drop probability is given by

$$p_k = \varphi \left( \frac{b_k}{b_{\text{max}}} \right),$$

with $b_{\text{max}}$ equal to the maximum queue length and $\varphi(\cdot)$ a monotonically nondecreasing function equal to zero at zero and one at one.

With $\varphi$ we assume return within one time step with some random loss. The measurement $y_k$ is a counting process of the number of successful ACKs, i.e. the number of packets which arrive at the source at time $k$. With these conditions, the full system with state $x_k$, send rate $r_k$, and measurement $y_k$ can now be described as a Hidden Markov Model (HMM) [10], [11], [12].

$$\Pi_{k+1} = A(r_k)\Pi_k,$$

$$\Psi_k = C(r_k)\Pi_k,$$

$$x_k \sim \Pi_k,$$

$$y_k \sim \Psi_k,$$

$$A(r_k)[i,j] = P(x_{k+1} = i|x_k = j, r_k),$$

$$C(r_k)[i,j] = P(y_k = i|x_k = j, r_k).$$

In [1], the HMM filter was shown to perform well in simulation, and computation of quantifiers from information theory of the estimator performance agreed with our qualitative observations. Here, we study the effect of the control on increasing the information content of the outputs and show how this may be viewed as a controllability question. Our starting point is the existing concepts of controllability and reachability.

### III. Preliminaries

Before it is possible to define versions of controllability for stochastic systems, we require detailed characterization of the control signal for stochastic systems. Feedback is intuitively necessary to remove disturbances in the stochastic state evolution. On the other hand, we may also wish to consider control policies which do not use any feedback. These include open-loop control, vibrational control [13], where an unstable system is stabilized by a periodic control input determined off-line, and parameter-noise control [14], where an unstable system is stabilized via a stochastic parametric input sequence which is independent of the state. In between pure-feedback and pure-noise inputs are mixed policy algorithms which arise frequently as equilibrium solutions for game-theoretic problems.

To account for the variety of control policies, for a system

$$x_{k+1} = f(x_k, u_k, w_k),$$

where $x_0 \sim \Pi_0$ and $\{w_{k}\}_0^\infty$ are independent, we define an admissible control at each time $k$ to be a function $u_k = g_k(\{x_{k}\}_0^k, r_k, \Pi_0)$, where $r_k$ is independent of $\{x_{k}\}_0^k$ and $\{w_{k}\}_0^k$ and $g_k$ is measurable on the product $\sigma$-algebra generated by $\{x_{k}\}_0^k$ and $r_k$. The initial state probability law $x_0 \sim \Pi_0$ is user-specified and fixed, so it is trivially measurable. Note that the sequence $\{r_k\}_0^k$ may be interpreted as the contribution of measurement noise when a control policy operates in output feedback, as the randomized component of a mixed strategy, or as the parametric noise of a control studied in [14].

### IV. Linear Gaussian Case

We begin with the extension of the definitions of controllability/reachability for linear, deterministic systems to linear-Gaussian systems. Consider a linear system of the form

$$x_{k+1} = Ax_k + Bu_k + w_k,$$

with state $x_k \in \mathbb{R}^n$, control input $u_k \in \mathbb{R}^m$, and disturbance $w_k \in \mathbb{R}^r$. The initial state is normally distributed $x_0 \sim N(x_0, \Sigma_0)$ and the disturbances $w_k$ form a white noise, normally distributed sequence $w_k \sim N(0, \Sigma_k)$. Additionally, $w_k$ and $x_0$ are independent for every non-negative $k$. Denote by $\mathcal{F}_k$ the filtration generated by $\sigma(\{x_{k}\}_0^k)$, that is, the $\sigma$-algebra generated by the sequence of states from time 0 to $k$.

For convenience, we write

$$x_n = A^n x_0 + \mathcal{G}_{n-1} \mathcal{W}_{n-1} + \mathcal{Z}_{n-1},$$

where for times $k = 0, 1, \ldots, n$, $\mathcal{G}_{n-1}$ is the controllability matrix, $\mathcal{G}_{n-1}$ is the vector of control inputs, $\mathcal{Z}_{n-1} = [I_n, A, A^2, \ldots, A^{n-1}]$ with $I_n$ being the $n$-dimensional identity matrix, and $\mathcal{W}_{n-1}$ is the vector of disturbances.

Given an admissible control sequence $\mathcal{W}_{n-1}$, denote by $\Sigma_n(\mathcal{W}_{n-1})$ the covariance of state $x_n$ at time $n$. Denote by $\Sigma_n(0)$ the covariance of the state when the control input at each time is zero.

#### A. Controllability

**Definition 1 ([4]):** Let $w_k \equiv 0$ for $k = 0, 1, \ldots$. We say (3) is completely (linear-deterministic) controllable if for every point $\bar{x} \in \mathbb{R}^n$, there exists a control function which drives (3) from the initial state $x_0 = \bar{x}$ to $x_N = 0$ in finite time $N < \infty$. 

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1. Our model is an idealization; realistic values for $y_k$ and $c_k$ may not be suitable in the HMM realization.
As is well known, a necessary and sufficient condition for Definition 1 is
\[
\text{range}(A^n) \subseteq \text{range}(C_{n-1}). \tag{5}
\]
We extend Definition 1 using the state covariance. 

**Definition 2:** The system (3) is said to be completely linear-stochastic controllable if for every n-vector \( \xi \), either
\[
\xi^T x_n \quad \text{and} \quad x_0 \quad \text{are independent when } \mathcal{W}_{n-1} \equiv 0 \text{ or } \xi^T \Sigma_n(\mathcal{W}_{n-1}) \xi < \xi^T \Sigma_n(0) \xi, \tag{6}
\]
for some admissible \( \mathcal{W}_{n-1} \).

For convenience, assume in the following that \( x_0 = 0 \). The linear system (3) is completely linear-stochastic controllable if the matrix pair \((A, B)\) satisfies the deterministic conditions for controllability.

**Theorem 1:** The system (3) is completely linear-stochastic controllable for every \( \xi \), \( \Sigma \), and \( \{Q_k\}_{0}^{\infty} \) if and only if condition (5) holds.

**Proof:** Denote by \( C_{n-1}^{-1} \) the Moore-Penrose pseudo-inverse of \( C_{n-1} \). Fix an n-vector \( A \). Because the range of \( A^n \) is contained in the range of \( C_{n-1} \), there exists a \( \xi \) such that
\[
A^n x_0 = C_{n-1}^{-1} \xi.
\]

Define \( \mathcal{W}_{n-1} \equiv -C_{n-1}^{-1} C_{n-1} \xi \). The expression (4) reduces to
\[
x_n = A^n x_0 - C_{n-1}^{-1} C_{n-1} \xi + \mathcal{W}_{n-1},
\]
\[
= A^n x_0 - C_{n-1}^{-1} \xi + \mathcal{W}_{n-1},
\]
\[
= A^n x_0 - A^n x_0 + \mathcal{W}_{n-1},
\]
\[
= \mathcal{W}_{n-1}.
\]

Due to (8), for an n-vector \( \xi \), if \( \xi^T A^n x_0 \neq 0 \) almost surely, then \( \xi^T \Sigma_n(\mathcal{W}_{n-1}) \xi < \xi^T \Sigma_n(0) \xi \). On the other hand, if \( \xi^T A^n x_0 \) is 0 almost surely, then the unforced response of (4) is
\[
\xi^T x_n = \xi^T \mathcal{W}_{n-1}.
\]
Since the disturbances \( \mathcal{W}_{n-1} \) and the initial state are independent, it follows that \( \xi^T x_n \) and \( x_0 \) are independent in the unforced system.

**Only if part:** In the first case, if \( \xi^T x_n \) and \( x_0 \) are independent in the unforced system, then \( E[\xi^T x_n x_0^T] = 0 \) so
\[
0 = \xi^T A^n E[x_0 x_0^T] + \xi^T \mathcal{W}_{n-1} E[\mathcal{W}_{n-1} x_0^T],
\]
\[
= \xi^T A^n \Sigma_0.
\]
In particular, let \( \Sigma_0 > 0 \); then \( \xi^T A^n = 0 \), i.e. \( \xi \) is not in the range of \( A^n \).

On the other hand, when \( \xi^T x_n \) and \( x_0 \) are not independent in the unforced system, we have \( \xi^T A^n \neq 0 \). To see this, note that if \( \xi^T A^n = 0 \) – as argued above – \( \xi^T x_n \) and \( x_0 \) are independent because \( x_0 \) and \( \mathcal{W}_{n-1} \) are independent. Combining \( \xi^T A^n \neq 0 \) with the condition that there is an admissible control \( \mathcal{W}_{n-1} \) for which \( \xi^T \Sigma_n(\mathcal{W}_{n-1}) \xi < \xi^T \Sigma_n(0) \xi \) we are required to show \( \xi^T \mathcal{W}_{n-1} \neq 0 \).

For the contradiction argument, fix \( \xi \) so that \( \xi^T A^n \neq 0 \) but \( \xi^T \mathcal{W}_{n-1} = 0 \). Then for any admissible control \( \mathcal{W}_{n-1} \), we have by substitution into (4),
\[
\xi^T x_n = \xi^T A^n x_0 + \xi^T \mathcal{W}_{n-1},
\]
\[
\Rightarrow \xi^T \Sigma_n(\mathcal{W}_{n-1}) \xi = \xi^T A^n \Sigma_0 \xi + \xi^T \mathcal{W}_{n-1} \mathcal{W}_{n-1}^T \xi
\]
\[
= \xi^T \Sigma_n(0) \xi,
\]
where \( \mathcal{Q}_{n-1} \) is the block diagonal matrix containing \( Q_k \), \( k = 0, 1, \ldots, n - 1 \). The equality in (9) is a contradiction; therefore, \( \xi^T \mathcal{W}_{n-1} \neq 0 \).

The condition,
\[
\xi^T x_n \quad \text{and} \quad x_0 \quad \text{are independent when } \mathcal{W}_{n-1} \equiv 0, \tag{10}
\]

is not obvious. One interpretation is that stable directions in the state-space forget the initial condition in finite time, regardless of the initial state and noise distributions, \( x_0, \Sigma_0, \{Q_k\}_{0}^{\infty} \); additionally, an important case where \( \xi x_n \perp \{x_k\}_{0}^{\infty} \) when \( A \equiv 0 \) – is contained in (10). Nevertheless, it may be possible to relax (10) and derive a result like Theorem 1 with appropriate modifications to the assumptions in the theorem. The selection of (10) allows more direct generalization to nonlinear systems where the origin may not have a clear physical interpretation, for instance in some finite-state Markov models.

### B. Reachability

**Definition 3 ([4]):** Let \( x_0 \) be known and \( w_k \equiv 0 \) for \( k = 0, 1, \ldots \). We say (3) is completely (linear-deterministic) reachable if for every point \( \bar{x} \in \mathbb{R}^n \), there exists a control function which drives (3) from the initial state \( x_0 \) to \( x_N = \bar{x} \) in finite time \( N < \infty \).

Definition 3 is equivalent to the familiar condition that the rank of \( [B \ AB \ldots \ A^{n-1}B] \) must be maximal.

We extend Definition 3 to linear-Gaussian systems in much the same way we extended Definition 1. For a fixed selection of \( x_r \), define the state error at time \( n \) as \( \bar{x}_n \equiv x_n - x_r \). Denote by \( \bar{\Sigma}_n(\mathcal{W}_{n-1}) \) the second moment of \( \bar{x}_n \),
\[
\bar{\Sigma}_n(\mathcal{W}_{n-1}) \equiv E[\bar{x}_n \bar{x}_n^T],
\]
subject to control sequence \( \mathcal{W}_{n-1} \). Similarly, denote by \( \bar{\Sigma}_n(0) \) the second moment of \( \bar{x}_n \) arising from the unforced system.

**Definition 4:** The system (3) is said to be completely linear-stochastic reachable for every n-vector \( \xi \) and any selection of \( x_r \), either
\[
\xi^T x_r = \xi^T (A^n x_0 + \mathcal{W}_{n-1}) \quad \text{a.s. or } \tag{11}
\]
\[
\xi^T \bar{\Sigma}_n(\mathcal{W}_{n-1}) \xi < \xi^T \bar{\Sigma}_n(0) \xi, \tag{12}
\]
for some admissible \( \mathcal{W}_{n-1} \).

Again, for clarity, assume \( \bar{x}_0 = 0 \).

**Theorem 2:** The system (3) is completely linear-stochastic reachable for every \( \bar{x}_0, \Sigma_0, \) and \( \{Q_k\}_{0}^{\infty} \) if and only if the controllability matrix \( \mathcal{W}_{n-1} \) is full rank.

**Proof:** We adopt the notation used in the proof of Theorem 1.

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If part: Select $\mathcal{U}_{n-1} = \mathcal{C}_{n-1}^+(x_r - A^nx_0)$. Since $\mathcal{C}_{n-1}$ is full rank by assumption, (4) yields

$$\tilde{x}_n = A_{n-1}W_{n-1}.$$  

The unforced-system response is

$$\xi^T\tilde{x}_n = \xi^TA^nx_0 - \xi^Tx_r + \xi^TA_{n-1}\tilde{W}_{n-1}.$$  

Suppose $\xi^TA^n x_0 \neq \xi^Tx_r$ on a set of non-zero probability. Then, for some $\delta > 0,$

$$\xi^T\Sigma_n(0)\xi\geq \delta^2 P((\xi^T(A^n x_0 - x_r))^2 > \delta) + \xi^T\mathcal{A}_{n-1}\tilde{Q}_{n-1}\mathcal{A}_{n-1}\xi,$$

since we may choose $\delta > 0$ sufficiently small so that the probability in (14) is positive. Thus, either $\xi^T A^n x_0 = \xi^T x_r$ almost surely, or $\xi^T\Sigma_n(0)\xi > \xi^T\Sigma_n(\mathcal{U}_{n-1})\xi.$

Only if part: We show the contrapositive, i.e. if $\mathcal{C}_{n-1}$ is not full rank, then there exists $\xi$ such that $\xi^T\mathcal{C}_{n-1} = 0.$ Fix $x_0, x_r, \{Q_k\}_{0}^{n-1}$ and choose $x_r$ so $\xi^T x_r \neq \xi^T (A^n x_0 + \mathcal{A}_{n-1}\mathcal{W}_{n-1})$ almost surely. The error version of (4) has the form

$$\xi^T\tilde{x}_n = \xi^T A^n x_0 - \xi^T x_r + \xi^T\mathcal{C}_{n-1}\mathcal{W}_{n-1} + \xi^T\mathcal{A}_{n-1}\tilde{W}_{n-1},$$

with covariance

$$\xi^T\Sigma_n(\mathcal{W}_{n-1})\xi$$

$$= \xi^T A^n\Sigma_0A^T\xi + \xi^T x_r x_r^T \xi + \xi^T\mathcal{A}_{n-1}\tilde{Q}_{n-1}\mathcal{A}_{n-1}\xi,$$  

for any admissible control $\mathcal{W}_{n-1}.$ We have $\xi^T\Sigma_n(\mathcal{W}_{n-1})\xi = \xi^T\Sigma_n(0)\xi$ for every admissible control $\mathcal{W}_{n-1},$ which concludes the proof.

As with the role of (10) in Theorem 1, the condition (11) accommodates pathological selections of $\tilde{x}_0, \Sigma_0,$ and $\{Q_k\}_{0}^{n-1}$ where (12) fails. For instance, in the system

$$x_{k+1} = U_k,$$

with $x_0 \sim N(0,1),$ we should consider the origin reachable in spite of the absence of a control satisfying (12).

V. NONLINEAR EXTENSION

The extension of deterministic controllability and reachability analysis to stochastic, nonlinear systems is not new. It is appropriate to discuss the literature at this point because many of the concepts may be viewed as generalizations of the formulation presented in Sections IV-A and IV-B, and also because the Markov model for the network presents the same fundamental difficulties in both our linear definitions and in the definitions offered by other works.

The basic ideas in much of the existing literature stem from Sunahara, et al. [15] who, given an initial condition, define controllability as the capacity to steer the state variance to within an $\varepsilon$ neighborhood of the origin with a given probability.

Sunahara’s work is extended by Mahmudov and his collaborators [16], [17] in a series of papers, which include results for certain classes of nonlinear systems as well as systems with time delays. Perhaps because the literature mainly focuses on continuous-time systems, Mahmudov does not distinguish between controllability and reachability; in fact, his notion of controllability is more comparable to what we call reachability and for which Sunahara’s definition is a special case. Being a study of the extensions of the deterministic reachability definitions, Mahmudov’s analysis also recovers the better known reachability rank condition. Further extensions to wider classes of nonlinear systems have been approached by Muthukumar and Balasubramaniam [18]; here, as in earlier works, there is an emphasis on the idea that reachability describes the ability to steer the state closer to any desired point in the state space.

Ugrinovskii [19] approaches the controllability question from a different direction. His formulation concerns the robust control of linear uncertain systems and delineates controls which bound the $L_2$ norm of the state over a class of permissible uncertainties, which are described by their distributions. In this context, he shows necessary and sufficient conditions for complete controllability using game-type Ricatti differential equations and recovers the deterministic results under natural assumptions.

As with our own discussion, [15], [16], [17], [19] rely on the $L_2$ metric being a meaningful quantifier for distance between points in the state space. In finite-state systems such as the HMM (1.2), because the indexing of the state space is non-unique the $L_2$ distance between two states becomes dependent on the choice of indexing. Thus, common formulations of stochastic controllability and reachability, which rely on the reduction of the $L_2$ distance between the controlled state and a desired terminal value, are not sensible without possibly vague assumptions that the $L_2$ distance of the state indices being representative of some physical closeness of the states.

Within the literature that specifically refers to stochastic reachability, we point out work by Bujorianu [20] and by Lygeros and collaborators [21], [22], [23]. There are close connections to earlier work by [24], though to the authors’ knowledge, this connection is not acknowledged. In [21], [20], [24], the reachability question for a measurable set $S$ is whether there exist state trajectories which enter the measurable set,

$$\exists \omega \in \Omega \text{ such that for some } t \in [0,T], \ x_t(\omega) \in S$$

where $\Omega$ is the sample space. Bujorianu and Lygeros examine the probability that the hitting time of the set,

$$T_S \triangleq \inf\{t > 0 | x_t \in S\}$$

2In continuous-time, time-invariant, deterministic systems, the two are the same.
is less than a user-specified time, while Zabczyk requires
that the expected value of the hitting time be finite. Our
introduction of a nominal system as a point of compar-
ison refines their formulation slightly by setting apart those
systems which require control action to enter sets $S$ from
systems which can enter any set $S$ via a realization of the
disturbance process.

A. Information Theory

As with the stochastic nonlinear definition of observability
in [1], we borrow tools from information theory to address
the difficulties of the finite-state network model. The relative
entropy [25] provides a quantity which generalizes the $L_2$
distance.

Definition 5: Given two probability mass (density) func-
tions, $p$ and $\hat{p}$, for random variable $X$, the relative
entropy between them is

$$D(p \parallel \hat{p}) = E_p \left[ \log \frac{p(X)}{\hat{p}(X)} \right], \quad (15)$$

where the subscript on $E_p$ denotes that the expectation
is taken using the measure (density) $p$. By convention,
$0 \log(0/0) = 0$, $0 \log(0/q) = 0$ and $0 \log(p/0) = \infty$. More
generally, if probability measure $P$ is absolutely continuous
with respect to probability measure $\hat{P}$, the relative entropy
may be defined in terms of the Radon-Nikodym derivative,

$$D(P \parallel \hat{P}) = E_{\hat{P}} \left[ \log \frac{dP}{d\hat{P}} \right]. \quad (16)$$

Recall that $D(P \parallel \hat{P}) \geq 0$ with equality if and only if $P = \hat{P}$.

For our intents, the relative entropy acts much like a
distance between two probability distributions. As is well
known, the relative entropy is not a true metric as it is
not symmetric and does not obey the triangle inequality,
though we do not require these properties in our analyses.
A further benefit of using the relative entropy is that it
remains invariant to the indexing of the space described by
the probability mass function. That is, let $x_1, x_2 \in \Omega$, $x_1 \sim p_1$
and $x_2 \sim p_2$ and consider a bijection $f : \Omega \rightarrow \Omega$. Define
distributions $f(x_1) \sim p'_1$ and $f(x_2) \sim p'_2$. Then $D(p_1 \parallel p_2) = D(p'_1 \parallel p'_2)$.

B. Controllability

In deterministic linear systems, controllability defines the
existence of a control which can move the system from some
initial state to the origin in the absence of noise, or close to
the origin otherwise. This meaning is not well defined for the
Markov model (1) as the origin is in general not well-defined.
We consider two possible relaxations of the definition.

Rather than demanding that the state be stabilized at the
origin, we may instead simply require that the state be
stabilized at some equilibrium point or distribution which is
reasonable in some sense, e.g. finite. This fits within the
theory of time-invariant Markov chains, since it is usual to
define the stability of such systems as the irreducibility of
the probability transition matrix $A$ in (1) – with $u_k$ fixed –
or, equivalently, the ergodicity of the process. Complications
arise in attempting to extend these ideas to time-varying
systems, that is for general $u_k$ within the class of admissible
controls. Addressing ergodicity then requires some concept of
stationarity of $u_k$; on the other hand, since we allow $u_k = g_k(x_k, \{y_k\}_{k=0}^\infty; r_k, \Pi_0)$ to be a function of an exogenous random
signal $r_k$, which need not be stationary, the requirement of
stationarity does not fit well with the class of controllers we
would like to examine.

Thus, we weaken the concept of controllability further; we
only require that either that there is an admissible control for
which the controlled system differs from the nominal system
or that the nominal system forgets its initial condition. In
the context of the network example, the estimator generated
by the HMM filter is the conditional state probability law
$\hat{\Pi}_{N|N}(\cdot) \equiv P(x_N \in \cdot | \{y_k\}_{k=0}^\infty)$. It is of interest to ask whether
$\hat{\Pi}_{N|N}$ can be controlled, as the ability to control the con-
tional state distribution entails the ability to control its
tropy, which determines the information content of the
output signal [25], [1]. In the former case, if $\hat{\Pi}_{N|N}$ of the
controlled system differs from that of the nominal system,
then control provides a means to change the entropy of the
con$^3$, while in the latter, the HMM filter is stable [26],
that is, the estimator eventually forgets mismatches between
the user-specified distribution for $x_0$ and the true prior
distribution; this coincides with a version of observability
due to van Handel [26].

To address the general definition of controllability, con-
ider a nonlinear system of the form

$$x_{k+1} = f(x_k, u_k, w_k), \quad x_0 \sim \Pi_0, \quad (17)$$

where $w_k$ is an independent, identically distributed noise
sequence, $x_0$ is independent of $w_k$ for $k = 0, 1, \ldots$, and $u_k$
is an admissible control.

For notational clarity, given (17) and a measurable func-
tion $\ell(x_k)$, denote by $P_t$ the probability law for $\ell(x_k)$ subject
to an admissible control sequence $\{u_k\}_{k=0}^{t-1}$. Denote by $\ell_{k:0}$
the state $x_k$ subject to a fixed control sequence denoted as
$u_t \equiv 0, t = 1, 2, \ldots, k-1$ and let $P_t^{0}$ be its probability law.

Definition 6: Call a measurable function $\ell(x_k)$ (nonlinear-
stochastic) controllable if there exists an admissible control
sequence $\{u_k\}_{k=0}^{t-1}$ such that

$$D(P_t \parallel P_t^{0}) > 0, \quad (18)$$

or $\ell(x_{k:0})$ and $x_0$ are independent. Call (17) completely
(nonlinear-stochastic) controllable if every measurable func-
tion $\ell$ is controllable.

Note that $\hat{\Pi}_{N|N}$ in the HMM example is a function of both
the sequence of states $\{x_k\}_{k=0}^\infty$ and randomness in the
output process; nevertheless, the concept of controllability may be
applied to $\hat{\Pi}_{N|N}$ in a parallel fashion to that in Definition 6
for $\ell$.

The next result demonstrates the connection between Def-
initions 2 and 6.

$^3$It is possible that the control only decreases the informativeness of the
outputs, though this is easy to identify numerically.
Theorem 3: If the system (3) is completely linear-stochastic-controllable for every \( \bar{x}_0, \Sigma_0, \) and \( \{Q_k\}_{0}^{n-1} \), then it is completely nonlinear-stochastic-controllable.

Proof: Due to Theorem 1,

\[
\text{range}(A^n) \subset \text{range}(C_{n-1}),
\]

where, as before, \( C_{n-1} \) denotes the controllability matrix. Since \( D(P_t \parallel P^n_t) = 0 \) if and only if \( P_t = P^n_t \), it suffices to show that if (19) holds, then there is some Borel set \( S \) in the range of \( \ell \) such that \( P(\ell(x_n) \in S) \neq P(\ell(x_0) \in S) \) for an appropriately chosen control signal.

For the contradiction argument, suppose that \( \ell(x_n) \) and \( x_0 \) are not independent and \( P(\ell(x_n) \in S) = P(\ell(x_0) \in S) \) given any Borel set \( S \) in the range of \( \ell \) and control input. Let \( S^- \triangleq \{x_n: \ell(x_n) \in S\} \). Adopting the notation of (4), we have

\[
P(\ell(x_n) \in S) = P(A^n x_0 + C_{n-1} \mathcal{U}_{n-1} + \mathcal{W}_{n-1} \in S^-),
\]

\[
= P(A^n x_0 + C_{n-1} \mathcal{W}_{n-1} \in s + \mathcal{W}_{n-1} \in S^-),
\]

\[
P(x_0 \in S'),
\]

where \( S' \triangleq \{s: s + C_{n-1} \mathcal{W}_{n-1} \in S^-\} \). Let \( \Xi \) form a basis for \( \mathbb{R}^n \). Then

\[
P(x_0 \in S') = P\left( \left\{ x_0 : \sum_{\xi \in \Xi} \xi^T x_0 = \xi^T s, s \in S' \right\} \right).
\]

We suggest without proof\(^4\) that

\[
P(x_0 \in S') = P(x_0 \in S') \Rightarrow S' = S^- \text{ a.s.}
\]

If \( S' = S^- \) almost surely, then for each \( \xi \in \Xi, P(x_0 \in S') = P(x_0 \in S') \) implies either \( \xi^T \mathcal{W}_{n-1} = 0 \) or \( \{s : s \in S'\} = (-\infty, \infty) \) almost everywhere. For \( \xi \in \Xi \), \( \xi^T A^n = 0 \) by (19), so

\[
\xi^T x_0 = \xi^T C_{n-1} \mathcal{W}_{n-1} = 0.
\]

For \( \xi \) in the latter case,

\[
\left\{ x_0 : \xi^T x_0 = \xi^T s, s \in S' \right\} = (-\infty, \infty), \text{ a.e.}
\]

Since this holds for any Borel set \( S \) in the range of \( \ell \), (20-21) imply that \( \ell(x_0) \) and \( x_0 \) are independent, which is a contradiction.

One may recover the linear definition from the nonlinear version by considering the class of functions \( \ell(x_k) = \xi^T T x_k \). This also demonstrates the significance of (21).

In [1], a function \( \varphi \) of the state \( x_k \) was said to be unobservable from an output sequence \( \{y_k\}^N_0 \) for \( k, N > 0 \) if \( \varphi(x_k) \) and \( \{y_k\}^N_0 \) were independent. With Definition 6, we may draw the connection to controllability.

Theorem 4: A scalar measurable function \( \varphi \) of the state \( x_k \) is unobservable from \( \{y_k\}^N_0 \) for input sequence \( \{u_k\}^N_0 \) and any \( k, N > 0 \) if and only if \( D(\Pi^{\varphi}_{k-1} \parallel \Pi^{\varphi}_{k|N}) = 0 \) almost surely for any \( k \geq 0 \). Thus, if the estimator for \( \varphi \) with distribution \( \Pi^{\varphi}_{k|N} \) is controllable, then \( \varphi(x_k) \) is observable.

\(^4\)The additional steps do not contribute much to the discussion.

Proof: First, let \( D(\Pi^{\varphi}_{k-1} \parallel \Pi^{\varphi}_{k|N}) = 0 \) almost surely\(^5\). We have \( \Pi^{\varphi}_{k-1} = \Pi^{\varphi}_{k|N} \) almost surely. Thus, the conditional probability

\[
P(\varphi(x_k) \in \cdot | \{y_k\}^N_0) = P(\varphi(x_k) \in \cdot)
\]

almost surely, which suffices to show independence.

On the other hand, if \( \varphi(x_k) \) is unobservable from \( \{y_k\}^N_0 \), then \( \varphi(x_k) \) and \( \{y_k\}^N_0 \) are independent. Thus,

\[
P(\varphi(x_k) \in \cdot | \{y_k\}^N_0) = P(\varphi(x_k) \in \cdot)
\]

almost surely, which implies

\[
D(\Pi^{\varphi}_{k-1} \parallel \Pi^{\varphi}_{k|N}) = 0 \text{ a.s.}
\]

Connections to network congestion control: The network congestion control problem lends context to Theorem 4. The distinction between controllability of the estimator and controllability of the state (being estimated) is essential. For instance, suppose the router capacity \( c_k \) is governed by a Markov process with an identity transition probability matrix, \( c_k \sim \text{Markov}(\Pi_c, P_c) \), where \( P_c = I \) and no element of \( \Pi_c \) is equal to one, and let the range of \( c_1 \) be contained in the range of the source send rate \( r_k \) (i.e. the source can saturate the router capacity). The capacity \( c_1 \) is uncontrollable; it does not forget its initial condition and no control law changes the distribution of \( c_k \) for any \( k \). On the other hand, the estimator of \( c_k \) defined by the random variable \( \hat{c}_{k|N} \sim \mathcal{N}(c_k, \{y_k\}^N_0) \) is controllable so long as the source computer saturates the router capacity, \( r_k > c_k \). As a result, by Theorem 4, \( c_k \) is observable.

One may also ask under what circumstance the estimator becomes uncontrollable. One circumstance is if the admissible set of values for \( r_k \) is smaller than the range of \( c_k \). E.g. suppose \( r_k \equiv 1 \) and let \( c_k \sim \text{Markov}(\Pi_c, P_c) \).

\[
\Pi_c = \begin{bmatrix}
\frac{1}{2} & \frac{1}{2} \\
\frac{1}{2} & \frac{1}{2}
\end{bmatrix},
\]

\[
P_c = \begin{bmatrix}
0 & 0 & 0 \\
0 & 0 & 1 \\
0 & 1 & 0
\end{bmatrix},
\]

with range \( (c_k) = \{1, 2, 3\} \). Since the minimal network capacity exceeds the maximal admissible send rate, the source computer has no control over the estimator of \( c_k \), or equivalently, is unable to reduce the uncertainty in the estimate of \( c_k \). This intuition forms the basis of current implementations of network congestion control laws, which intentionally increase \( r_k \) to the point of saturating \( c_k \) in order to determine the available network capacity.

C. Reachability

Our extension of reachability to nonlinear stochastic systems follows a similar approach. Let \( \ell_d \) be a point in the range of a measurable function \( \ell \) and denote a sequence of measures \( \{P_{d,m}\}_{m=1}^\infty \) with mean \( \ell_d \) which converge to \( P_{d}(\cdot) = 1_{\ell_d}(\cdot) \). The terms \( P_t, P^n_t, \) and \( x_{d,t} \) remain as previously defined. In the following analysis, it will be convenient to have that \( P_{t} \) is absolutely continuous with respect to \( P_{d,m} \) and

\(^5\)Almost surely with respect to the probability law of \( \{y_k\}^N_0 \)
\(P_{d,m}\) is absolutely continuous with respect to \(P^0_\ell\) for every finite \(m\) (i.e. \(P_\ell \ll P_{d,m} \ll P^0_\ell\)); we assume this whenever a sequence \(\{P_{d,m}\}_{m=1}^\infty\) is specified.

**Definition 7:** Call the point \(\ell_d\) in the range of a measurable function \(\ell(x_k)\) (nonlinear-stochastic) reachable if there is an admissible control sequence \(\{u_t\}_{t=0}^{k-1}\) such that for some sequence \(\{P_{d,m}\}_{m=1}^\infty\) with first moment \(\ell_d\) and which converges to \(P_d(\cdot) = 1_{\ell_d}(\cdot)\), and for every \(m > M\) where \(M < \infty\),

\[
D(P_{d,m} \mid P_\ell) < D(P_{d,m} \parallel P^0_\ell) \quad (23)
\]
or

\[
\ell(x_k,0) = \ell_d \text{ almost surely.} \quad (24)
\]

Our consideration of measurable functions has connections to the examination of reachability of sets in [20], [21], [22], [23], [24], as the indicator functions of sets are the building blocks of measurable functions.

It is unknown whether a connection like Theorem 3 exists between the linear and nonlinear definitions of stochastic reachability.

**Conclusion**

We have used the analysis of a network congestion control approach to motivate the development of stochastic controllability and reachability concepts in such a fashion as to preserve their connection to underlying deterministic system properties and to extend their applicability to stochastic systems, such as the HMM network model, where linearity and the concept of a state origin are absent. Distributional properties achievable via the control signal are the basis for the analysis, which leads rather naturally to information theoretic distance-like measures to characterize reachability. It also leads us, as in [1] to treat explicitly certain pathological situations in the definitions. The focus of the analysis lies in understanding how to characterize the capacity of a control law to preserve stochastic state observability.

**References**


