Formation Control of Directed Multi-Agent Networks based on Complex Laplacian

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Abstract—Real graph Laplacians are of great importance in consensus of multi-agent systems. This paper introduces complex graph Laplacians as a new tool to study the formation control problem in the plane. It is shown that complex graph Laplacians are of equally great importance for planar formation control like real Laplacians for consensus. First, complex graph Laplacians are used to characterize planar formations under given topology of networked agents. Second, complex graph Laplacians are used to derive local and distributed control strategies for asymptotically achieving formations. This paper explores the relations between graph topology, complex Laplacians, and planar formations, and obtains several graphical and algebraic conditions for realizability of spatial formations. Both simulation and experiment results are provided to illustrate our results.

I. INTRODUCTION

In recent years, multi-agent systems (MAS) have attracted much research attention from various disciplines of engineering and science due to their interdisciplinary nature and broad potential applications. Collective behaviors arise from simple local interaction rules to permit sophisticated cooperation of the group that would never be achieved by individual members. Current research on MAS aims to provide deep understanding of group coordination and cooperation and inspire the design of distributed autonomous multi-agent systems.

Graph Laplacian plays a very important role in coordination and cooperation of multi-agent systems, in particular, for the consensus problem. Local interaction laws based on real graph Laplacian for networked multiple agents would lead to a collective behavior of consensus regarding the state of interest. It is shown in [3], [9], [10], [15] and [16] that consensus emerges if and only if the real graph Laplacian has a simple eigenvalue, or equivalently the control graph is rooted (having a root from which every node is reachable). Thus, the kernel (null space) of the real graph Laplacian is exactly the consensus subspace.

However, complex graph Laplacians corresponding to graphs with edges attributed with complex weights have not come into notice in the study of multi-agent systems. This paper introduces complex graph Laplacian as a new tool to study the formation control problem of spatial multi-agent systems in the plane. It is shown that complex graph Laplacians are equally important in coordination and cooperation of multi-agent systems, particularly for planar formations.

First, we show that a complex graph Laplacian can be used to characterize a spatial formation subject to four degrees of freedom (rigid-body translation, rotation and scaling). That is, its kernel defines a spatial formation of four degrees of freedom if and only if it has zero eigenvalues with both algebraic and geometric multiplicity two. We also provide a graphical connectivity condition for the feasibility of such characterization for planar formations subject to rigid-body translation, rotation and scaling. Second, we show that a planar formation can be achieved via a lightweight local and distributed control law based on the complex graph Laplacian that defines the planar formations. But unlike real graph Laplacians, complex graph Laplacians distribute their eigenvalues everywhere in the complex plane and thus may lead to instability of the system with respect to its equilibrium formation. To tackle this challenge, we show that an invertible complex diagonal matrix exists under certain conditions to re-assign the eigenvalues of the complex graph Laplacian while this procedure does not change the control structure and also keeps the good property of distributed and local implementability of the control strategy. This problem is related to the well-known multiplicative inverse eigenvalue problem [6] and we provide a simple algorithm to find such an invertible complex diagonal matrix which we call a stabilizing matrix. Moreover, we show that with two virtual leaders, the agents globally asymptotically converge to a specific formation with its location, orientation and scale determined by the virtual leaders.

The work is supplementary to the current study of formation control and provides a new approach based on complex graph Laplacians. Compared to the formation control strategies based on real graph Laplacians together with input bias [5], [8], [11], [14] which require a common sense of direction, the proposed approach based on complex graph Laplacians for formation control eliminates the requirement of global knowledge of a common direction. Also, compared with the ideas of using rigidity and gradient descent control laws for pattern formation [2], [7], [13], [17], though they all use local knowledge of relative information, globally asymptotical stability is not able to be ensured, while our proposed approach based on complex graph Laplacians ensures globally asymptotical stability, requires less links to define a formation, and can simply handle directed multi-agent networks.

Notations: \( \mathbb{C} \) and \( \mathbb{R} \) denote the set of complex and real numbers, respectively. \( \iota = \sqrt{-1} \) denotes the imaginary unit. \( 1_n \) represents the \( n \)-dimensional vector of ones and \( I_n \) denotes the identity matrix of order \( n \).

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II. PRELIMINARIES AND PROBLEM SETUP

A. Graph theory

A directed graph \( G = (V, E) \) consists of a non-empty node set \( V = \{1, 2, \cdots, n\} \) and an edge set \( E \subseteq V \times V \). An edge of \( G \) is denoted by a pair of nodes \((j, i)\). A walk \( p \) in a graph \( G \) is an alternating sequence

\[
p : v_1v_2v_3\cdots v_k
\]

of nodes \( v_i \) and edges \( e_i \) such that \( e_i = (v_i, v_{i+1}) \) for every \( i = 1, 2, \cdots, k - 1 \). If the nodes of a walk \( p \) are distinct, \( p \) is then called a path, in which \( v_1 \) and \( v_k \) are called terminal nodes and other nodes are called internal nodes. In the following, we let \( N_i \) denote the neighbor set of node \( i \), i.e., \( N_i = \{j : (j, i) \in E\} \).

Next, we introduce two notions that are important to the development of our results.

**Definition 2.1:** For a directed graph \( G \), a node \( v \) is said to be 2-reachable from a non-singleton subset of nodes \( \{u_1, \ldots, u_k\} \) if there exists a path from a node in \( \{u_1, \ldots, u_k\} \) to \( v \) after removing any one node except \( v \).

**Definition 2.2:** A directed graph \( G \) is said to be 2-rooted if there exists a subset of two nodes, from which every other node is 2-reachable. These two nodes are called roots of the graph.

Finally, we introduce the notion of complex Laplacian for a directed graph. A complex Laplacian \( L \) of a directed graph \( G \) is defined as follows:

\[
L(i, j) = \begin{cases} 
-w_{ij} & \text{if } i \neq j \text{ and } j \in N_i, \\
0 & \text{if } i \neq j \text{ and } j \notin N_i, \\
\sum_{j \in N_i} w_{ij} & \text{if } i = j,
\end{cases}
\]

where \( w_{ij} \in \mathbb{C} \) is a complex weight attributed on edge \((j, i)\).

B. Problem setup

We consider a group of \( n \) agents in the plane. The positions of \( n \) agents are denoted by complex numbers \( z_1, \ldots, z_n \in \mathbb{C} \). We use a directed graph \( G \) of \( n \) nodes to represent the sensing graph in which an edge \((j, i)\) indicates that agent \( i \) can measure the relative position between agent \( i \) and agent \( j \).

Each agent \( i \) is of single-integrator kinematics and takes a local information based linear control strategy

\[
\dot{z}_i = \sum_{j \in N_i} w_{ij}(z_j - z_i), \quad i = 1, \ldots, n, \tag{1}
\]

where \( w_{ij} \) is a complex weight to be designed attributed on edge \((j, i)\).

Denote the aggregate vector \( z = [z_1, z_2, \ldots, z_n]^T \in \mathbb{C}^n \). Then the overall dynamics of the \( n \) agents under the local control strategy (1) is

\[
\dot{z} = -Lz, \tag{2}
\]

where \( L \) is a complex Laplacian of \( G \).

In the paper we will study the following problems.

**P1:** Under what graphical and/or algebraic conditions of \( G \), does there exist a local control of form (1) such that the \( n \) agents can achieve a planar formation?

**P2:** For a given graph \( G \), how is a local control of form (1) designed such that the \( n \) agents asymptotically achieve a planar formation?

The local control law (1) looks identical to the consensus algorithm when the weights are real. However, as we will show later, it requires different connectedness for the purpose of formation control and it is more challenging in designing proper complex weights to ensure asymptotic stability.

III. MAIN RESULT

In this section, we first present a new characterization for planar formations in terms of complex Laplacian of directed graphs. Next, we analyze the existence of a local distributed control law for formations and provide an approach for the design. Finally, we derive a special result for formations with leaders.

A. Characterization of planar formations

We begin with a directed graph \( G = (V, E) \) with its node set \( V = \{1, 2, \ldots, n\} \). Along with this graph we bring in \( n \) points in the plane represented by complex numbers

\[
\xi_1, \xi_2, \ldots, \xi_n \in \mathbb{C}
\]

and we denote the ordered \( n \)-tuple of these points by \( \xi = [\xi_1, \xi_2, \cdots, \xi_n]^T \in \mathbb{C}^n \), a formation basis in the plane. Throughout the paper, it is assumed that

\[
A1: \quad \xi_i \neq \xi_j \quad \text{for any } i \neq j
\]

meaning that no two points overlap each other. Then \( G(\xi) \) denotes a graph-like structure in the plane, which has nodes at the points \( \xi \) and a line with an arrow pointing from \( \xi_j \) to \( \xi_i \) just when \((j, i) \in E\). Such a structure is called a framework. It is a realization of the graph at certain points in the plane. We shall not fix the reference coordinate frame and the scale of the formation. Thus the formation has four degrees of freedom subject to rigid-body rotation, translation, and scaling, and it can be defined as

\[
F_\xi = \{c_1\mathbf{1}_n + c_2\xi, \quad \text{where } c_1, c_2 \in \mathbb{C}\}.
\]

All the formations of the same framework \( G(\xi) \) have the same shape.

Next we present a characterization for planar formations of a framework \( G(\xi) \) in terms of associated complex Laplacian \( L \) of the directed graph \( G \). It says when certain algebraic conditions of the directed graph hold, all the formations of the framework \( G(\xi) \) are exactly corresponding to the kernel of \( L \), which represents a kind of constraints of the \( n \) points in the plane.

**Theorem 3.1:** For a framework \( G(\xi) \) satisfying A1, \( F_\xi = \ker(L) \) if and only if \( L\xi = 0 \) and \( \text{rank}(L) = n - 2 \).

**Proof:** (Sufficiency) Because \( L\xi = 0 \), \( L \) has zero eigenvalue with an associated eigenvector \( \xi \). Moreover, according to the definition of Laplacian, \( L\mathbf{1}_n = 0 \) where \( \mathbf{1}_n \) is a vector of all 1 entries, so \( \mathbf{1}_n \) is another eigenvector
corresponding to the zero eigenvalue. The two eigenvectors $\xi$ and $1_n$ are linearly independent because $\xi_i \neq \xi_j$ by assumption A1. Moreover, since $\text{rank}(L) = n - 2$, it is known that $L$ has only two zero eigenvalues. Thus the kernel of $L$ is
\[ \{ c_1 1_n + cz : c_1, c_2 \in \mathbb{C} \}. \]

(Necessity) Suppose on the contrary that $L\xi \neq 0$. Then $\xi \in F_\xi$ does not belong to $\ker(L)$, a contradiction. On the other hand, suppose $L\xi = 0$ but $rank(L) \neq n - 2$. This case means $rank(L) < n - 2$. Then it follows that the kernel of $L$ is of at least 3-dimensional. Hence there must be a vector $\eta$ in $\ker(L)$ but not in $F_\xi$.

In order that a complex Laplacian $L$ can be used to characterize planar formations $F_\xi$, $\xi$ has to be in the kernel of $L$. Next, we present a necessary topological condition so that $\text{rank}(L) = n - 2$.

**Theorem 3.2:** For a framework $\mathcal{G}(\xi)$ satisfying A1, if $\text{rank}(L) = n - 2$ for $L$ satisfying $L\xi = 0$, then $\mathcal{G}$ is 2-rooted.

**Proof:** We prove it in a contrapositive form. Suppose that $\mathcal{G}$ is not 2-rooted. Since $L\xi = 0$ and $L1_n = 0$, there must be two rows of $L$, say the $q$th and $p$th rows, can be transformed to zero vectors by elementary row transformation. Choose the two nodes $p$ and $q$ corresponding to the two rows as roots. As $\mathcal{G}$ is not 2-rooted, some nodes not in $\{p, q\}$ are not reachable from $\{p, q\}$ after removing a node. Without loss of generality, suppose after removing node $k$, there exist a subset $\mathcal{W}$ of nodes that are not reachable from any root and a subset $\mathcal{W}'$ of nodes that are reachable from one of the two roots. It is certain that the two roots are contained in $\mathcal{W}$ and the nodes in $\mathcal{W}'$ are not reachable from any node in $\mathcal{W}$ after removing node $k$. Suppose the total number of nodes in $\mathcal{W}'$ is $m$. If necessary, relabel the nodes in $\mathcal{W}'$ as $1, \cdots, m$, change the label of node $k$ to $m + 1$, and relabel the nodes in $\mathcal{W}$ as $m + 2, \cdots, n$. Then the Laplacian $L$ after relabeling satisfies $L(i, j) = 0$ for $i \in \mathcal{W}$ and $j \in \mathcal{W}'$. That is, $L$ is of the following form

\[ \begin{bmatrix} L_w & l & 0 \\ * & * & * \end{bmatrix} \]

where $L_w \in \mathbb{C}^{m \times m}$ and $l \in \mathbb{C}^m$. Denote the formation basis $\xi$ after relabeling by $[\xi_\alpha^T, \xi_b^T]^T$ where $\xi_\alpha \in \mathbb{C}^{m+1}$ and $\xi_b \in \mathbb{C}^{(n-m-1)}$. According to the definition of $L$, then we have

\[ [L_w \ l] 1_{m+1} = 0 \quad \text{and} \quad [L_w \ l] \xi_\alpha = 0. \]

As $1_{m+1}$ and $\xi_\alpha$ are linearly independent by assumption, then there exists a row in $[L_w \ l]$ which can be transformed into zero vector with elementary row transformation. Recall that another two rows with the indices corresponding the two roots can be transformed into zero vectors. Therefore, $\text{rank}(L) \leq n - 3$.

**Remark 3.1:** When the graph becomes undirected, a rigid graph is also 2-rooted. It means that our approach requires less edges in achieving a formation compared with the work using rigidity theory. An in-depth discussion refers to [12].

**B. Local information based distributed formation control**

From Theorem 3.1, we know that in order to achieve a desired formation $F_\xi$ using the sensing graph $\mathcal{G}$ and the local control law (2), it has to be held that $\text{rank}(L) = n - 2$ and $L\xi = 0$ so that every formation in $F_\xi$ is an equilibrium state of (2). Due to Theorem 3.2, it is known that the sensing graph $\mathcal{G}$ has to be 2-rooted as otherwise there is no solution for the problem. Therefore, under the assumption that the graph $\mathcal{G}$ is 2-rooted, each agent $i$ can select the complex weights on edges $(j, i)$, where $j \in N_i$, such that

\[ \sum_{j \in N_i} w_{ij}(\xi_j - \xi_i) = 0. \] (3)

Then, $L\xi = 0$ holds. Moreover, generically $\text{rank}(L) = n - 2$.

However, it is not certain that the complex Laplacian $L$ for arbitrarily chosen complex weights has all eigenvalues of non-negative real parts. If $-L$ for arbitrarily chosen complex weights is not stable, then does there exist an $L$ such that the system (2) is asymptotically stable with respect to the formations? The problem is addressed below.

Firstly, notice that when pre-multiplying an invertible diagonal matrix $D = \text{diag}\{d_1, \cdots, d_n\}$ to $L$, the overall system becomes

\[ \dot{z} = -DLz \]

and the equilibrium states are preserved since the pre-multiplication of $D$ does not change the kernel.

Secondly, when pre-multiplying an invertible diagonal matrix $D$ to $L$, the complex weights on edges having heads at agent $i$ are multiplied by a nonzero complex number $d_i$. Therefore, the interaction rule is still locally implementable using only relative position information and is distributed.

Consequently, if $-L$ is unstable, then the design problem becomes the discovery of $D$ such that $-DL$ is stable. We show in the next result the existence of an invertible diagonal matrix $D$ such that $DL$ has all other eigenvalues with positive real parts in addition to two zero eigenvalues and thus $\dot{z} = -DLz$ is asymptotically stable with respect to $\ker(L)$. Such a matrix $D$ is called a stabilizing matrix.

**Theorem 3.3:** If there is a permutation for $L$ such that the leading principal minors up to the $(n - 2)$th order are nonzero, then a stabilizing matrix $D$ for system $\dot{z} = -Lz$ exists.

The proof requires the following result related to the multiplicative inverse eigenvalue problem.

**Theorem 3.4 ([11]):** Let $A$ be an $n \times n$ complex matrix all of whose leading principal minors are nonzero. Then there is an $n \times n$ complex diagonal matrix $D$ such that all the roots of $DA$ are positive and simple.

**Proof of Theorem 3.3:** By the condition in the theorem, there is a permutation matrix $P$ (equivalent to relabel the nodes) such that

\[ LP = PLP^T = \begin{bmatrix} B_1 & B_2 \\ B_3 & B_4 \end{bmatrix} \]

in which the leading principal minors of $B_1 \in \mathbb{C}^{(n-2) \times (n-2)}$ are nonzero, and $B_2$, $B_3$ and $B_4$ are of appropriate dimension. Denote $D = \text{diag}\{d_1, \cdots, d_n\}$, $D_1 = \text{diag}\{d_1, \cdots, d_{n-2}\}$ and $D_2 = \text{diag}\{d_{n-1}, d_n\}$. Write $DL_P$ as $M(d_{n-1}, d_n)$ to emphasize that it is a function of variables
Clearly
\[
M(0, 0) = \begin{bmatrix}
D_1B_1 & \cdots & D_1B_4 \\
D_2B_5 & \cdots & D_2B_8
\end{bmatrix} = \begin{bmatrix}
D_1B_1 & D_1B_2 \\
0 & 0
\end{bmatrix}.
\]
So \(\lambda I - M(0, 0) = \lambda^2|I - D_1B_1|\) and the nonzero eigenvalues of \(DL_P\) are just those of \(D_1B_1\). According to Theorem 3.4, there exists a matrix \(D_1\) such that all the eigenvalues of \(D_1B_1\) can be made in the open right half complex plane. Moreover, since the eigenvalues of \(M(d_{n-1}, d_n)\) are continuous functions of \(d_{n-1}\) and \(d_n\), it follows that for all \(d_{n-1}\) and \(d_n\) in the neighborhood of the origin, \(n - 2\) eigenvalues of \(M(d_{n-1}, d_n)\) are still in the open right half complex plane. On the other hand, recall that rank\((L) = n - 2\), so is \(L_P\) and \(DL_P\). That means, an invertible complex diagonal matrix \(D\) is obtained such that the system \(\dot{z} = -DLz\) has two zero eigenvalues and all the others have positive real parts.

With the existence of a stabilizing matrix ensured by Theorem 3.3, we are now ready to present a numerical approach to find a stabilizing matrix \(D = \text{diag}[d_1, \cdots, d_n]\). Denote \(L_{(1\sim i)}\) the sub-matrix formed by the first \(i\) rows and columns of \(L\). If the zero eigenvalues of \(L_{(1\sim i)}\) can not be reassigned by pre-multiplying any diagonal matrix \(\text{diag}(d_1, \cdots, d_i)\), we call them fixed zero modes.

\[\text{Algorithm 3.1:}\]

\begin{algorithmic}
\For {\text{for } i = 1, \ldots, n} 
\State find \(d_i\) to assign the eigenvalues of \(\text{diag}(d_1, \cdots, d_i)L_{(1\sim i)}\) in the open right half complex plane except for the fixed zero modes.
\EndFor
\end{algorithmic}

Next we give some interpretation on why \(D\) can be found from Algorithm 3.1. Without loss of generality, assume that the nodes are re-labeled so that the leading principal minors of \(L\) from the first order to the \((n - 2)\)th order are nonzero. First, \(d_1\) can be easily found such that \(d_1L_{(1)}\) lies in the open right half complex plane. Next we suppose that \(\text{diag}(d_1, \cdots, d_i)L_{(1\sim i)}\) for \(2 \leq i < n - 2\), has all eigenvalues lying in the open right half complex plane and then show that we are able to find \(d_{i+1}\) such that \(\text{diag}(d_1, \cdots, d_{i+1})L_{(1\sim i+1)}\) has all eigenvalues lying in the open right half complex plane. Denote
\[N(d_{i+1}) = \text{diag}(d_1, \cdots, d_{i+1})L_{(1\sim i+1)}.
\]
Note that \(N(0) = \text{diag}(d_1, \cdots, d_0)L_{(1\sim i+1)}\) has one zero-eigenvalue and \(i\) eigenvalues which are exactly the same as the eigenvalues of \(\text{diag}(d_1, \cdots, d_i)L_{(1\sim i)}\) and are in the open right half complex plane. Then the continuity of eigenvalues with respect to \(d_{i+1}\), it follows that there must be \(d_{i+1}\) such that \(N(d_{i+1})\) has all eigenvalues lying in the open right half complex plane.

In conclusion, the procedure is quite simple for the design of local control to solve the formation control problem. The premise is that the sensing graph \(G\) is 2-rooted.

**Step 1:** Each agent arbitrarily chooses \(w_{ij}\) for \(j \in N_i\) to satisfy (3).

**Step 2:** Verify whether rank\((L) = n - 2\). If no, go back to Step 1.

**Step 3:** Verify whether there is a permutation for \(L\) such that the leading principal minors up the \((n - 2)\)th order are nonzero. If no, go back to Step 1.

**Step 4:** Run Algorithm 3.1 to find \(d_i\),

Then the local and distributed control law
\[
\dot{z}_i = d_i \sum_{j \in N_i} w_{ij}(z_j - z_i), \quad i = 1, \ldots, n
\]
can make the \(n\) agents achieve a planar formation \(F_\xi\) subject to rigid-body translation, rotation, and scaling.

**C. Formations with leaders**

In the preceding subsection, we have presented an approach for the design of a local and distributed control law to achieve planar formations. However, due to the leaderless coordination mechanism, the attained formation is subject to rigid-body translation, rotation, and scaling, depending on the initial states of the \(n\) agents. If instead there are two (or more than two) leaders or virtual leaders, then the shape, location, and orientation of the final formation will be uniquely determined.

We consider a group of \(n - 2\) agents together with two virtual leaders labeled \(n - 1\) and \(n\), which can be landmarks or beacons at designated locations. Suppose the two virtual leaders never access the relative position information of others, which means, these two nodes in the sensing graph \(G\) do not have any incoming edge from others. Again, assume the sensing graph \(G\) is 2-rooted. Then in this setup, the two roots must be the two virtual leaders and thus the Laplacian is of the following form
\[
L = \begin{bmatrix}
L_f & L_l \\
0_{2 \times n} & 0_{2 \times 2}
\end{bmatrix}.
\]
Again, each agent \(i\) takes the local control
\[
\dot{z}_i = \sum_{j \in N_i} w_{ij}(z_j - z_i), \quad i = 1, \ldots, n - 2
\]
with \(w_{ij}\) being chosen to satisfy (3) for a given formation basis \(\xi\).

Thus from Theorem 3.3, we reach the following corollary.

**Corollary 3.1:** If all leading principal minors of \(L_f\) are nonzero, then there exists \(d_i \in \mathbb{C}\) such that under the local control law
\[
\dot{z}_i = d_i \sum_{j \in N_i} w_{ij}(z_j - z_i), \quad i = 1, \ldots, n - 2,
\]
the \(n - 2\) agents (together with two virtual leaders) asymptotically achieve the target formation \(c_1 \mathbf{1}_n + c_2 \xi\) with \(c_1\) and \(c_2\) uniquely determined by the two virtual leaders.

**IV. SIMULATIONS AND EXPERIMENTS**

In this section, we present both simulation and experiment results to illustrate the success of our proposed schemes.
A. Simulation

In the simulation, we consider a group of 6 agents without leaders. The framework \( G(\xi) \) is depicted in Fig. 1, for which \( \xi = [2i, -1 + i, -1 - i, -2i, 1 - i, 1 + i]^T \) where \( i \) is the imaginary unit.

![Fig. 1. A framework \( G(\xi) \) of 6 agents.](image)

Firstly, it can be checked that the graph \( G \) is 2-rooted. For instance, \( \{1, 4\} \) can be chosen as the subset of two roots and every other node is 2-reachable from \( \{1, 4\} \). Then arbitrary complex weights can be selected to satisfy \( L\xi = 0 \), for example,

\[
L = \begin{bmatrix}
-2 & 0 & 1 - 3i & 0 & 1 + 3i & 0 \\
-2 & 3 - i & -1 + i & 0 & 0 & 0 \\
0 & -1 & 2 - i & -1 + i & 0 & 0 \\
0 & 0 & -1 & 1 - i & i & 0 \\
0 & 0 & 0 & -2 & 3 - i & -1 + i \\
-2 & 0 & 0 & 0 & -1 - i & 3 + i \\
\end{bmatrix}.
\]

It is evaluated that rank(\(L\)) = 4 which verifies Theorem 3.1. By checking the eigenvalues of \( L \), it is obtained that the eigenvalues of \( L \) are \(-1.1329 - 0.1665i, 4.2119 + 0.0022i, 3.9210 - 1.8358i, 3 - i, 0, 0 \). So the local control law (1) with the above complex weights is not able to make the agents asymptotically converge to a formation.

Secondly, by Theorem 3.3 it can be checked that a stabilizing matrix \( D \) exists. Utilizing Algorithm 3.1, a feasible stabilizing matrix can be found, e.g.,

\[
D = \text{diag}\{-1, 1, 1, 1, 1, 1\}.
\]

Thus, the eigenvalues of \( DL \) have positive real parts except two fixed zero eigenvalues. The simulated trajectories of six agents with random initial positions are shown in Fig. 2.

B. Experiment

The proposed algorithm is implemented on a group of mobile robots—Wowwee’s Rovio. The Rovio robot contains a true-track beacon with which it can localize itself based on the NorthStar Localization System [4]. It also has Wi-Fi connectivity (802.11b and 802.11g) so that it can be controlled through wireless communications. The robot is equipped with three omni-directional wheels and thus it can move freely in the plane like a point mass. For our experiment, six Rovio robots are used. We use a central computer to get all the locations of Rovio robots but only utilize the relative position information to control the motion of each robot for the purpose of mimicking distributed and local implementation of the algorithm. The moving direction and moving speed are quantized from the continuous control signal calculated according to our derived control law \( \dot{z} = -DLz \) presented in the simulation. The experimental results (final formation achieved) of six Rovio robots are shown in Fig. 3 while the trajectories of the robots are recorded in Fig. 4 via the NorthStar Localization System. It can be seen from Fig. 3 and 4 that the experimental results provide also very similar desired formation to the simulation one in Fig. 2 though there exist localization errors for the robots in the experiment.

V. CONCLUSIONS AND FUTURE WORK

The paper studies the formation control problem by introducing a new tool—complex Laplacian. With this framework, a directed multi-agent network is addressed. Algebraic
and/or graphical conditions are obtained to show whether a formation can be achieved by a complex Laplacian based control law. Unlike real Laplacians that have all eigenvalues in the closed right half complex plane, complex Laplacians may distribute their eigenvalues in the whole complex plane, which may lead to instability of the system. For this technical challenge, we also show that under certain condition there exists a complex Laplacian ensuring globally asymptotical stability of the system. In addition, we show that the agents can globally asymptotically reach a specific formation with the location, orientation and scale determined by two virtual leaders that could be landmarks or beacons.

There are a broad list of issues and ideas that have not been fully explored yet, but may lead to further progress. For example, how to find optimal complex Laplacian in order to improve certain system performance? How to switch complex Laplacian based control laws for the purpose of collision avoidance and connectivity maintenance?

**ACKNOWLEDGMENT**

The work is supported by National Nature Science Foundation of China (No. 61273113).

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