Rolling motions of pseudo-orthogonal groups

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Abstract— The classical definition of a rolling map, describing the rolling motion, without slip or twist, of one Euclidean submanifold over another of the same dimension, as given in Sharpe [8], is generalized for the situation when the embedded space is equipped with a pseudo-Riemannian metric and applied to derive the kinematic equations for the constrained rolling motion of a connected pseudo-Riemannian orthogonal group over its affine tangent spaces at a point. The kinematic equations are solved explicitly when the curve along which the first manifold rolls is a geodesic. We also show that rolling motions along a curve with non-holonomic constraints of no-twist and no-slip parallel transport, and derive formulas for the tangent and normal parallel transport of a vector along geodesics. Finally, we make a brief reference on how rolling motions can be used to generate smooth interpolating curves on pseudo-orthogonal groups.

I. INTRODUCTION

Nonholonomic systems have attracted much attention in control literature due to their numerous applications in physics and engineering problems. Nonholonomic constraints are usually analyzed from the point of view of sub-Riemannian geometry. If the manifold is only equipped with a pseudo-Riemannian metric (nondegenerate and indefinite), we will be in the presence of problems in sub-pseudo-Riemannian geometry. A system consisting of a pair of n-dimensional pseudo-Riemannian manifolds rolling on each other without slip or twist fits into that category and poses many theoretical challenges and interesting control problems.

Our paper is devoted to studying a particular problem of sub-pseudo-Riemannian type and arises from the kinematic problem of rolling, without slip or twist, a connected pseudo-orthogonal group over its affine tangent space at a point, when they are considered embedded in a pseudo-Riemannian manifold of bigger dimension. Rolling motions are isometries in the embedding space, in particular preserving length of curves, which result from the constrained action of the isometry group of that space. The approach and results presented in this paper were inspired by those in Hüper and Silva Leite [3] for the Euclidean rolling of orthogonal groups. Other works concerning rolling motions of particular Euclidean submanifolds are [5], [2] and [10]. Generalizations of rolling for abstract Riemannian manifolds are studied in [4], and an intrinsic approach that doesn’t require any embedding appears in [1]. The case of Lorentzian rolling spheres, treated in [6], is also a sub-pseudo-Riemannian problem.

In this article, we start with the definition of rolling, an adaptation of the classical definition for rolling Euclidean manifolds given in [8]. The kinematic equations for rolling a connected pseudo-orthogonal group are derived and solved completely when rolling along geodesics. The kinematic equations can be seen as a control system evolving on a subgroup of the isometry group of the embedding space, where the choice of controls corresponds to the choice of rolling curves. It is proven that rolling performs parallel transport of tangent and normal vectors along the curve of contact and explicit formulas for parallel transport along geodesics are derived. To conclude, we mention how rolling maps may be successfully used to generate interpolating curves on manifolds.

II. BASIC NOTIONS OF PSEUDO-RIEMANNIAN MANIFOLDS

For details concerning pseudo-Riemannian manifolds (also called semi-Riemannian manifolds), we refer to O’Neill [7]. Here, we recall some important facts from [7] that will be necessary for further developments.

A pseudo-Riemannian manifold is a smooth manifold \( \mathcal{M} \) furnished with a metric tensor \( \mathring{g} \) (a symmetric nondegenerate \((0, 2)\) tensor field on \( \mathcal{M} \) of constant index). The common value \( \nu \) of the index \( \mathring{g}_p \) is called the index of \( \mathcal{M} \) and \( 0 \leq \nu \leq \dim(\mathcal{M}) \). If \( \nu = 0 \), each \( \mathring{g}_p \) is then a positive definite inner product on \( T_p \mathcal{M} \) and \( \mathcal{M} \) is a Riemannian manifold. If \( \nu = 1 \) and \( \dim(\mathcal{M}) \geq 2 \), \( \mathcal{M} \) is called a Lorentzian manifold. If \( (\mathcal{M}, \mathring{g}) \) is a pseudo-Riemannian manifold and \( V_p \in T_p \mathcal{M} \), then:

- \( V_p \) is spacelike if \( \mathring{g}(V_p, V_p) > 0 \) or \( V_p = 0 \);
- \( V_p \) is timelike if \( \mathring{g}(V_p, V_p) < 0 \);
- \( V_p \) is lightlike if \( \mathring{g}(V_p, V_p) = 0 \) and \( V_p \neq 0 \).

The set \( C(p) \) of all lightlike vectors in \( T_p \mathcal{M} \) form te lightcone of \( \mathcal{M} \) at \( p \).

Let \( M \) be a submanifold of a pseudo-Riemannian manifold \( (\mathcal{M}, \mathring{g}) \) and \( \imath : M \hookrightarrow \mathcal{M} \) the inclusion map. Then \( M \) is a pseudo-Riemannian submanifold of \( \mathcal{M} \) if the pullback metric \( g = \imath^*(\mathring{g}) \) is a metric tensor on \( M \). If \( M \) is equipped with the induced metric \( g \), then \( \imath \) is an isometric embedding. In subsequent sections, we use \( \langle , \rangle \) as an alternative notation for \( g \).

Let \( M \) be a pseudo-Riemannian submanifold of \( \mathcal{M} \) (write \( M \subset \mathcal{M} \)), and \( p \in M \). Each tangent space \( T_p M \) is, by definition, a nondegenerate subspace of \( T_p \mathcal{M} \). Consequently,
$T_p \overline{M}$ decomposes as a direct sum

$$T_p \overline{M} = T_p M \oplus (T_p M)\perp \tag{1}$$

and $(T_p M)\perp$ is also nondegenerate. Vectors in $(T_p M)\perp$ are said to be normal to $M$, while those in $T_p M$ are, of course, tangent to $M$. Similarly, a vector field $Z$ on $\overline{M}$ is normal (respectively tangent) to $M$ provided each value $Z_p$, for $p \in M$ belongs to $(T_p M)\perp$ (respectively $T_p M$). Projecting orthogonally at each $p \in M$ onto the tangent and normal subspaces gives maps called the tangential and normal projections

$$\pi^\top : T\overline{M}|_M \to TM \quad \pi^\perp : T\overline{M}|_M \to (TM)\perp \quad X \mapsto X^\top \quad X \mapsto X^\perp .$$

If $X, Y$ are vector fields on $M$, we can extend them to $\overline{M}$, apply the ambient connection $\nabla$ (the Levi Civita connection with respect to $g$) and then decompose at points of $M$ to get

$$\nabla_X Y = (\nabla_X Y)^\top + (\nabla_X Y)^\perp . \tag{2}$$

It turns out that the following Gauss formula holds for vector fields $X$ and $Y$ (tangent) to $M$:

$$\nabla_X Y = \nabla_{X^\top} Y + (\nabla_{X^\perp} Y)^\perp , \tag{3}$$

where $\nabla$ is the connection with respect to $g$. The normal term is usually also denoted by $\Pi(X, Y)$ and called the the second fundamental form of $M$, in opposition to $\nabla$ which is the first fundamental form on $M$. It measures the difference between the intrinsic connection on $M$ and the ambient (or extrinsic) connection on $\overline{M}$.

If $M \subset \overline{M}$, $t \mapsto \gamma(t)$ is a curve in $M$ and $V$ is a smooth vector field tangent to $M$ along $\gamma$, the Gauss formula (3) along the curve $\gamma$ reduces to

$$\frac{dV}{dt} = \nabla_{\gamma^\top} V + \Pi(\dot{\gamma}, V) , \tag{4}$$

where $\frac{d}{dt}$, $\left(\frac{d}{dt}\right)$, denote extrinsic (intrinsic) covariant derivative along $\gamma$. $\frac{dV}{dt}$ is also called the tangent covariant derivative along $\gamma$, to distinguish from the normal covariant derivative, hereafter denoted by $\frac{\partial}{dt}$. A (tangent) vector field $V$ along $\gamma$ is said to be a tangent parallel vector field along $\gamma$ if $\frac{dV}{dt} = 0$.

Usually, the geometry of $M \subset \overline{M}$ is considered that of vectors tangent to $M$. There is, however, an analogous geometry of vectors normal to $M$.

If $X$ is a tangent vector field to $M$ and $Z$ is a normal vector field to $M$, we have (as in (2))

$$\nabla_X Z = (\nabla_X Z)^\top + (\nabla_X Z)^\perp . \tag{5}$$

The normal connection of $M \subset \overline{M}$ is the function $\nabla^\perp$ that, to each pair $(X, Z)$ of smooth vector fields, $X$ tangent to $M$ and $Z$ normal to $M$, assigns a vector field $\nabla_X^\perp Z$ normal to $M$, defined by $\nabla_X^\perp Z = (\nabla_X Z)^\perp$.

An analogous to the Gauss formula (3) holds for vector fields $X$ tangent to $M$ and $Z$ normal to $M$:

$$\nabla_X Z = \nabla_X^\top Z + \nabla_X^\perp Z . \tag{6}$$

The (tangent) connection $\nabla$ was adapted to tangent vector fields along curves in $M \subset \overline{M}$, to produce the identity 4. Similarly, he normal connection $\nabla^\perp$ can also be adapted as follows to normal vector fields along curves on $M$. If $Z$ is a normal vector field along a curve $\gamma$ on $M$, then its normal covariant derivative $\frac{D}{dt} Z$ is defined to be the normal component of its $\overline{M}$ covariant derivative $\nabla^\perp \gamma Z$, and the following holds

$$\nabla^\perp \gamma Z = \nabla^\perp \gamma Z^\top + \frac{D}{dt} Z \tag{7}$$

tangent to $M$ normal to $M$.

A normal vector field $Z$ along $\gamma$ is said to be a normal parallel vector field along $\gamma$ if $\frac{D}{dt} Z \equiv 0$. The following holds, both for tangent and for normal parallel vector fields along curves in $M$.

**Lemma 2.1:** Let $t \in [a, b] \mapsto \gamma(t)$ be a curve in $M \subset \overline{M}$.

1. If $Y_0$ is a vector tangent to $M$ at $\gamma(a)$, there is a unique tangent parallel vector field $Y$ along $\gamma$ such that $Y(a) := Y(\gamma(a)) = Y_0$.
2. If $Z_0$ is a vector normal to $M$ at $\gamma(a)$, there is a unique normal vector field $Z$ along $\gamma$ such that $Z(a) = Z_0$.

With the notations above, if $\gamma(a) = p$ and $\gamma(b) = q$, the function

$$T_p M \to T_q M \quad Y(a) \mapsto Y(b) \tag{8}$$

is called the tangent parallel translation of $Y_0$ along $\gamma$, from the point $p$ to the point $q$. Similarly,

$$T_p^\perp M \to T_q^\perp M \quad Z(a) \mapsto Z(b) \tag{9}$$

is called the normal parallel translation of $Z_0$ along $\gamma$, from the point $p$ to the point $q$.

Both, tangent and the normal parallel translations are linear isometries. Consequently, tangent (respectively normal) parallel translation of a tangent (respectively normal) frame gives a tangent (respectively normal) parallel frame field along $\gamma$. A curve $t \mapsto \gamma(t)$ in $M$ is a geodesic if its velocity vector field is parallel along $\gamma$, i.e., $\nabla^\perp \gamma V(t) = 0$. The theory of semi-Riemannian geometry guarantees that a geodesic starting at $p_0$ with initial velocity $Y_0$ is locally unique and the causal character of a geodesic is that of its initial velocity vector.

**III. $G\ell(n)$ as a PSEUDO-RIEMANNIAN MANIFOLD**

For any matrix $A \in G\ell(n)$ and $J = \text{diag}(I_{n-m}, -I_m)$ ($0 \leq m \leq n$) define

$$A^J := J^T A^T J.$$
$Gl(n)$ may be equipped with the indefinite inner product
\[ (A, B)_J := \text{trace}(A^T B). \] (10)
For $m \neq 0$ this is not an inner product in the usual sense. It is bilinear and symmetric, but the positive-definite condition is just replaced by the weaker condition of non-degeneracy. When $m = 0$, then $J = I$ and $(A, B)_J = \text{trace}(A^T B)$ induces a Riemannian metric on $Gl(n)$. There is a natural isomorphism between $(Gl(n), \langle \cdot, \cdot \rangle_J)$ and $\mathbb{R}^{n^2}$ equipped with the indefinite inner product
\[ \langle x, y \rangle_J = x^T J y, \]
where $J = \text{diag}(I_n - k, -I_k)$ and $k = 2m(n - m)$. This is easily seen if to each matrix $A \in Gl(n)$ partitioned as
\[ A = \begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \end{bmatrix}, \]
(11)
where $A_1$ and $A_4$ are of order $n - m$ and $m$ respectively, one associates a vector resulting from a convenient shuffled vectorization. The isomorphism is defined by
\[ Gl(n) \rightarrow \mathbb{R}^{n^2}, \]
\[ A = \begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \end{bmatrix} \rightarrow a = \begin{bmatrix} \text{vec} A_1 \\ \text{vec} A_4 \\ \text{vec} A_2 \\ \text{vec} A_3 \end{bmatrix}, \]
(12)
where $\text{vec} X$ denotes the vectorization of the matrix $X$ formed by stacking the columns of $X$ into a single column vector. A simple calculation shows that for $A, B \in Gl(n)$ and corresponding vectors $a, b \in \mathbb{R}^{n^2}$ the following holds:
\[ (A, B)_J = \langle a, b \rangle_J. \]

A. Pseudo-orthogonal groups
To each matrix $J = \text{diag}(I_n - m, -I_m)$ one can associate a matrix Lie group
\[ H = \{ X \in Gl(n) | X^T J A = J \}, \]
usually called a pseudo-orthogonal group, also denoted by $O(n - m, m)$. The matrices in this group have determinant $\pm 1$, and those with determinant $1$ form the Lie subgroup $SO(n - m, m)$, which is not connected if $m \neq 0, n$. Each matrix in $O(n - m, m)$ can be decomposed in the form
\[ R = \begin{bmatrix} R_1 & R_2 \\ R_3 & R_4 \end{bmatrix}, \]
where $R_1$ and $R_4$ are invertible matrices of order $n - m$ and $m$ respectively, and $O(n - m, m)$ splits into four disjoint sets indexed by the signs of the determinants of $R_1$ and $R_2$ ($|R_1|$ and $|R_4|$), in this order. The connected component containing the identity matrix, hereafter denoted by $SO^+(n - m, m)$, is defined by
\[ SO^+(n - m, m) = \{ R \in SO(n - m, m) : |R_1| > 0, |R_4| > 0 \}. \]
The Lie algebra of $H$ (also of $SO^+(n - m, m)$), equipped with the Lie bracket $[A, B] = AB - BA$, is defined by
\[ L = \{ A \in Gl(n) | A^T J = -JA \}. \]
The vector space
\[ L_s = \{ A \in Gl(n) | A^T J = JA \}, \]
is also associated to $J$ and is closed under the symmetric product $\{ A, B \} = AB + BA$.
The following properties concerning $H$, $L$ and $L_s$ can be easily checked.
\[ \text{Proposition 3.1:} \]
1) $[L, L] \subset L$; $[L, L_s] \subset L_s$; $[L, L_s] \subset L$.
2) $\{ L, L_s \} \subset L_s$; $\{ L_s, L \} \subset L$; $\{ L_s, L_s \} \subset L_s$.
3) If $A \in L$, then $A^{2n+1} \in L$ and $A^{2n} \in L_s$, $\forall k \in \mathbb{N}_0$.
4) If $A \in L_s$, then $A^k \in L_s$, $\forall k \in \mathbb{N}_0$.
5) $Gl(n) = L \oplus L_s$. More precisely, every $A \in Gl(n)$ admits a unique decomposition as
\[ A = \frac{-A - A^T}{2} + \frac{A + A^T}{2}, \]
with $\frac{A - A^T}{2} \in L$ and $\frac{A + A^T}{2} \in L_s$.
6) If $X \in H$ and $A \in L (L_s)$, then $X^{-1} AX \in L (L_s)$.
7) If $t \rightarrow X(t)$ is a smooth curve in $H$, then
\[ \dot{X}X^{-1} \in L, \quad X^{-1} \dot{X} \in L. \]
The next proposition contains some results also involving the pseudo-Riemannian metric $\langle \cdot, \cdot \rangle_J$ and are easily derived.
\[ \text{Proposition 3.2:} \]
1) $\langle AX, XB \rangle_J = \langle AX, BX \rangle_J = \langle A, B \rangle_J$, for all $X \in H, A, B \in Gl(n)$;
2) $\langle [A, B], C \rangle_J = \langle [B, C], A \rangle_J$, for all $A, B, C \in L$;
3) $\langle L, L_s \rangle_J = 0$.

B. The connected pseudo-orthogonal group
Hereafter $G$ denotes the connected component, containing the identity of the Lie group $H$ defined in the previous subsection, that is,
\[ G := SO^+(n - m, m). \]
$G$ can be embedded in the $n^2$-dimensional pseudo-Riemannian manifold $(Gl(n), \langle \cdot, \cdot \rangle_J)$ of index $m$. Not all submanifolds of a pseudo-Riemannian manifold are pseudo-Riemannian. For instance, the hyperbolic $n$-sphere is embedded in the pseudo-space $\mathbb{R}^n_1$ but the restriction of the pseudo-metric to its tangent spaces is positive definite, thus defining a Riemannian metric. This doesn’t happen for our Lie group $G$, as the following shows.
\[ \text{Proposition 3.3:} \]
$\text{(SO}^+(n - m, m), \langle \cdot, \cdot \rangle_J)$ is a pseudo-Riemannian submanifold of $(Gl(n), \langle \cdot, \cdot \rangle_J)$. 

\textbf{Proof:} We have to show that the restriction of $\langle \cdot, \cdot \rangle_J$ to each tangent space to $G = \text{SO}^+(n - m, m)$ is nondegenerate, not positive definite, and that the index of $TXG$ is the same for every $X \in G$. The later statement is easily seen to be true from the first invariant property of $\langle \cdot, \cdot \rangle_J$ in Proposition 3.2. To prove non-degeneracy, consider the partition (VI) for matrices in $so(n - m, m)$, so satisfying $A^T J = -JA$. 

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It turns out that every matrix \( A \in \mathfrak{so}(n-m, m) \) can be partitioned as
\[
A = \begin{bmatrix} A_1 & A_2 \\ A_2^T & A_4 \end{bmatrix},
\]
where \( A_1 \) and \( A_2 \) are skewsymmetric of order \( n-m \) and \( m \) respectively. Consequently, \( \forall A, B \in \mathfrak{so}(n-m, m) \),
\[
\langle A, B \rangle_J = -\text{trace}(AB) = -(\text{trace}(A_1B_1) + \text{trace}(A_2B_2^T) + \text{trace}(A_3B_4))
\]
Consequently, \( \langle A, B \rangle_J = 0, \forall B \in \mathfrak{so}(n-m, m) \) implies that \( A = 0 \), that is, \( \langle ., . \rangle_J \) is nondegenerate. Finally, for every \( A \in \mathfrak{so}(n-m, m) \),
\[
\langle A, A \rangle_J = \text{trace}(A_1^T A_1) + \text{trace}(A_4^T A_4) - 2\text{trace}(A_2 A_2).
\]
Since \( A_1 \) and \( A_4 \) are skewsymmetric, the first two terms are \( \geq 0 \), while the third term is \( \leq 0 \). So, \( \langle ., . \rangle_J \) defines a true pseudo-Riemannian metric on \( G = SO^+(n-m, m) \).

The tangent space to \( G \) at a point \( P_0 \in G \), its affine space at \( P_0 \), its orthogonal complement with respect to \( \langle ., . \rangle_J \), and the lightcone of \( G \) at the point \( P_0 \), can be characterized respectively by
\[
\begin{align*}
T_{P_0}G &= \{ P_0A \mid A \in \mathcal{L} \}; \\
T_{P_0}^\text{aff}G &= \{ P_0 + P_0A \mid A \in \mathcal{L} \}; \\
(T_{P_0}G)^\perp &= \{ P_0A \mid A \in \mathcal{L}\setminus\{0\} \} \text{ and } \text{trace}(A^2) = 0 \\
\mathcal{C}(P_0) &= \{ P_0A \mid A \in \mathcal{L}\setminus\{0\} \} \text{ and } \text{trace}(A^2) = 0
\end{align*}
\]
We are interested in rolling the Lie group \( G \) over the affine tangent space at a point \( P_0 \). The intersection of these two manifolds reduces to the point \( P_0 \) only if \( m = 1 \), but it always belong to the lightcone of \( G \) through \( P_0 \), as shown next.

**Proposition 3.4:** For \( G = SO^+(n-m, m) \) and \( P_0 \in G \),
\[
G \cap T_{P_0}^\text{aff}G = \{ P_0 + P_0A \mid A \in \mathfrak{so}(n-m, m) \} \text{ and } A^2 = 0 \}
\]

Moreover, \( G \cap T_{P_0}^\text{aff}G = P_0 \) only if \( m = 1 \).

**Proof:** Let \( X \in T_{P_0}^\text{aff}G \). Then \( X = P_0 + P_0A \), for some \( A \in \mathfrak{so}(n-m, m) \). If \( X \in G \subset H \), then \( X^TJX = J \). Since the last equation is also satisfied by \( J \) and \( P_0 \), we may write
\[
X^TJX = J \iff (P_0^T + A^T P_0^T)J(P_0 + P_0A) = J \iff A^2 = 0.
\]
Consequently, \( G \cap T_{P_0}^\text{aff}G = \{ P_0 + P_0A \mid A \in \mathcal{L} \} \text{ and } A^2 = 0 \}. \]

**IV. ROLLING PSEUDO-RIEMANNIAN MANIFOLDS**

**A. Definition of rolling map**

The classical definition of a rolling map, as appeared in Sharpe [8], describes how two manifolds of the same dimension, \( M_1 \) and \( M_2 \), both isometrically embedded in the same Euclidean space \( \mathbb{R}^N \), roll on each other without slip or twist. Such a rolling motion is described by the action of the group of isometries in Euclidean space \( \mathbb{R}^N \), which preserve orientations, \( SE(N) = SO(N) \ltimes \mathbb{R}^N \). This definition has been extended and used in the situation when the two manifolds are isometrically embedded in a general Riemannian (or pseudo-Riemannian) manifold \( M \), and adapted to describe intrinsic rolling, for instance in [1]. This definition is easily extended to describe the rolling motion of a manifolds \( M_1 \) over a manifold \( M_2 \), isometrically embedded in a pseudo-Riemannian manifold \( \overline{M} \), replacing \( SE(N) \) by the group \( G \) of orientation preserving isometries of the ambient space, and taking orthogonality with respect to the pseudo-metric of the embedding space. For the sake of simplicity, we assume that \( \overline{M} = \mathbb{R}^N_k \) and denote he action of \( G \) on \( \overline{M} \) by \( \circ \).

Before presenting the definition we introduce some useful notations as in [3]. Let \( t \mapsto \chi(t) \) be a smooth curve on \( \overline{M} \), so that, for any point \( p \in \overline{M} \), \( \chi(t) \circ p \) makes sense. Also, let \( x : (-\epsilon, \epsilon) \rightarrow \overline{M} \) is a smooth curve satisfying \( \dot{x}(0) = \xi \).

Now define
\[
\chi(t) \circ x := \left( \frac{d}{d\sigma} (\chi(\sigma) \circ x) \right)_{\sigma=t} ,
\]
\[
\chi(t) \circ \chi^{-1}(t) \circ x := \left( \frac{d}{d\sigma} (\chi(s) \circ \chi^{-1}(t) \circ y(\sigma)) \right)_{\sigma=t} ,
\]
\[
\chi(t) \circ \chi^{-1}(t) \circ y(\sigma) := \left( \frac{d}{d\sigma} (\chi(t) \circ \chi^{-1}(t) \circ y(\sigma)) \right)_{\sigma=0}.
\]

We are ready to present the definition of a rolling map of \( M_1 \) over \( M_2 \), (without slip or twist). *0.1 cm

**Definition 4.1:** A map
\[
\chi : [0, \tau] \rightarrow \overline{G}, \quad t \mapsto \chi(t)
\]
satisfying the following conditions 1)-3), \( t \in [0, \tau] \), is called a rolling map of \( M_1 \) on \( M_2 \) without slipping or twisting.

1) (Rolling conditions) There exists a (piecewise) smooth curve \( \alpha \) on \( M_1 \) such that
   a) \( \chi(t) \circ \alpha(t) \in M_2 \)
   b) \( T_{\chi(t)\circ\alpha(t)}(\chi(t) \circ \alpha(t)) = T_{\chi(t)\circ\alpha(t)}M_2 \)

\( \alpha \) is called the rolling curve (on \( M_1 \)) and \( t \mapsto \alpha_{\text{dev}}(t) := \chi(t) \circ \alpha(t) \) is called the development of \( \alpha \) on \( M_2 \).

2) (No-slip condition) \( \chi(t) \circ \chi(t^{-1}) \circ \alpha_{\text{dev}}(t) = 0 \)

3) (No-twist conditions)
   a) (Tangential part)
   \( \chi(t) \circ \chi(t^{-1}) \circ T_{\alpha_{\text{dev}}(t)}M_2 \subset (T_{\alpha_{\text{dev}}(t)}M_2)^\perp \)
   b) (Normal part)
   \( \chi(t) \circ \chi(t^{-1}) \circ (T_{\alpha_{\text{dev}}(t)}M_2)^\perp \subset T_{\alpha_{\text{dev}}(t)}M_2 \).
Rolling maps for the rotation group SO(n) (does corresponding to the Riemannian situation when \( J = I \)) have been studied in [3]. This manifold is naturally embedded in Euclidean space \( \mathbb{R}^{n \times n} \), through the vec-isomorphism of “stacking columns”. So, rolling motions of these manifolds can be seen as rigid motions in the ambient space. In order to keep the matrix structure, SO(n) is embedded in \( \mathfrak{gl}(n) \) equipped with the Euclidean metric \( \langle A, B \rangle = \text{trace}(A^T B) \). We follow this approach for rolling the connected component containing the identity of the pseudo-orthogonal groups.

V. ROLLING THE LIE GROUP \( SO^+(n-m,m) \)

Assume that \( M_1 = G = SO^+(n-m,m) \) is rolling, without slip or twist, over \( M_2 = T_{P_0}G \), both embedded in \( (\mathfrak{gl}(n), <.,.>_J) \).

Following the approach in [3], we derive the kinematic equations for this pseudo-Riemannian system without destroying the matrix structure of the manifolds involved. This requires to define the appropriate action of the group of isometries of the embedding space

The following statements can easily be checked. The Lie group \( G \times G \) acts transitive on \( G \) via

\[
\sigma : (G \times G) \times G \to G
\]

\[
((U, W), R) \mapsto URW^{-1}
\]

(18)

and for each \( U, W \in G \), the action \( \sigma \) defines a map

\[
\sigma_{U,W} : G \to G
\]

\[
R \mapsto URW^{-1}.
\]

(19)

On \( \mathcal{G} = G \times G \times \mathfrak{gl}(n) \) define group operations

\[
(U_2, W_2, X_2) \circ (U_1, W_1, X_1) :=
\]

\[
(U_2U_1, W_2W_1, U_2X_1W_2^{-1} + X_2),
\]

(20)

and

\[
(U, W, X)^{-1} := (U^{-1}, W^{-1}, -U^{-1}WX).
\]

(21)

The identity element in \( \mathcal{G} \) is \((I, I, 0)\).

**Proposition 5.1:** The Lie group \( \mathcal{G} \) also acts on \( \mathfrak{gl}(n) \) via

\[
\mathcal{G} \times \mathfrak{gl}(n) \to \mathfrak{gl}(n)
\]

\[
((U, W, X), Z) \mapsto UZW^{-1} + X.
\]

(22)

Now, let \( P_0 \) be an arbitrary point in \( G \). Due to the transitive action \( \sigma \) of \( G \times G \) on \( G \), any curve \( t \mapsto \alpha(t) \) in \( G \), starting at \( P_0 \) (at \( t = 0 \)) may be defined on the interval \([0, \tau]\) as

\[
\alpha : [0, \tau] \to G
\]

\[
t \mapsto \sigma_{U,W}(P_0) = U(t)P_0W^{-1}(t),
\]

(23)

where \( U \) and \( W \) are curves in \( G \) satisfying \( U(0) = W(0) = I \).

Our main objective now is to find conditions on

\[
\chi : [0, \tau] \to \mathcal{G} = G \times G \times \mathfrak{gl}(n)
\]

\[
t \mapsto \chi(t) = (U^{-1}(t), W^{-1}(t), X(t)),
\]

(24)

which ensure that this map satisfies all the conditions in the definition of rolling, for \( M_1 = G, M_2 := T_{P_0}G \), rolling curve

\[
\alpha(t) = U(t)P_0W^{-1}(t),
\]

(25)

and its development

\[
\alpha_{dev}(t) = \chi(t) \circ \alpha(t)
\]

\[
= U^{-1}(t)\alpha(t)W(t) + X(t)
\]

(26)

\[
= P_0 + X(t).
\]

Note that, since \( \chi(t) = (U^{-1}(t), W^{-1}(t), X(t)) \in \mathcal{G} \), the action of \( \chi(t) \) on points of \( G \) is the restriction to \( G \) of the action (22). That is, for any \( P \in G \)

\[
\chi(t) \circ P = U^{-1}(t)PW(t) + X(t).
\]

(27)

We also have

\[
\dot{\chi}(t) = \left( \dot{U}^{-1}(t), \dot{W}^{-1}(t), \dot{X}(t) \right);
\]

\[
(\chi(t))^{-1} = \left( U(t), W(t), -U(t)X(t)W^{-1}(t) \right).
\]

(28)

Before proceeding with the requirements that \( \chi \) must satisfy the non-slip and the non-twist conditions, we rewrite the relations (14)-(16), so that they are ready to be used in the present situation. In what follows, \( Y \) is any point in \( \mathfrak{gl}(n) \) and \( \eta \) any vector in \( \mathfrak{gl}(n) \).

\[
\dot{\chi} \circ Y = \dot{U}^{-1}YW + U^{-1}YW\dot{W} + \dot{X},
\]

(29)

\[
(\dot{\chi} \circ \chi^{-1}) \circ Y = \dot{U}^{-1}U(Y - X) + (Y - X)W^{-1}\dot{W} + \dot{X},
\]

(30)

\[
(\dot{\chi} \circ \chi^{-1}) \circ \eta = \dot{U}^{-1}U\eta + \eta W^{-1}W.
\]

(31)

**Lemma 5.1:** The **no-slip condition** takes the form:

\[
\dot{X} = \frac{\Omega_U}{2}P_0 + \frac{\Omega_W}{2}, \quad \Omega_U, \Omega_W \in \mathcal{L}.
\]

(32)

**Proof:** According to Definition 4.1, \( \chi \) defined by (24) must satisfy

\[
(\dot{\chi}(t) \circ \chi^{-1}(t)) \circ \alpha_{dev}(t) = 0, \quad \text{for all } t,
\]

(33)

or, equivalently,

\[
\dot{\chi}(t) \circ \alpha(t) = 0, \quad \text{for all } t.
\]

But, according to (29), we may write (omitting the \( t \) dependence, for the sake of simplicity)

\[
\dot{\chi} \circ \alpha = 0 \iff \dot{\chi} \circ (U\dot{P}_0W^{-1}) = 0
\]

\[
\iff U^{-1}\dot{U}P_0W^{-1}W + U^{-1}U\dot{P}_0W^{-1}W + \dot{X} = 0
\]

\[
\iff -U^{-1}U\dot{P}_0 + P_0W^{-1}W + \dot{X} = 0.
\]

(34)

Setting

\[
U^{-1}\dot{U} := \frac{\Omega_U}{2} \in \mathcal{L},
\]

(35)

\[
W^{-1}\dot{W} := -\frac{\Omega_W}{2} \in \mathcal{L},
\]

we obtain

\[
\dot{X} = \frac{\Omega_U}{2}P_0 + \frac{\Omega_W}{2}, \quad \Omega_U, \Omega_W \in \mathcal{L}.
\]

(36)
Lemma 5.2: The no-twist conditions take the form:
\[ \Omega_UP_0 = P_0\Omega_W. \]  
(37)

Proof: According to the definition of rolling map, the conditions
\[ (\chi \circ \chi^{-1}) \circ T_{\alpha_{dev}, M_2} \subset (T_{\alpha_{dev}, M_2})^\perp, \]
(38)
and
\[ (\chi \circ \chi^{-1}) \circ (T_{\alpha_{dev}, M_2})^\perp \subset T_{\alpha_{dev}, M_2}, \]
(39)
must hold for \( M_2 = T_{aff}^{dev}G \). Using (31), the non-twist conditions is equivalent to
\[ \dot{U}^{-1}U\xi + \xi W^{-1}\dot{W} \in (T_{\alpha_{dev}, M_2})^\perp, \forall \xi \in T_{\alpha_{dev}, M_2}, \]
(40)
\[ \dot{U}^{-1}U\eta + \eta W^{-1}\dot{W} \in T_{\alpha_{dev}, M_2}, \forall \eta \in (T_{\alpha_{dev}, M_2})^\perp. \]
(41)
But, due to the second rolling condition, \( T_{\alpha_{dev}, M_2} \simeq T_{P_0}M_2 \) (and similarly for the corresponding orthogonal spaces). So, \( \xi \in T_{\alpha_{dev}, M_2} \) is of the form \( \xi = P_0\Psi \), for some \( \Psi \in L \) and, similarly, any vector \( \eta \in (T_{\alpha_{dev}, M_2})^\perp \) is of the form \( \eta = P_0S \), for some \( S \in L_s \). Consequently, the tangential part of the no-twist condition is equivalent to requiring that the matrix \( P_0^{-1}(U^{-1}UP_0\Psi + P_0\Psi W^{-1}W) \) belongs to \( L \), for all \( \Psi \in L \), while the normal part requires that the matrix \( P_0^{-1}(U^{-1}UP_0S + P_0SW^{-1}W) \) is in \( L \) for all \( S \in L_s \).

After some calculations, where the definitions of \( G, L \) and \( L_s \) are used we reached the following conclusion. The tangential condition reduces to
\[ [P_0^{-1}\Omega_UP_0 - \Omega_W, \Psi] = 0, \]
for all \( \Psi \in L \), which is equivalent to having \( P_0^{-1}\Omega_UP_0 = \Omega_W \). It turns out, however, that if this condition is satisfied, the normal condition holds as well. Therefore, the no-twist condition reduces to the single equation \( \Omega_UP_0 = P_0\Omega_W \).

The kinematic equations for rolling \( G \) over \( T_{aff}^{dev}G \) are now easily derived from the non-holonomic restrictions of no-split and no-twist. Taking into consideration (35) and introducing the ”control function”
\[ t \mapsto \Omega(t) := \Omega_U(t) = P_0\Omega_W(t)P_0^{-1} \in L, \]
(42)
the following holds.

Theorem 5.1: The following set of differential equations, where \( X \in GL(n) \), \( U, W \in G \), are the Kinematic equations for our rolling motion.
\[
\begin{align*}
\dot{X}(t) &= \Omega(t)P_0 \\
\dot{U}(t) &= \frac{1}{2}U(t)\Omega(t) \\
\dot{W}(t) &= \frac{1}{2}W(t)P_0^{-1}\Omega(t)P_0 
\end{align*}
\]  
(43)
The matrix function \( t \mapsto \Omega(t) \in L \) plays the role of a control function, since the motion is entirely determined by the choice of \( \Omega \). Controllability properties of this system are still under investigation.

Remark 5.1: Note that \( X \) plays the role of a ”translation”, while \( U,W \) define a ”rotational” motion. Also, the translational velocity \( \dot{X} \) remains in \( T_{P_0}G \) during the entire rolling motion.

We can now state the main result.

Theorem 5.2: If \( (X, U, W) \) is the solution of the kinematic equations (43), corresponding to a particular choice of the control function \( \Omega \) and satisfying \( (X(0), U(0), W(0)) = (0, I, I) \), then \( t \mapsto \chi(t) = (U^{-1}(t), W^{-1}(t), X(t)) \) is a rolling map for \( SO^+(n-m,m) \), in the sense of Definition 4.1. along the rolling curve \( t \mapsto \alpha(t) = U(t)P_0W^{-1}(t) \) with development \( t \mapsto \alpha_{dev}(t) = P_0 + X(t) \in M_2 \).

Remark 5.2: Note that when \( m = 0 \) the last result reduces to that in [3] for the rotation group \( SO(n) \).

Corollary 5.1: If \( \Omega(t) = \Omega \) is a constant matrix, then the solution of the kinematic equations with initial conditions \( (X(0), U(0), W(0)) = (0, I, I) \) is
\[
\begin{align*}
X(t) &= (t\Omega)P_0, \\
U(t) &= e^{\frac{t\Omega}{2}}, \\
W(t) &= P_0^{-1}e^{-\frac{t\Omega}{2}}P_0, \\
t \mapsto \alpha(t) &= e^{\frac{t\Omega}{2}}P_0P_0^{-1}e^{\frac{t\Omega}{2}}P_0 = e^{t\Omega}P_0, 
\end{align*}
\]  
(44)
and is a geodesic on \( G \), passing through \( P_0 \) at \( t = 0 \). Consequently, \( \alpha_{dev}(t) = P_0 + X(t) = P_0 + t\Omega P_0 \) is also a geodesic in \( T_{aff}^{dev}G \), passing through \( P_0 \) at \( t = 0 \).

The geodesics \( \alpha \) and \( \alpha_{dev} \) are spacelike if \( \langle \Omega, \Omega \rangle_J = -trace(\Omega^2) > 0 \), timelike if \( \langle \Omega, \Omega \rangle_J < 0 \) and lightlike if \( \langle \Omega, \Omega \rangle_J = 0 \).

Remark 5.3: If follows from the previous example that if \( \Omega(t) = \Omega \) is a constant matrix satisfying \( \Omega^2 = 0 \), then \( t \mapsto \Omega(t) = (I - \frac{1}{2}\Omega, P_0(I + \frac{1}{2}\Omega)P_0^{-1}, t\Omega P_0) \) is a rolling map with rolling curve \( t \mapsto \alpha(t) = (I + t\Omega)P_0 \in G \cap T_{aff}^{dev}G \). But, in this case, the development curve \( t \mapsto \alpha_{dev}(t) = (I + t\Omega)P_0 \) coincides with \( \alpha \). That is, when the rolling curve belongs to \( G \cap T_{aff}^{dev}G \), the corresponding rolling map \( \chi(t) = (I + t\Omega, I - tP_0\Omega P_0, t\Omega P_0) \) moves \( G \) and \( T_{aff}^{dev}G \) inside \( G \cap L(n) \) always keeping invariant every point of \( \alpha \). However, \( \chi(t) = (I, I, 0) \) is another (trivial) rolling map along \( \alpha \). This fact brings up an interesting issue about uniqueness of the rolling map corresponding to a rolling curve, which in the Euclidean situation always holds and in the present situation (when the rolling curve lies in a lightlike cone) may not. This is still under investigation.

A. Rolling performs parallel transport

We will show that parallel transport along a curve in \( G \) can be accomplished by rolling (without slip or twist) along that curve. This is valid, both for tangent and normal parallel transport.
Proposition 5.2: Let \( t \mapsto \chi(t) = (U^{-1}(t), W^{-1}(t), X(t)) \) be a rolling map for \( G = \text{SO}^+(n-m,m) \), with rolling curve \( t \mapsto \alpha(t) \) satisfying \( \alpha(0) = P_0 \), and \( Y_0P_0 \in T_{P_0}G \), then
\[
Y(t) = \chi^{-1}(t) \circ (Y_0P_0) = U(t)Y_0P_0W^{-1}(t)
\]
defines the unique tangent parallel vector field along \( t \mapsto \alpha(t) \), satisfying \( Y(0) = Y_0P_0 \).

Proof: The initial condition is satisfied since \( U(0) = W(0) = I \). The rolling curve is defined by \( \alpha(t) = U(t)P_0W^{-1}(t) \) and the tangent space to \( G \) at each point \( \alpha(t) \) can be parameterized by \( \{U(t)\Psi P_0W^{-1}(t) : \Psi \in \mathcal{L} \} \). Similarly, \( \{U(t)SP_0W^{-1}(t) : S = S^\top \in \mathbb{R}^{n \times n} \} \) parameterizes the normal space at \( \alpha(t) \). So, since \( Y_0 \in \mathcal{L}, Y(t) \) defines a vector field along the rolling curve, and to prove that it is indeed parallel along \( t \mapsto \alpha(t) \), it is enough to show that \( \dot{Y}_t \) lives in the normal space to \( \alpha(t) \), for all \( t \). Using the kinematic equations (43), we may write
\[
\begin{align*}
\dot{Y}_t &= U(t)Y_0P_0W^{-1}(t) + U(t)Y_0P_0W^{-1}(t) \\hfill (46) \\
&= \frac{1}{2}(U(t)\Omega(t)Y_0P_0W^{-1}(t) + U(t)Y_0P_0W^{-1}(t) + U(t)\Omega(t)P_0W^{-1}(t)) \\
&= U(t)\frac{1}{2}(\Omega(t)Y_0 + \Omega(t)P_0)W^{-1}(t).
\end{align*}
\]
Since the matrix \( \Omega(t)Y_0 + \Omega(t)P_0 \) is always in \( \mathcal{L} \), we conclude that \( \nabla_{\dot{\alpha}(t)}Y(t) \equiv 0 \).

Proposition 5.3: Let \( t \mapsto \chi(t) = (U^{-1}(t), W^{-1}(t), X(t)) \) be a rolling map for \( G = \text{SO}(n-m,m) \) (or \( G = \text{Sp}(m, \mathbb{R}) \)), with rolling curve \( t \mapsto \alpha(t) \) satisfying \( \alpha(0) = P_0 \), and \( Z_0P_0 \in T^G_{P_0} \), then
\[
Z(t) = \chi^{-1}(t) \circ (Z_0P_0) = U(t)Z_0P_0W^{-1}(t)
\]
defines the unique normal parallel vector field along \( t \mapsto \alpha(t) \), satisfying \( Z(0) = Z_0P_0 \).

The proof uses arguments similar to the above and is, for this reason, omitted.

Example 5.1: If the rolling curve is a geodesic, then \( U(t) = e^{t\frac{\Omega}{2}}, W(t) = P_0^{-1}e^{-t\frac{\Omega}{2}}P_0 \), so the parallel translation of \( Y_0P_0 \), along the rolling geodesic is given by
\[
Y(t) = e^{t\frac{\Omega}{2}}Y_0P_0e^{-t\frac{\Omega}{2}P_0} = e^{t\frac{\Omega}{2}Y_0}e^{t\frac{\Omega}{2}P_0}
\]
(49)

The formula (49) appears in [3], p. 482, for \( J = I \) and in [9], p.22, for the case of \( J = I = P_0 \).

VI. GENERATING INTERPOLATION CURVES ON \( G \)

Given a set of distinct points \( P_i \in G \), matrices \( \Omega_0 \) and \( \Omega_k \) in \( \mathcal{L} \) and fixed times \( t_i \), where
\[
0 = t_0 < t_1 < \cdots < t_{k-1} < t_k = \tau,
\]
consider the following:

Problem 6.1: Find a \( C^2 \)-smooth curve \( \gamma : [0, \tau] \to G \) satisfying interpolation conditions:
\[
\gamma(t_i) = P_i, \quad 1 \leq i \leq k - 1,
\]
and boundary conditions:
\[
\gamma(0) = P_0, \quad \gamma(\tau) = P_k, \\
\dot{\gamma}(0) = \Omega_0P_0, \dot{\gamma}(\tau) = \Omega_kP_k.
\]

A solution for this problem when the manifold \( G \) is the unit sphere \( S^n \), the rotation group \( \text{SO}(n) \) or Grassmann manifolds was presented in [3], using rolling motions. The same technique works for other manifolds, as long as one knows how to roll along geodesics. We have no space to give more details, but for the sake of completeness include the explicit formula for the interpolating curve that solves Problem 6.1.

Theorem 6.1: The curve \( t \mapsto \gamma(t) \) defined by
\[
\gamma(t) := \chi(t)^{-1}\left(\phi^{-1}(\beta(t) - \alpha_{\text{dev}}(t) + p_0) + \alpha_{\text{dev}}(t) - p_0\right)
\]
(53)
solves Problem 6.1.

Here, \( \chi \) is a rolling map of \( G \) on the affine tangent space at \( P_0 \) with development curve \( \alpha_{\text{dev}} \), \( \phi \) is a suitable local diffeomorphism \( G \to T^G_{P_0}G \) and \( \beta \) an interpolating curve in the affine tangent space. In order that the generation of these interpolation curves is computationally efficient, \( \phi \) and \( \chi \) have to be chosen appropriately. For instance, one needs to be able to solve explicitly the kinematic equations of rolling. This can be accomplished if rolling along a geodesic curve. We refer to [3] for details about this procedure.

REFERENCES