Delay-Dependent State-Feedback $\mathcal{H}_\infty$ Control for Nonlinear Stochastic Systems with Time-Varying Delays

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Abstract—This paper is concerned with the state-feedback $\mathcal{H}_\infty$ control problem for a class of stochastic nonlinear systems with time-varying state delays, where both the disturbance and the white noise are considered. To synthesize the investigated system, the parameter-dependent storage functional is firstly constructed. By means of completing the square, the delay-dependent controller is designed such that the closed-loop system is exponentially stable in mean square and the prescribed disturbance attenuation level is achieved. Finally, the developed delay-dependent conditions are further specialized to synthesize a type of special delayed nonlinear systems by solving several linear matrix inequalities (LMIs) and a bilinear matrix inequality (BMI).

I. INTRODUCTION

It is well known that the work in [1] laid a foundation for the $\mathcal{H}_\infty$ control problem for linear systems. Afterwards, based on the dissipative theory [2]–[4], many researchers aimed to establish the unified framework of the $\mathcal{H}_\infty$ control problem for deterministic nonlinear systems. The state-feedback $\mathcal{H}_\infty$ control problem for nonlinear systems can be found in [5]–[7] and the corresponding treatments for output-feedback cases are referred to [8]–[11]. The output-feedback controller parameterization problem was studied in [10].

In recent years, the study of the $\mathcal{H}_\infty$ control and filtering problems for stochastic nonlinear systems has received increasing attention. In [12] the state-feedback $\mathcal{H}_\infty$ control and estimation problems were treated as nonlinear stochastic minimax dynamic games; moreover, the equivalent results to deterministic nonlinear systems were developed. The state-feedback $\mathcal{H}_\infty$ control problem for discrete-time stochastic nonlinear systems was addressed in [13] and the corresponding continuous-time case was investigated in [14] and [15]. The work in [14] analyzed the time-invariant case, and [15] dealt with the time-varying case. As an extended result for works in [13] and [15], the output-feedback $\mathcal{H}_\infty$ control problem was studied in [16], and the sufficient conditions were proposed. The related $\mathcal{H}_\infty$ filtering problem was considered in [17]. And the stochastic nonlinear systems with Markovian jump was studied in [18], where both the state-feedback and output-feedback $\mathcal{H}_\infty$ control problems were covered.

Recently, the time-delay problem induced in practical engineering systems has stirred tremendous interest in the control and filtering areas [19], [20], and a lot of works have been contributed in the literature. For the $\mathcal{H}_\infty$ control problem for linear delayed systems, see, e.g. [21]–[24]; for the corresponding $\mathcal{H}_\infty$ filtering problem, refer to, e.g. [25]–[28] and the references therein. In general, the approaches can be categorized into delay-independent ones and delay-dependent ones. Since delay-dependent conditions are less conservative than the delay-independent ones, more attention has been focused on the delay-dependent approaches; see, e.g., [29]–[32].

It is worth noting that the nonlinear systems with time delays are more challenging compared to linear time-delay systems. Therefore, when it comes to the $\mathcal{H}_\infty$ control and filtering problem for nonlinear delayed systems, there are relatively small amount of results available in the literature, and especially, most works are concerned with special types of nonlinear systems. For example, the stabilization problem for a class of triangular structural time-delay nonlinear systems was investigated in [33]; the delay-dependent guaranteed cost control for delayed systems with nonlinear parameter perturbations was studied in [34]; the $\mathcal{H}_\infty$ filtering problem was addressed for stochastic delayed systems with sector-bounded nonlinearities in [35]; the delay-dependent stability analysis was reported for a class of delayed systems with nonlinearities constrained in a given polytopic region in [36]; the delay-independent/delay-dependent conditions for polynomial systems were presented in [37]. For a general class of stochastic nonlinear systems with time-varying delays, both the delay-dependent and delay-independent results have been reported in our recent work in [38], but the proof details of the delay-dependent results for a class of stochastic nonlinear systems are not explicitly provided. This paper is to study the state-feedback $\mathcal{H}_\infty$ control problem for such systems and provide the detailed derivations.

The rest of the paper is organized as follows. The system introductions and preliminaries are presented in Section II. The delay-dependent conditions for the synthesis of the state-feedback $\mathcal{H}_\infty$ controller are developed in Section III and the conclusion remarks are addressed in Section IV.

For the convenience, we use the following notations. The superscripts “$T$” and “$-1$” stand for matrix transposition and matrix inverse, respectively. $\mathbb{R}^n$ denotes the $n$-dimensional Euclidean space. $\mathbb{R}^{n \times m}$ is the set of $n \times m$ real matrices. The notation $P > 0$ ($P \geq 0$) means that $P$ is real symmetric and positive definite (positive semi-definite). $\| \cdot \|$ refers to the Euclidean norm for vectors and induced 2-norm for matrices. $\mathbb{E}$ stands for the mathematical expectation operator. Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P})$ be a complete probability
space with a filtration \( \{ F_t \}_{t \geq 0} \) containing all \( \mathcal{P} \)-null sets and being right continuous. \( L^2([0, \infty], \mathbb{R}^n) \) is the space of the nonanticipative stochastic processes \( y(t) \) with respect to filtration \( F_t \) satisfying \( \| y(t) \|^2 \leq E \int_0^\infty \| y(t) \|^2 dt < \infty \). \( C^{2,1}(\mathbb{R}^n \times \mathbb{R}; \mathbb{R}) \) is a class of functions \( V(x, t) \) second-order differentiable with respect to \( x \) and first-order differentiable with respect to \( t \). The space of continuously differentiable functions \( \phi : [-T, 0] \to \mathbb{R}^n \) with finite norm \( \| \phi \|_\tau = \sup_{-T \leq t \leq 0} \| \phi(t) \| \) is denoted by \( \mathcal{D} \). \( x(t) \in \mathbb{D} \) is a segment of the function \( x(\cdot) \) given by \( x_t(\theta) = x(t + \theta), \forall \theta \in [-T, 0] \).

II. Problem statement and preliminaries

Consider the following time-delay nonlinear stochastic system:

\[
\begin{align*}
\dot{z}(t) &= [f_1(x(t), x(t - \tau(t)))x(t) \\
&\quad + f_2(x(t), x(t - \tau(t)))x(t - \tau(t)) \\
&\quad + g_1(x(t), x(t - \tau(t)))u(t) \\
&\quad + g_2(x(t), x(t - \tau(t)))v(t)]dt \\
&\quad + q_1(x(t), x(t - \tau(t)))x(t)dw_1(t) \\
&\quad + q_2(x(t), x(t - \tau(t)))v(t)dw_2(t) \\
\end{align*}
\]

where \( x(t) \in \mathbb{R}^n \) is the system state; \( z(t) \in L^2([0, \infty], \mathbb{R}^n) \) is the regulated output; \( v(t) \in \mathbb{R}^n \) is the external disturbance which belongs to \( L^2([0, \infty], \mathbb{R}^n) \). \( W(t) = [\omega^T_1(t), \omega^T_2(t)]^T \) is the standard Wiener process (Brownian motion) defined on the complete probability space \( (\Omega, \mathcal{F}, \{ F_t \}_{t \geq 0}, \mathcal{P}) \) with the natural filter \( F_t \) generated by \( W(\cdot) \) up to time \( t \), where \( \omega_1(t) \) and \( \omega_2(t) \) are independent \( \mathbb{R}^1 \)-valued and \( \mathbb{R}^1 \)-valued Wiener processes, respectively. \( u(t) \in L^2([0, \infty], \mathbb{R}^n) \) is the control input, which is an adapted process with respect to \( \{ F_t \}_{t \geq 0} \). Furthermore, \( \tau(t) \) is the time-varying delays satisfying \( 0 \leq \tau(t) \leq T \leq \infty, \forall t > 0 \), \( \phi(\cdot) \in \mathcal{D} \), is the system initial function. Besides, the following assumptions are made. All the functions in (1) are assumed to be Borel measurable relative to the appropriate space, i.e., \( f_1 : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^{n \times n} \), \( f_2 : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^{n \times n} \), \( g_1 : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^{n \times n} \), \( g_2 : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^{n \times n} \), \( q_1 : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^{n \times n} \), \( q_2 : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^{n \times n} \), \( h : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^{n \times n} \).

**Assumption 1:** \( x_0 = x(0) = 0 \) is the unique equilibrium point for system (1) with \( u(t) \equiv 0 \) and \( v(t) \equiv 0 \).

In Assumption 1, the constraint for the equilibrium point is reasonable. In fact, if the equilibrium point is \( x_0 = x(t_1) \neq 0 \), then we can define an auxiliary equilibrium point \( \bar{x}_0 = x(t) - x(t_1) \), which is consistent with Assumption 1.

**Assumption 2:** system (1) possesses a unique strong solution.

Assumption 2 is a prerequisite for the analysis of the stochastic stability for the equilibrium and the synthesis of the state-feedback \( \mathcal{H}_\infty \) controller. For more details, the reader is referred to [39], [40] and [41].
functions if it does not cause any confusion in the rest of this paper.

A. Delay-dependent conditions for the controller design

For delayed linear system, the delay-dependent condition is usually established by using model transformations; see, e.g. [19], [20], but for delayed nonlinear stochastic systems, it is very difficult to perform model transformations due to the nonlinearity. Here, we take a novel parameter-dependent storage functional to establish the delay-dependent sufficient conditions, which are stated in the following theorem.

Theorem 1: For system (1), and a given attenuation $\gamma > 0$, suppose that the function $h(x(t), x_r)$ is norm-bounded, and if there exist three symmetric positive-definite functions $Q(x, x_r) \in C^{2,1}([\mathbb{R}^n \times \mathbb{R}^n]; \mathbb{R}^n \times \mathbb{R}^n)$, $P_1 \in \mathbb{R}^n \times \mathbb{R}^n$ and $P(\theta(x)) \in C^{2,1}([\mathbb{R} \times \mathbb{R}^n]; \mathbb{R}^n \times \mathbb{R}^n)$, which are have bounded eigenvalues for all $x$ and $x_r$, such that

$$\gamma^2 I - q_2^T(x, x_r)(Q(x, x_r) + TP_1)q_2(x, x_r) > 0,$$

(3)

where,

$$\Pi_1 \equiv \begin{bmatrix} \Pi_1 & \Pi_2 & \Pi_3 \\ * & \Pi_4 & \Pi_5 \end{bmatrix} \leq 0,$$

(4)

Proof: Take the storage functional as follows:

$$V(x(t), t) = x^T(t)Q(x, x_r)x(t)$$

$$+ \int_{t-\tau}^t (x(s) - x(s)ds)$$

(5)

is the state-feedback $H_\infty$ controller such that H1) and H2) hold.

By some algebraic manipulations, the infinitesimal operator $\mathcal{L}(x, t)$ can be evaluated as

$$\mathcal{L}(x, t) = \frac{\|u(t)\|^2}{2} - \frac{\|v(t) - u(t)\|^2}{2} \Theta(t)$$

$$+ \int_{t-\tau}^t (x(s) - x(s)ds)$$

(6)

where $\Theta(t) \equiv \int_{t-\tau}^t P_1 ([x(t) - x(s)]ds)$, $u(t) \equiv \Theta^{-1}(t)g_1^T[\Gamma(x, t) + Q(x, x_r)x]$ and $\Theta(t) \equiv \gamma^2 I - q_2^T [Q(x, x_r) + \tau(t)P_1]q_2$.

By applying $\dot{t} \leq \mu$ and $\tau(t) \leq T$ to (6), the following holds:

$$\mathcal{L}(x, t) \leq \|u(t)\|^2 - \frac{\|v(t) - u(t)\|^2}{2} \Theta(t)$$

$$+ \int_{t-\tau}^t (x(s) - x(s)ds)$$

(7)

which is $\leq [2x^T Q(x, x_r) + \tau(t)P_1]x$.

By considering this result as well as the control input $u(t) = u^*(t)$ in (5), we have $\mathcal{L}(x, t) \leq \|x(t)\|^2 - \gamma^2 \|v(t)\|^2$. Applying Dynkin's formula yields $E(V(x(T), T)) = V(x(0), 0) + \int_0^T \mathcal{L}(x(t), t)dt$.
\[ (dx(t) = (A + E_1 F(x, x_\tau)) H x(t) dt + [(A_1 + E_2 F(x, x_\tau) H)x(t) + G u(t) + G_1 v(t)] dt + (S + E_3 F(x, x_\tau) H)x(t) d\omega_1(t) + S_1 v(t) d\omega_2(t), \]

\[ z(t) = \begin{bmatrix} H_1(x(t) \\ H_2(x(t) \\ \ldots \end{bmatrix}, \forall t \in [-T, 0], \]

where \( A \in \mathbb{R}^{n \times n}, A_1 \in \mathbb{R}^{n \times m}, E_1, E_2, E_3 \in \mathbb{R}^{n \times m}, H \in \mathbb{R}^{n \times n}, G \in \mathbb{R}^{n \times m}, G_1 \in \mathbb{R}^{n \times n}, S_1 \in \mathbb{R}^{n \times n}, S \in \mathbb{R}^{n \times n} \) are constant matrices. \( F(x, x_\tau) \in \mathbb{R}^{m \times n} \) is the norm-bounded uncertainty, satisfying \( F^T(x, x_\tau) F(x, x_\tau) \leq I \).

To synthesize the state-feedback \( H_\infty \) controller for the system in (10), we have the following corollary.

**Corollary 1:** For the system in (10), if there exist three positive-definite matrices \( Q = Q^T \in \mathbb{R}^{n \times n}, P_1 = P_1^T \in \mathbb{R}^{n \times n} \) and \( P_2 = P_2^T \in \mathbb{R}^{n \times n} \) and three positive scalars \( \beta_1, \beta_2, \alpha \), such that the following matrix inequalities hold:

\[ \gamma^2 I - S_1^T(Q + TP_1)S_1 \geq \alpha I, \]  

\[ \begin{bmatrix} \bar{\Pi}_1 & \bar{\Pi}_2 & S^T(T + P_1) & \beta_1 H^T \\ P_1 & \Pi_1 & 0 & 0 \\ * & * & \bar{\Pi}_6 & 0 \\ * & * & * & -\beta_1 I \\ * & * & * & * \\ * & * & * & * \\ * & * & * & * \\ * & * & * & * \\ \end{bmatrix} \leq 0, \]  

\[ \begin{bmatrix} \bar{\Pi}_3 & \Pi_2 & \Pi_3 & S^T(Q + TP_1) & \beta_1 H^T \\ Q E_1 & 0 & Q E_2 & Q G_1 & 0 \\ \end{bmatrix} \begin{bmatrix} \beta_2 H^T & 0 & 0 \\ P_1 E_1 & 0 & P_1 E_2 & P_1 G_1 \\ E_3 & 0 & 0 & 0 \\ -\beta_1 I & 0 & 0 & 0 \\ * & -\beta_2 I & 0 & 0 \\ * & * & -\alpha I & 0 \\ \end{bmatrix} \leq 0, \]  

where \( \bar{\Pi}_1 = A^T Q + QA + H_1^T H - QGG^T Q, \bar{\Pi}_2 = QA_1, \bar{\Pi}_3 = A^T P_1 - QGG^T P_1, \Pi_4 = -(1 - \mu) F_2, \bar{\Pi}_5 = A_2^T P_1, \bar{\Pi}_6 = -P_1 G^T P_1 \). Then the control input

\[ u(t) = -G^T [Q x + P_1 \int_{t-\tau(t)}^{t} (x(s) - x(s))ds] \]

is the state-feedback \( H_\infty \) controller such that H1) and H2) are satisfied.
It is noted that the similar result has been reported in our work [38], but the complete proof is not available in that work.

Proof: Take a storage functional \( V(x(t), t) \) as
\[
V(x(t), t) = x^T(t)Qx(t) + \int_{t-\tau(t)}^{t} [x(t) - x_t(\theta)]^T P_1 x(t) - x_t(\theta))d\theta.
\]
According to Theorem 1, the condition (3) and (4) can be strengthened as
\[
\gamma^2 I - S_1^T(Q + hP_1)S_1 \geq \alpha I, \quad (14)
\]
where
\[
\Psi_1 = (A + E_1 F(x, x_r)H)^T Q + (A + E_1 F(x, x_r)H)Q G_1 \alpha^{-1} G_1^T - QG^T Q + H_1^T H_1 \times (Q + TP_1)(A + E_1 F(x, x_r)H),
\]
\[
\Psi_2 = Q(A_1 + E_2 F(x, x_r)H),
\]
\[
\Psi_3 = (A + E_1 F(x, x_r)H)^T P_1 + Q G_1 \alpha^{-1} G_1^T P_1 - QG^T P_1,
\]
\[
\Psi_4 = - (1 - \mu)P_2,
\]
\[
\Psi_5 = (A_1 + E_2 F(x, x_r)H)^T P_1,
\]
\[
\Psi_6 = P_3 G_1 \alpha^{-1} G_1^T P_1 - P_3 G^T P_1.
\]
By applying Schur Complement to (15), we have
\[
\begin{bmatrix}
\Psi_7 & \Psi_2 \\
\Psi_4 & \Psi_5
\end{bmatrix}
\begin{bmatrix}
S + E_3 F(x, x_r)H \\
0
\end{bmatrix}
\leq 0,
\quad (16)
\]
where
\[
\Psi_7 = Q(A + E_1 F(x, x_r)H) + (A + E_1 F(x, x_r)H)^T Q + H_1^T H_1 + Q G_1 \alpha^{-1} G_1^T - QG^T Q.
\]
To deal with the uncertainty, we rewrite (16) as
\[
\Theta + N_1^T F^T(x) M_1^T + M_1 F(x, x_r)N_1
\]
\[
+ N_2^T F^T(x) M_2^T + M_2 F(x, x_r)N_2 \leq 0,
\quad (17)
\]
where
\[
\Theta =
\begin{bmatrix}
\Psi_7 - 2Q E_1 F(x, x_r)H & \Psi_2 - Q E_2 F(x, x_r)H \\
* & \Psi_4 \\
* & * \\
* & * \\
* & * \\
\Psi_3 - H_1^T F^T(x) E_1^T P_1 & S^T \\
\Psi_5 - H_1^T F^T(x) E_2^T P_1 & 0 \\
\Psi_6 & 0
\end{bmatrix}
\]
and
\[
N_1 = [H, 0, 0, 0], \quad M_1^T = \begin{bmatrix} E_1^T Q, 0, E_1^T P_1, E_1^T \end{bmatrix},
\]
\[
N_2 = [0, H, 0, 0], \quad M_2^T = \begin{bmatrix} E_2^T Q, 0, E_2^T P_1, 0 \end{bmatrix}.
\]
Lemma 1 to (17), there exist two positive scalars \( \beta_1 \) and \( \beta_2 \) such that
\[
\Theta + \beta_1 N_1^T N_1 + \beta_2^{-1} M_1^T M_1^T
\]
\[
+ \beta_2 N_2^T N_2 + \beta_2^{-1} M_2^T M_2^T \leq 0.
\]
By employing Schur Complement again, we can obtain
\[
\begin{bmatrix}
\Theta_1 & N_1^T \\
* & -\beta_1^{-1} I
\end{bmatrix}
\begin{bmatrix}
M_1 & N_2^T \\
* & M_2
\end{bmatrix}
\leq 0.
\quad (18)
\]
Note that the inequality (18) can be further rewritten as
\[
\begin{bmatrix}
\Theta_1 & N_1^T \\
* & -\beta_1^{-1} I
\end{bmatrix}
\begin{bmatrix}
M_1 & N_2^T \\
* & M_2
\end{bmatrix}
\leq 0.
\quad (19)
\]
where
\[
\begin{bmatrix}
\Theta_1 & N_1^T \\
* & -\beta_1^{-1} I
\end{bmatrix}
\begin{bmatrix}
M_1 & N_2^T \\
* & M_2
\end{bmatrix}
= \begin{bmatrix}
2QG_1 \\
0
\end{bmatrix},
\]
and \( \Phi_7 = -(Q + TP_1)^{-1} \). In the light of Schur Complement, we further have
\[
\begin{bmatrix}
\Theta_1 & N_1^T \\
* & -\beta_1^{-1} I
\end{bmatrix}
\begin{bmatrix}
M_1 & N_2^T \\
* & M_2
\end{bmatrix}
\leq 0.
\quad (20)
\]
Finally, the inequality (12) is obtained readily by pre-multiplying and post-multiplying \( \Lambda \) and \( \Lambda^T \) to (20), respectively, where \( \Lambda = \text{diag}(I, I, I, I, (Q + TP_1), \beta_1 I, I, \beta_2 I, I, \alpha I) \). According to the expression in (5), the state-feedback \( \mathcal{H}_\infty \) controller is evaluated as (13). One the other hand, the state-feedback \( \mathcal{H}_\infty \) controller can exponentially stabilize the system in (10) with \( v(t) \equiv 0 \) in mean square. This can be readily derived by following the same line of that in Theorem 1. The proof is completed. ■

IV. CONCLUSIONS

In this paper, we have investigated the state-feedback \( \mathcal{H}_\infty \) control problem for a class of nonlinear stochastic systems with time-varying delays. The delay-dependent sufficient conditions have been developed for the design of the state-feedback \( \mathcal{H}_\infty \) controller. The LMI and BMI-based conditions have also been derived for a type of special nonlinear stochastic systems.
REFERENCES


