Improving Transient Performance of Signal Transformation Approach

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Abstract—A method is proposed to significantly improve the transient performance of Signal Transformation Approach (STA). For triangular reference signals, the proposed method is shown to be equivalent to a recently proposed hybrid control method, known as Impulsive State Multiplication (ISM), if the compensator contains at least two integrators. Effectiveness of the proposed method in tracking of arbitrary waveforms other than the triangular signals is also investigated. Relevant results governing SISO LTI state-space systems with more than one integrator are also obtained.

I. INTRODUCTION

Accurate tracking of triangular references is of critical importance in raster scanning applications such as scanning probe microscopy [5], optical scanners [15], selective laser sintering [16], and emerging probe-based data storage devices [7]. The high frequency content of a triangular reference signal forces an ordinary feedback control system to have a high closed-loop bandwidth in order to achieve an acceptable tracking performance. However, a high closed bandwidth may be restricted due to actuator bandwidth limitations, plant uncertainties, unmodeled dynamics, and sensor noise.

Control systems with low closed-loop bandwidths are particularly important in applications such as nanopositioning [5], where the impact of measurement noise on the controlled variable is to be reduced to a minimum. However, the tracking performance of a control system is deteriorated as the control bandwidth is reduced. This often leads to a compromise between tracking and noise performance, which can be addressed using a number of approaches, such as Command Pre-shaping [10] and Iterative Learning Control [15]. However, the former is not suitable for time-varying references and relies on accurate knowledge of plant parameters and the latter requires a large number of computations and iterations to converge.

In [8], the concept of signal transformation Approach (STA) was put forward as a novel method for tracking of triangular waveforms, and it was experimentally tested on a nanopositioning system. The method showed significant steady-state performance improvement compared with an ordinary feedback control system having a similar control bandwidth [1], [3]. However, the transient performance of STA is known to deteriorate as the scan speed is increased.

An alternative low bandwidth control strategy, known as Impulsive State Multiplication (ISM) Control, has been recently proposed, for tracking of piecewise linear (affine) references such as triangular waveforms [11]-[14]. A key motivation for the introduction of this method is its improved transient performance relative to the STA [12]. To implement an ISM controller, a combination of feedforward and feedback is used, where switching gains are incorporated just around the compensator in the feedback loop with no bias signals [12].

In this paper, a method is introduced to improve the transient performance of STA. The resulting method, Initialized Signal Transformation Approach (ISTA), is shown to be exactly equivalent to ISM even during the transient, as long as the compensator includes more than one integrator. Performances of the two methods are illustrated by simulations. It is shown that the proposed ISTA method can be equally applied to improve the transient in a recently developed STA method for a more general class of reference signals [2], where the ISM method is not applicable. Moreover, SISO LTI compensators that include at least two integrators and their high frequency gains are not equal to −1, are studied in some detail.

II. SIGNAL TRANSFORMATION APPROACH

In STA for triangular waveform tracking, transformation mappings are incorporated in the feedback loop around the plant such that the compensator just needs to vanish the tracking error of a ramp signal, while the plant output follows the triangular reference. Based on Internal Model Principle in control [6], the compensator needs at least two integral actions to accurately vanish the tracking error of the ramp signal. For plants whose dominant behavior can be approximated by a second order linear stable model with unity DC gain and no zeros, one can easily design a loop with adequate stability margins using a strictly proper second-order compensator that includes two integrators (see [1], Section 5.2). Incorporation of signal transformation blocks in an ordinary feedback control system with a triangular reference, as shown in Fig. 1(a,b), can significantly improve the steady-state tracking performance if the following conditions are satisfied [1]:

1) The plant is stable, has unity low frequency gain, and no dominant zeros.
2) The compensator contains at least two integrators.
3) The closed-loop ordinary system in Fig. 1(a) meets acceptable stability margins.
4) The closed-loop bandwidth of the ordinary system is much lower than that of the plant.
The fundamental frequency of the triangular reference signal is sufficiently low to meet the stability criterion \( \gamma = \frac{\lambda_{\text{max}}\left(\tilde{A}\right)}{\lambda_{\text{min}}\left(\tilde{A}\right)} < 1 \) with a reasonable margin.\(^1\)

However, due to a low closed-loop bandwidth and/or a poor stability margin for a fast triangular reference, the performance of transient response in STA may not be acceptable [8]. In the next section, a remedy to the aforementioned problem is offered.

### III. INITIALIZED SIGNAL TRANSFORMATION APPROACH (ISTA)

One approach to improving the transient performance in STA is to initialize the compensator states to appropriate values instead of starting with zero initial conditions. Let us consider the following discussion before determining the appropriate initial conditions.

Due to the transformation blocks in Fig. 1(b), the double integral compensator attempts to track the ramp reference signal \( r \) and to generate a ramp signal at its output \( v \). After the transient, the compensator states keep following ramp and/or step trajectories. These trajectories depend on both the plant and compensator parameters. We assume that the initial states of the plant cannot be manipulated and only its low frequency gain is known. The plant is stable, has a unity DC gain, and a bandwidth much higher than that of the closed-loop system. Hence, if we replace the plant with a unity DC gain stable system with infinite bandwidth, the transient performance will not significantly change. With this approximation, we will only use stability and DC gain knowledge of the plant to calculate appropriate initial states of the compensator.

Replacing the combination of the plant and its surrounding signal transformation mappings in Fig. 1(b) with a unity gain, the resulting system is still stable due to conditions 3 and 4 in Section II, and provides a good approximation of the compensator behavior in the STA system. Hence, any initial state values that can improve the transient performance of the ordinary feedback system in Fig. 1(c) will have a similar impact on the STA system.

One can easily check that the state and input matrices of the closed-loop system in Fig. 1(c) are \( A_{cl} = A_c - \frac{B_cC_p}{1+D_c} \) and \( B_{cl} = \frac{B_c}{1+D_c} \). Hence, the response of the compensator in Fig. 1(c) can be expressed as:

\[
X_c(s) = (sI - A_{cl})^{-1} \left[ x_c(0) + \frac{B_c a_0}{T(1 + D_c)s^2} \right] (1)
\]

where \( x_c(0) \) is the initial state of the compensator and \( \frac{a_0}{s} \) is the slope the ramp input signal \( r \). The existence of the inverse matrix in (1) is ensured by the foregoing stability and bandwidth conditions in Section II. It is also assumed that the high frequency gain of the compensator is not \( -1 \), so the denominator in Eq. (1) is well defined. We use partial fraction expansion techniques to decompose the right hand side of (1) into transient and steady-state components as:

\[
(sI - A_{cl})^{-1} V(s) = \frac{V_1}{s} + \frac{V_2}{s^2} (2)
\]

where \( V_1 = -A_{cl}^{-2} \frac{B_c a_0}{T(1+D_c)} \) and \( V_2 = -A_{cl}^{-1} \frac{B_c a_0}{T(1+D_c)} \) are constant vectors. Equating the right hand side of (1) with (2), one can easily show that vector \( V(s) \) in (2) is constant as well. In this way, the overall response of the system in Fig. 1(c) with initial conditions can be written as:

\[
X_c(s) = \underbrace{(sI - A_{cl})^{-1} \left[ x_c(0) + \frac{A_{cl}^{-2}B_c a_0}{T(1+D_c)s^2} \right]}_{\text{Transient}} - \underbrace{(sI + A_{cl})A_{cl}^{-2}B_c a_0}{T(1+D_c)s^2} (3)
\]

\[
\lim_{t \to \infty} X_c(t) = -\frac{(tI + A_{cl}^{-1})A_{cl}^{-1}B_c a_0}{T(1+D_c)} (4)
\]

To remove the transient response of the approximate ordinary system in Fig. 1(c), we may set the initial states of the compensator equal to the right hand side of Eq. (4) at \( t = 0 \). This choice exactly vanishes the first term of the overall state response in Eq. (3), which represents the transient part of the approximate system. In this way, the following initial state

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\(^1\)Here, \( \tilde{A} \) is \( E\tilde{T} \), where \( E = e^{A T} \) and \( \tilde{T} = [-I_{n_p}, 0; 0, I_{n_c}] \) with \( I_{n_p} \) and \( I_{n_c} \) referring to identity matrices with dimensions \( n_p \) and \( n_c \), corresponding to those of the plant and compensator, respectively. Matrix \( A = [A_p - B_p D_c C_p; B_p C_p; -B_c C_p, A_c] \) is the state matrix of the closed-loop system in Fig. 1(a). Although the compensator feedthrough term \( D_c \) is zero in [1], however, one can readily check that Theorem 2 in [1] still holds with nonzero values of \( D_c \), if \( A \) and \( B \) refer to the state and input matrices of the closed-loop system in Fig. 1(a) (\( B = [B_p D_c; B_c] \)).

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Fig. 1. (a) An ordinary feedback control system. (b) Incorporation of signal transformation blocks in the ordinary feedback system for tracking of triangular reference (STA). (c) Approximation of the compensator behavior in Fig 1(b).
vector is obtained as an appropriate candidate for transient improvement in the STA system.

\[ x_c(0) = -\frac{A_{cl}^2 B_c a_0}{T(1 + D_c)} \]  

(5)

The following section demonstrates how the transient performance is improved in STA with the proposed initialization of the compensator states.

A. Simulation Results

The selected plant has a unity DC gain, a zero at \(-7000\) rad/sec, and poles at \(-5000, -5000, \) and \(-10^4\) rad/sec. The state space matrices of the compensator, which contains two integral actions, are as:

\[
A_c = \begin{bmatrix} -40 & 0 & 0 \\ 8 & 0 & 0 \\ 0 & 2 & 0 \end{bmatrix},
B_c = \begin{bmatrix} 64 \\ 0 \\ 0 \end{bmatrix},
D_c = 0; \quad C_c = \begin{bmatrix} 0.78125 \\ 39.453125 \\ 78.125 \end{bmatrix}^T.
\]

(6)

With this configuration, the closed-loop bandwidth is only 35Hz, which is much less than the plant bandwidth of 560 Hz. For a 100 Hz triangular reference with unity amplitude of \(a_0 = 1\), using Eq. (5), the appropriate initial states of the compensator are \(X_0 = [0 \ 1.28 - 0.6464]^T\). The plant initial conditions are assumed zero. The resulting tracking errors of the STA system with compensator initial states set to the appropriate and zero values are shown in Fig. 2 with labels ISTA and STA, respectively. Clearly, the proper selection of compensator initial conditions can significantly improve the transient performance without altering the steady-state response.

Remark: In this example, we did not consider any disturbances, noises, or plant uncertainties for brevity. However, STA method can provide adequate robustness against these issues when an appropriate robustifying feedback loop is wrapped around the original plant and the loop is treated as the plant in Fig. 1(b) (see Sec. 4 in [1], Sec. V in [4], and Sec. X in [2]). Since the compensator’s initial conditions do not affect the robust performance of STA system, the same robustifying method can be equally applied to ISTA method.

IV. EQUIVALENCE OF ISTA AND IMPULSIVE STATE MULTIPLICATION (ISM)

In this section, we show that ISTA is equivalent to a recently developed hybrid control method, the so-called Impulsive State Multiplication (ISM) [11]–[14], if at least two integrators are included in the compensator. The relationship between STA and ISM was presented in [12], [13] but the important effect of compensator initialization was not addressed. In addition, ISM only deals with affine references, while ISTA can also handle the non-affine ones. To show that ISM is equal to ISTA for triangular references, we start from ISTA block diagram in Fig. 1(b). Using straightforward signal flow graph rules, as the elements within the two signal transformation blocks \(\Phi\) are identical, they can be moved after the addition block that generates the error signal \(e\). The signal transformation block \(\Phi^{-1}\) outputs the triangular signal \(x_d\) if it is fed by the ramp signal \(r(t) = a_0 t/T\). Hence, the bias signal \(b_1(t)\) can be expressed in the following form:

\[ b_1(t) = x_d(t) - g_1(t)r(t). \]

(7)

Thus, the ISTA system can be described as shown in Fig. 3(a). The compensator with initial condition (5) can be replaced by its replica with zero initial conditions whose output is augmented by the ramp signal, as shown in Fig. 3(b). To show this, one can use the superposition principle and treat the initial state and the compensator input signal \(e(t)\) as independent sources that actuate the compensator. To show that the zero input response of the compensator to initial condition (5) is the ramp signal \(r(t)\), we refer to the discussion in section III concerning the system in Fig. 1(c). With the initial state of (5) in Fig. 1(c), the output signal \(y(t)\) exactly follows the ramp reference signal \(r(t)\), all the time, and compensator input \(e(t)\) is identically zero, as if the compensator input is open. This ensures that the zero input response of the compensator to initial condition (5) is the ramp signal \(r(t)\). Replacing the compensator in Fig. 3(a) by its equivalent in Fig. 3(b), one can easily obtain a reduced form for ISTA, as shown in Fig. 3(c).

The switching gain signals \(g_1(t)\) and \(g_2(t)\) have a square wave profile that switches between +1 and \(-1\), described as [1]:

\[ g_1(t) = g_2(t) = (-1)^i(t) \]

(8)

where \(i = \frac{t}{T}\), and \(T\) is the half period of the triangular reference signal, depicted in Fig. 1(b). Defining \(q_i := g_2^{-1}\), \(t_i := iT\), and using Theorem 3 in [12], we conclude that the part surrounded by the dotted line in Fig. 3(c) is equivalent to the following ISM system, that describes the relationship...
between input $w(t)$ and output $z(t)$:

$$
\begin{align*}
    \dot{X}(t) &= A_i X(t) + B_i w(t), \quad \text{if} \quad t \neq t_i \\
    X(t_i) &= \frac{q_i}{q_{i-1}} X(t_{i-1}) = -X(t_{i-1}) \\
    z(t) &= C_i X(t)
\end{align*}
$$

where $i = 1, 2, 3, \ldots$. Hence, with the triangular reference, ISTA is equivalent to ISM approach if the compensator has at least two poles at the origin. The foregoing equivalence between ISTA and ISM was also justified by simulations, where the ISM approach produced exactly the same tracking error profile as that of the ISTA, all the time.

V. APPLICATION TO ARBITRARILY SHAPED REFERENCES

In this section, we illustrate by simulations that the proposed ISTA method improves the transient performance in the recently developed STA method for arbitrary references, reported in [2]. This STA method also works based on transformation mappings that can convert the reference signal to a ramp signal $r = a_0 t$ and vice versa, and has the same structure as shown in Fig. 1(b). However, as the reference signal $x_{\text{ref}}(t)$ is not necessarily triangular, the internal details of the transformation mappings are generally more complicated than those of the triangular references [2]. In this method, under conditions almost similar to those stated in Sec. II, the response of the compensator can also be approximated by the feedback loop shown in Fig. 1(c). Hence, the analysis in Sec. III is still valid, if the ramp slope of $a_0 t$ is replaced by $a_0$. Thus, the transient will be improved, if the compensator’s initial state is set as:

$$
x_{\text{c}}(0) = -\frac{A_i^{-2} B_i a_0}{1 + D_c}.
$$

To show this, we consider a sixth-order model of the x-axis of a piezoelectric tube stage, which has the undesirable resonant behavior shown by the un-damped frequency and unit step responses in Fig. 4. The model has a unity DC gain, zeros at $-308 \pm 6150i$, and poles at $-1885s - 103s \pm 5176i$, $-317s + 6338i$, and $-9425 rad/sec$. The un-damped response of the tube can be alleviated using the following controller:

$$
C_{\text{PPF}}(s) = 0.4 \frac{s + 1}{s^2 + 2 \times 0.34 s + 1}.
$$

The damping loop structure, which is shown inside the dashed rectangle in Fig. 5, has the step and frequency responses shown in Fig. 4, and adequate stability margins of $8 \text{ dB}$ and $-49 \text{ deg}$. The positive DC gain of the damping controller is consistent with the popular method of Positive Position Feedback (PPF) for vibration suppression in flexible structures [9]. In STA methods, the DC gain of the plant block in Fig. 1(b) should be unity. However, the DC gain of the damped tube is not unity and can change if the tube’s DC gain varies with time. Hence, we add a PI control loop to the damped tube, as shown inside the dotted rectangle in Fig. 5, to provide a robust unity DC gain for the plant block in STA structure (STA plant) as well as rejection of constant disturbances. The PI controller loop has stability margins of $7 \text{ dB}$ and $47 \text{ deg}$, and provides a bandwidth of $548 \text{ Hz}$ and step and frequency responses shown in Fig. 4 for the STA plant. The overall structure of the control system
in this example is shown in Fig. 5, where the reference signal \( x_d(t) \) is a periodic non-triangular signal, which was arbitrarily selected in a period of \( T_1 + T_2 \) as:

\[
x_d(t) = \begin{cases} 
\alpha \sinh \left( \frac{t - 2a}{\beta_1} \right), & t \in [0, T_1) \\
-\alpha \sinh \left( \frac{t - T_1 - 2a}{\beta_2} \right), & t \in [T_1, T_1 + T_2) 
\end{cases}
\]

where \( \alpha = \frac{a_0}{10.7397} \), \( \beta_1 = \frac{T_1}{0.7397} \), \( t \in \{1, 2\} \), and \( a_0 \) is the amplitude of the reference. The slope of the selected reference signal varies between 0.05 to 5.37 times the slope of a triangular signal with the same amplitude and time intervals. Hence, tracking of the selected reference signal is much harder than a corresponding triangular signal. Using the above explicit expressions of the reference signal, the mappings \( \Phi \) and \( \Phi^{-1} \) in Fig. 5 are designed to convert the reference to a ramp and vice versa, respectively, according to the procedure mentioned in [2]. The STA compensator is a double integrator whose state space matrices are in the following form:

\[
A_c = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}; 
B_c = \begin{bmatrix} 0 \\ 1 \end{bmatrix}; 
C_c = \begin{bmatrix} 180 \\ 120 \end{bmatrix}^T; D_c = 0.
\]

This compensator provides stability margins of 21.5 dB and 85 deg, and a closed-loop bandwidth of 21 Hz for the ordinary feedback system (from the reference to the plant output \( x \) with the mappings replaced by unity gains). Assuming a unity amplitude for the reference \( (a_0 = 0) \) and using (13) and (10), the compensator’s initial state is obtained as \( [-0.0037, 0.005]^T \). The reference signal with \( T_1 = 0.2 \) s and \( T_2 = 0.3 \) s is shown in Fig. 6 along with the controlled plant output using the proposed ISTA method, where unit step signals are applied as input and output disturbances at \( t = 0.4 \) s and \( t = 0.8 \) s, respectively. The resulting tracking error with ISTA method is shown in Fig. 7, along with that of the STA method, which uses zero values as the initial state of the compensator. Clearly, ISTA method considerably improves the transient performance of STA method without affecting the performances of disturbance rejection and steady-state tracking error. Time histories of plant inputs are shown in Fig. 8 for a demonstration of adequate control efforts in ISTA and STA methods.

Using the same reference signal and no disturbances, we finally compare the tracking error of ISTA method, which has a bandwidth of 21 Hz, with that of an ordinary control system. The ordinary system is the PI control system indicated by the dotted line in Fig. 5, which has a bandwidth of 548 Hz from \( u \) to \( x \), and almost the same noise rejection performance as the ISTA. The results, shown in Fig. 9, clearly demonstrate the superiority of ISTA over an ordinary control system.

VI. FORMULAS GOVERNING SISO LTI SYSTEMS WITH MORE THAN ONE INTEGRATOR

In this section we derive relationships that govern linear-time-invariant (LTI) systems that have at least two poles at the origin, such as the compensator considered in this work. Due to the double integral action, the steady-state response of the system in Fig. 1(c), subject to stability, is exactly equal to the ramp reference \( R(s) = \frac{a_0}{s^2} \). Hence, if we premultiply the steady-state part of the state response (3) by \( C_c \) to obtain the steady-state output, the result should be the same as the
reference and we obtain:

\[- \frac{C_c (sI + A_c) A_c^{-2} B_c}{(1 + D_c)} = 1, \forall s\]  \hspace{1cm} (14)

The above equality, which holds for all values of $s$, establishes that the following relationships are satisfied:

\[- \frac{C_c A_c^{-1} B_c}{1 + D_c} = 1, \quad C_c A_c^{-2} B_c = 0.\]  \hspace{1cm} (15)

According to Section IV, the zero input response of the compensator to initial condition (5) is also exactly equal to the ramp signal $R(s) = \frac{1}{s^2}$. The zero input response can also be expressed as $C_c (sI - A_c)^{-1} x_c(0)$. Thus we may conclude that the following equality holds for all values of $s$:

\[- \frac{C_c (sI - A_c)^{-1} A_c^{-2} B_c}{1 + D_c} = \frac{1}{s^2}, \forall s.\]  \hspace{1cm} (16)

**Remark:** Although we assumed that the closed-loop matrix $A_c = A_c - \frac{B_c C_c}{1 + D_c}$ is stable, the only conditions needed for Equations (14)-(16) to hold are that $A_c$ has at least two poles at the origin and $D_c \neq -1$. Considering the compensator as a general SISO LTI state-space system with more than one integrator and a high frequency gain not equal to $-1$, we conclude that Equations (15) and (16) are valid if $A_c$, $B_c$, $C_c$, and $D_c$ assume state-space matrices of any SISO LTI dynamics satisfying the foregoing condition.

**VII. Conclusion**

The transient performance of signal transformation approach (STA) can be significantly improved by suitable adjustment of compensator’s initial states. For triangular references, it was shown that the improved STA method is equivalent to impulsive state multiplication (ISM) when at least two integral actions are included in the compensator. The proposed method also improves the transient performance of STA method for arbitrary references, where ISM is not applicable. Some equations were also obtained that LTI SISO systems satisfy, if they include at least two poles at the origin and their feedthrough terms are not $-1$.

**References**


