Iterative Ensemble Control Synthesis for Bilinear Systems

Anatoly Zlotnik† and Jr-Shin Li‡

Abstract—An emerging area in mathematical control theory called Ensemble Control constitutes a class of problems that require the manipulation of an uncountably infinite collection of structurally identical dynamical systems, which are indexed by a parameter set, by applying a common open-loop control. Our investigation is motivated by compelling engineering problems in areas including quantum control and sensorless robotic manipulation that involve ensembles of bilinear systems for which analytical control laws are infeasible in practice or do not exist. While controllability of such systems has been investigated, constructive control design methods remain elusive. We introduce an iterative fixed-point method for optimization-free synthesis of ensemble controls for these systems. At each step, the bilinear ensemble system is approximated by a time-varying linear ensemble system, and the next control iterate is synthesized using a method for linear systems based on the singular value decomposition (SVD). The procedure converges in the given examples to a control function that accomplishes the desired state transfer.

I. INTRODUCTION

All scientific experiments and engineering applications are complicated by uncertainty or variation in system model parameters, for which known control techniques are often unable to completely compensate. This issue is especially challenging when the control task must be accomplished without feedback, whether the control function must transfer a single control system between states of interest without sensitivity to an uncertain parameter set, or steer a possibly uncountable collection of structurally identical systems with variation in common parameters between states that may also depend on the parameters. The latter category of problems is motivated by practical factors that arise in nuclear magnetic resonance (NMR) spectroscopy and imaging (MRI) as well as the broader field of quantum control, and has given rise to a new area of mathematical control theory called ensemble control [1]. These rapidly progressing technologies require the manipulation of very large ensembles of quantum systems, e.g., on the order of Avogadro’s number \(6 \times 10^{23}\), whose states cannot be measured during the transfer, and whose dynamics are subject to dispersion in parameters such as frequency. The performance of the necessary controls must be insensitive to parameter variation across the ensemble, as well as to inhomogeneity in the applied radiofrequency (RF) control field [2], [3], [4]. Furthermore, the techniques developed for ensemble control can be applied to robust sensorless manipulation, which is of interest in machining and manipulation tasks where feedback is unavailable and the performance must be immune to uncertainty in model parameters [5]. Such problems are largely neglected in the literature on robust control, which is oriented on the use of feedback [6], [7], [8].

The examination of ensemble control begins with the notion of ensemble controllability, which determines the existence of controls that achieve various types of state transfers for an ensemble system of interest. It has been shown that a bilinear system evolving on \(\text{SO}(3)\) called the Bloch system, which represents the transition over time of the bulk magnetization of a sample of nuclear spins, is ensemble controllable [1], [3]. More fundamentally, the necessary and sufficient conditions for ensemble controllability of finite-dimensional time-varying linear systems have been derived [9], and extended to stochastic systems [10]. These conditions depend on the singular system of the operator that characterizes the system dynamics, which can in turn be used to represent the minimum norm control that accomplishes the transfer as an infinite sum of weighted eigenfunctions [11].

The controllability conditions for general nonlinear ensemble control problems are unknown, and analytical control design methods remain a challenging problem, although analytical solutions exist for a few specific systems [12]. Our goal in this paper is to develop constructive numerical methods for synthesizing optimal ensemble controls, rather than to focus on issues of controllability. A recent optimization-based approach is a pseudospectral method that translates an optimal control problem in function space into a finite discrete nonlinear programming problem [13], [14], [15]. This method has been effective for a variety of ensemble control problems, but relies heavily on nonlinear programming techniques due to its inherent computational complexity [16]. Therefore, a need exists for optimization-free, stable, and computationally efficient numerical methods for synthesizing ensemble controls that can accomplish diverse state transfers for a variety of systems. We have previously introduced a method based on the SVD that constructs minimum norm ensemble controls for finite-dimensional time-varying linear systems [17], as well as stochastic linear systems [10]. In this approach the singular system of the Fredholm integral operator of the first kind that characterizes the dynamics of a linear ensemble system is approximated using the SVD, and the optimal control is approximated as a weighted sum of singular vectors.

Although the former approach is effective for linear ensemble systems, the most compelling practical ensemble control problems in quantum control and robotics involve
systems of bilinear form. A natural approach to such problems is to apply methods for linear systems iteratively to linearized bilinear systems to produce a scheme for successive approximations. Such techniques exist for feedback control design for finite-time, free endpoint problems with quadratic cost functionals and bilinear dynamic constraints, and are based on iterative solution of Riccati-type equations [18], [19]. In contrast, we are interested in designing open-loop control solutions to fixed-endpoint problems for bilinear systems such that the control performance is insensitive to system parameter variation, which requires a novel approach.

In this paper, we present an iterative fixed-point method for directly synthesizing ensemble controls for bilinear systems. At each step of the iteration, the bilinear ensemble system is approximated by a time-varying linear ensemble system, and the SVD-based method is used to synthesize the control used to generate the state trajectory estimate, about which the system is linearized in the following iteration. In Section II, we review preliminary results on the SVD-based method for ensemble control of linear systems. In Section III, we present our main result, namely, a fixed-point iteration scheme for bilinear systems, as well as an ensemble consisting of pairs of Bloch systems with time-varying coupling, and examine the issues of numerical stability and convergence. It is shown that the iteration converges for these systems to a control that accomplishes the desired state transfer with sufficiently low error. We proceed with a discussion of these results and prospects for future work, followed by concluding remarks in Section V.

II. PRELIMINARY RESULTS

The goal of ensemble control is to concurrently steer a continuum of dynamical systems, which are governed by internal and external dynamics that depend on a parameter varying over an indexing set, between states of interest by applying a common open-loop control to each system. In this section, we review the basic results that enable ensemble control synthesis for finite-dimensional time-varying linear systems.

Consider a family of dynamical systems indexed by a parameter \( \beta \) varying over a compact set \( K \), given by

\[
\dot{X}(t, \beta) = A(t, \beta)X(t, \beta) + B(t, \beta)u(t),
\]

where \( A(t, \beta) \in \mathbb{R}^{n \times n} \) and \( B(t, \beta) \in \mathbb{R}^{n \times m} \) have elements that are real \( L_\infty \) and \( L_2 \) functions, respectively, defined on a compact set \( D = [0, T] \times K \), and are denoted \( A \in L_\infty^{n \times n}(D) \) and \( B \in L_2^{n \times m}(D) \). The ensemble controllability conditions for the system (1) depend on the existence of an open-loop control \( u : [0, T] \to U \) that can steer the instantaneous state of the ensemble \( X(t, \cdot) : K \to M \) between any points of interest in the Hilbert space of functions on \( K \). Let \( H_T = L_2^m[0, T] \) denote the set of \( m \)-tuples, whose elements are real vector-valued square-integrable functions defined on \( 0 \leq t \leq T \), with an inner product defined by

\[
\langle g, h \rangle_T = \int_0^T g'(t)h(t)dt,
\]

where \( ^t \) denotes the transpose. Similarly, let \( H_K = L_2^m(K) \) be equipped with an inner product

\[
\langle p, q \rangle_K = \int_K p'(\beta)q(\beta)d\mu(\beta),
\]

where \( \mu \) is the Lebesgue measure. With well-defined addition and scalar multiplication, \( H_T \) and \( H_K \) are separable Hilbert spaces, where \( ||\cdot||_T \) and \( ||\cdot||_K \) denote the respective induced norms.

Definition 1: (Ensemble controllability) [9] We say that the family (1) is ensemble controllable on the function space \( H_K \) if for all \( \varepsilon > 0 \), and all \( X_0, X_F \in H_K \), there exists \( T > 0 \) and an open loop piecewise-continuous control \( u \in H_T \), such that starting from \( X(0, \beta) = X_0(\beta) \), the final state \( X(T, \beta) = X_T(\beta) \) satisfies \( ||X_T(\beta) - X_F(\beta)|| < \varepsilon \).

Note that the acceptable range of \( T \in (0, \infty) \) may depend on \( \varepsilon, K, \) and \( U \). Necessary and sufficient conditions have been determined for the ensemble controllability of finite-dimensional time-varying linear systems, and are based on the Fredholm integral operator that characterizes the system dynamics [11]. In particular, applying the variation of parameters formula to (1) yields

\[
X(T, \beta) = \Phi(T, 0, \beta)X_0(\beta) + \int_0^T \Phi(T, \tau, \beta)B(\tau, \beta)u(\tau)d\tau,
\]

where \( \Phi(t, 0, \beta) \) is the transition matrix for the system \( \dot{X}(t, \beta) = A(t, \beta)X(t, \beta) \). Setting \( \dot{X}(T, \beta) = X_F(\beta) \) results in the integral operator equation

\[
(卢)(\beta) = \int_0^T \Phi(0, \tau, \beta)B(\tau, \beta)u(\tau)d\tau = \xi(\beta),
\]

where \( \xi(\beta) = \Phi(0, T, \beta)X_F(\beta) - X_0(\beta) \). Ensemble control of linear systems can be reduced to the solution of this equation. A spectral decomposition, called the singular system, of the operator \( L \) is used to produce an infinite eigenfunction series expansion for \( u \in H_T \) of minimum norm that satisfies (5) with sufficient accuracy.

Definition 2: Singular System [20]: Let \( Y \) and \( Z \) be Hilbert spaces and \( L : Y \to Z \) be a compact operator. If \( (\sigma_n^2, \nu_n) \) and \( (\sigma_n^2, \mu_n) \) are eigensystems of \( LL^* \) and \( L^*L \), respectively, namely, \( LL^*\nu_n = \sigma_n^2\nu_n, \mu_n \in Y \), for \( \sigma_n > 0 \), and the two systems are related by the equations \( L\mu_n = \sigma_n\nu_n \) and \( L^*\nu_n = \sigma_n\mu_n \), we say that \( (\sigma_n, \mu_n, \nu_n) \) is a singular system of \( L \).

Suppose that \( (\sigma_n, \mu_n, \nu_n) \) is a singular system of \( L \) as in (5), which is compact [9]. The necessary and sufficient conditions for ensemble controllability of the system (1) are

\[
(i) \sum_{n=1}^{\infty} \frac{|\langle \xi, \nu_n \rangle_K|^2}{\sigma_n^2} < \infty, \quad (ii) \xi \in \overline{R(L)},
\]

3485
where $\mathcal{R}(L)$ denotes the closure of the range space of $L$. In addition, the control law
\[
u = \sum_{n=1}^{\infty} \frac{\langle \xi, \nu_n \rangle K}{\sigma_n} \mu_n
\] (7)
satisfies $\langle u, u \rangle_T \leq \langle u_0, u_0 \rangle_T$ for all $u_0 \in \mathcal{U}$ and $u_0 \neq u$, where $\mathcal{U} = \{ v \mid L v = \xi \}$. For further details we refer the reader to the original work on this subject [9], [11].

Given a numerical approximation to the singular system $(\sigma_n, \mu_n, \nu_n)$ for the operator $L$ of an ensemble controllable system, the series (7) can be truncated to synthesize an approximation to $\nu$ that results in $||X_T - X_F||_K < \varepsilon$ as desired. The singular system in Definition 2 is the infinite-dimensional analogue of the singular value decomposition (SVD) for matrices [21], hence a natural approach is to approximate the action of the compact operator $L : \mathcal{H}_T \rightarrow \mathcal{H}_K$ in equation (5) on a function $g \in \mathcal{H}_T$ by a matrix acting on a vector of sampled values of $g$. The SVD can then approximate the singular system of the operator, and thereby also the minimum norm ensemble control $u$, as follows.

Let $\{ \beta_j \}$ be a finite collection of points that are distributed uniformly throughout $T$ and indexed by $j = 0, 1, 2, \ldots, P - 1$, and let $\{ t_k \}$ be a collection of points that linearly interpolate the time domain $[0, T]$ for $k = 0, 1, \ldots, N - 1$, including endpoints, with $t_k - t_{k-1} = \delta$. Using this discretization, we make the quadrature approximation
\[
(L g)(\beta) = \int_0^T \Phi(0, t, \beta) B(t, \beta) g(t) dt
= \sum_{k=1}^{N-1} \left( \int_{t_k}^{t_{k+1}} \Phi(0, t, \beta) B(t, \beta) g(t) dt \right)
\approx \sum_{k=1}^{N-1} \delta \Phi(0, t_k, \beta) B(t_k, \beta) g(t_k).
\] (8)
The action of the operator $L$ on a function $g \in \mathcal{H}_T$ is approximated by the action of a block matrix $W \in \mathbb{R}^{np \times mN}$, with $n \times m$ blocks $W_{jk} = \delta \Phi(0, t_k, \beta_j) B(t_k, \beta_j)$, on a vector $\tilde{g} \in \mathbb{R}^{mN}$, with $N$ blocks $\tilde{g}_k = g(t_k)$ of dimension $m \times 1$. If the SVD of this matrix is $W = U \Sigma V^T$, and $\tilde{u}_j$ and $\tilde{v}_j$ are columns of $U$ and $V$, respectively, corresponding to the singular value $s_j$, then $W W' \tilde{u}_j = s_j^2 \tilde{u}_j$ and $W' \tilde{v}_k = s_k^2 \tilde{v}_k$. Therefore the SVD $(s_j, \tilde{u}_j, \tilde{v}_j)$ of the matrix $W$ approximates the singular system $(\sigma_j, \mu_j, \nu_j)$ of the operator $L$, where $\tilde{v}_j$ and $\tilde{u}_j$ are discretizations of $\mu_j$ and $\nu_j$, respectively. Now suppose that $\xi \in \mathbb{R}^{np}$ is given by $\xi_j = \xi(\beta_j)$ for a function $\xi \in \mathcal{H}_K$. Then the minimum norm solution $\hat{g}$ that satisfies $W \hat{g} = \hat{\xi}$ is given by $\hat{g}^* = W' \hat{\xi}$ where $W' \hat{\xi} = \hat{\xi}$ [22], so by basic properties of the SVD
\[
\hat{g}^* = \sum_{j=1}^{mq} \frac{\xi_j}{s_j} \tilde{u}_j \tilde{v}_j.
\] (9)
The components of the synthesized minimum norm control $\hat{u}^* = (\hat{u}_1^*, \ldots, \hat{u}_m^*)$ are therefore given by
\[
\hat{u}_k^* = \sum_{j=1}^{q} \frac{\xi_j}{s_k + m(j-1)} \tilde{u}_k^{m(j-1)}. \tag{10}
\]
The time and parameter discretizations $N$ and $P$ must be chosen such that $nP \leq mN$, so the pair $(W, \xi)$ represents an underdetermined system and therefore a minimum norm and not a least squares approximation problem. The number $q$ of eigenfunctions used in the approximation is limited by $q \leq P$. The major source of numerical error arises from computation of the SVD. In order to prevent errors from dominating the synthesized control, we choose $q$ in (9) such that the corresponding first and last singular values used satisfy $s_1/s_{mq} < 10^4$. For a discussion of numerical issues and examples of this technique, we refer the reader to our previous work [17].

III. MAIN RESULTS

Many important problems in control engineering with applications to robotics, medicine, and other fields require the manipulation of bilinear systems [13], [23], [24], [4], [25]. We focus here on time-invariant systems because our approach is easily generalized to the non-autonomous case. Consider the system
\[
\dot{X}(t, \beta) = A(\beta) X(t, \beta) + B(\beta) u(t), \tag{11}
\]
where $u = (u_1, \ldots, u_m)'$, $X = X(t, \beta)$, $A(\beta) \in \mathbb{R}^{n \times n}$ has elements that are real $L_\infty$ functions defined on a compact set $K$, and $B_1(\beta), \ldots, B_m(\beta) \in \mathbb{R}^{n \times m}$ and $B(\beta) \in \mathbb{R}^{n \times m}$ have elements that are real $L_2$ functions also defined on $K$. We denote $A \in L_\infty^{n \times n}(K), B_1, \ldots, B_m \in L_2^{n \times m}(K)$, and $B_0 \in L_2^{n \times m}(K)$. Ensemble controllability for the system (11) follows Definition 1. This system can be rewritten as
\[
\dot{X}(t, \beta) = A(\beta) X(t, \beta) + \sum_{j=1}^{n} x_j B_j(\beta) + B_0(\beta) \tag{12}
\]
where $X = (x_1, \ldots, x_n)'$ and $B_1, \ldots, B_n \in L_2^{n \times m}(K)$. Hence the one-step ensemble control synthesis scheme for linear systems described above can be extended to produce an iterative fixed-point method that constructs ensemble controls for bilinear systems by replacing at each iteration the autonomous inhomogeneous component with a time-varying homogeneous approximation involving the output of the previous step. Suppose that the ensemble (12) is controllable and is to be guided from $X(0, \beta) = X_0(\beta)$ to $X(T, \beta) = X_F(\beta)$. Given an estimate $(X^\alpha, u^\alpha)$ of the trajectory-control pair where $X^\alpha = (x^\alpha_1, \ldots, x^\alpha_n)'$, $u^\alpha = (u^\alpha_1, \ldots, u^\alpha_m)'$ and $\alpha$ is an iteration index, substituting $x_j(\beta, t) = x^\alpha_j(t, \beta)$ for $j = 1, \ldots, n$ in (12) yields the time-varying linear system
\[
\dot{X}(t, \beta) = A(\beta) X(t, \beta) + B^\alpha(t, \beta) u(t), \tag{13}
\]
where
\[
B^\alpha(t, \beta) = \sum_{j=1}^{n} x_j^\alpha(t, \beta) B_j(\beta) + B_0(\beta). \tag{14}
\]
The SVD-based method in Section II is used to synthesize a control $u_{\alpha+1}$ that solves the integral operator equation

$$(L^\alpha u)(\beta) = \int_0^T \Phi(0, \sigma, \beta)B^\alpha(\sigma, \beta)u(\sigma)d\sigma = \xi(\beta), \quad (15)$$

where $\xi(\beta) = \Phi(0, T, \beta)X_F(\beta) - X_0(\beta)$ characterizes the desired state transfer for the linearized system (13). This control is applied to (12) with $X(0, \beta) = X_0(\beta)$ to produce a new trajectory $X^{\alpha+1}$, and the next iterate $(X^{\alpha+1}, u^{\alpha+1})$, and the process is repeated until $E(\alpha) := \|u^{\alpha+1} - u^\alpha\|/\|u^\alpha\| < \gamma$ where $\gamma$ is a relative error tolerance.

In order to avoid the accumulation of numerical errors that cause the iteration to diverge, the entire procedure must be conducted using the discretization scheme (8) in Section II. Specifically, we define the vector $\hat{\xi} \in \mathbb{R}^{n \cdot P}$ with $n \times 1$ blocks $\hat{\xi}_j = \xi(\beta_j)$ that characterizes the desired state transfer. At the next iteration, a vector $\hat{u}^{\alpha+1} \in \mathbb{R}^{m \cdot N}$ with $m \times 1$ blocks $\hat{u}^{\alpha+1}_k = u(t_k)$ constitutes a discretization of the control. The action of $L^\alpha$ on a function $g \in \mathcal{H}_F$ is approximated by the action of a block matrix $W^\alpha \in \mathbb{R}^{n \cdot P \times m \cdot N}$, with $n \times m$ blocks $W^\alpha_{jk} = \delta \Phi(0, t_k, \beta_j)B^\alpha(t_k, \beta_j)$, on a vector $\hat{g} \in \mathbb{R}^{m \cdot N}$, with $N$ blocks $\hat{g}_k = g(t_k)$ of dimension $m \times 1$. Following the steps in Section II, the minimum norm solution $\hat{u}^{\alpha+1}$ that satisfies $W^\alpha \hat{u}^{\alpha+1} = \hat{\xi}$ is given by

$$\hat{u}^{\alpha+1} = \sum_{j=1}^{m} \frac{\hat{\xi}_j}{s_j} \hat{u}_j \quad (16)$$

where $(s_j, \hat{u}_j, \hat{\xi}_j)$ is the SVD of $W^\alpha$. We then require that the discrete approximation $X^{\alpha+1}$ of $X^{\alpha+1}$ is made on the same grid $\{\beta_j\} \times \{t_k\}$ that is used to synthesize $W^\alpha$ as the discrete approximation of $L^\alpha$. In particular, we define the quadrature approximation $X^{\alpha+1}_{jk} \approx X^{\alpha+1}(t_k, \beta_j)$ by

$$X^{\alpha+1}_{jk} = \Phi(t_k, 0, \beta_j)[X_0(\beta_j) + \mathcal{V}^\alpha_{jk}U^0_{\beta_j}] \quad (17)$$

where $\mathcal{V}^\alpha_{jk} = [W^\alpha_{j1} \ W^\alpha_{j2} \ldots \ W^\alpha_{jn}] \in \mathbb{R}^{n \times mk}$ and $U^0_{\beta_j} = [\hat{u}^0_{\beta_j}^1 \hat{u}^0_{\beta_j}^2 \ldots \hat{u}^0_{\beta_j}^m] \in \mathbb{R}^{mk \times 1}$.

If a control $u^* = (u^*_1, \ldots, u^*_m)'$ exists that accomplishes this transfer via the trajectory $X^* = (x^*_1, \ldots, x^*_n)'$, it follows that $u^*$ is a solution to the integral operator (5) with

$$B(t, \beta) = \sum_{j=1}^{n} x^*_j(t, \beta)B_1(\beta) + B_0(\beta). \quad (18)$$

In certain cases, the successive approximations $(\hat{X}^\alpha, \hat{u}^\alpha)$ converge to a fixed point $(\hat{X}^*, \hat{u}^*)$, which approximates a trajectory-control pair $(X^*, u^*)$ that accomplishes the desired transfer. When $B_i \equiv 0$ for $i = 1, \ldots, m$, the system (12) is linear in $X$ and the iteration terminates after a single step.

IV. EXAMPLES AND DISCUSSION

The canonical example of a bilinear control system of particular interest is the Bloch system [24], which describes the evolution of the bulk magnetization of a sample of nuclear spins immersed in a magnetic field, and which is given by

$$\dot{X} = \begin{bmatrix} 0 & -\omega & \varepsilon u_1 \\ \omega & 0 & -\varepsilon u_2 \\ -\varepsilon u_1 & \varepsilon u_2 & 0 \end{bmatrix} X, \quad (19)$$

where $X = (x_1, x_2, x_3)'$ is a unit vector in $\mathbb{R}^3$, $u = (u_1, u_2)'$ is a vector of control parameters, and $\omega \in [-\mu, \mu]$ and $\varepsilon \in [1 - \delta, 1 + \delta]$ represent dispersion in Larmor frequency and radio frequency (RF) inhomogeneity, respectively, for $\mu > 0$ and $\delta \in (0, 1)$. This system can be re-written as

$$\dot{X} = \begin{bmatrix} 0 & -\omega & 0 \\ \omega & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} X + \varepsilon \begin{bmatrix} x_3 \\ 0 \\ -x_1 \end{bmatrix} u. \quad (20)$$

It is a well established result that the Bloch system is ensemble controllable with respect to both $\omega$ and $\varepsilon$ [1], [2], [3], [26]. However, when formulated as in (13), the system (20) is of the form $\dot{X} = A(\omega)X + \varepsilon B^\alpha(t, \omega)u$, and applying the variation of parameters formula makes it clear that it is not ensemble controllable with respect to $\varepsilon$. This can also be shown using a Lie algebraic argument ([11], Example 1). This observation highlights a significant limitation of our
method, namely, that the linearization of the bilinear system must be ensemble controllable with respect to all uncertain parameters. We will let $\varepsilon \equiv 1$ be a constant, and apply the technique to successfully compensate for variation in the parameter $\omega$ to produce a broadband excitation pulse.

We now examine the synthesis of an ensemble control that achieves a $90^\circ$ transfer of the state from $X_0(\beta) = (0, 0, 1, 1)\,'$ to $X_F(\beta) = (1, 0, 0, 0)\,'$ for the Bloch system, which is a canonical step in NMR. The time interval chosen is $T = 1$. Figure 1 compares trajectories sampled from the ensemble with $\mu = 8$, which are produced when the ensemble control constructed for $\mu = 5$ is applied, with those produced when the optimal control for the nominal system with $\omega = 0$ is applied. It is noteworthy that the control performance for this broadband compensation pulse exceeds the specification. Figure 2 displays ensemble controls for several values of $\mu$, which achieve various broadband excitation profiles. In each case the transfer is accomplished with $v \equiv 0$, using only the control $u$. A plot of the relative change in successive control iterates $u^n$ and a log-log plot of the terminal error as a function of discretization $N$ are shown for several values of $\mu$ as well. These numerical convergence results indicate that the method is accurate and consistent for this example.

The method in Section III is also effective for controlling time-varying systems with multiple uncertain parameters. Consider an ensemble consisting of pairs of Bloch systems with time-varying coupling, with system dynamics given by

$$\dot{X} = \begin{bmatrix} 0 & -\xi & u_1 & 0 & -\rho t & 0 \\
\xi & 0 & -u_2 & \rho t & 0 & 0 \\
-u_1 & u_2 & 0 & 0 & 0 & 0 \\
0 & \rho (1-t) & 0 & 0 & -\lambda & u_1 \\
-\rho (1-t) & 0 & 0 & -u_1 & u_2 & 0 \\
0 & 0 & 0 & -u_1 & u_2 & 0 \end{bmatrix} X$$

(21)

where $X = (x_1, x_2, x_3, x_4, x_5, x_6)\,'$, and $u = (u_1, u_2)\,'$ is a vector of control parameters. Given initial and target states $X_0(\beta) = (0, 0, 1, 0, 0, 1)\,'$ and $X_F(\beta) = (1, 0, 0, 1, 0, 0)\,'$ and a time horizon $T = 1$ and discretization $N = 5000$, we find an ensemble control for interaction parameter $\rho = 0.1$ and frequency dispersions $\xi \in [-0.2, 0.2]$ and $\lambda \in [0.8, 1.2]$, and $P = 25$ samples. The resulting trajectories on $SO(3) \times SO(3)$, the ensemble control, and the log of the relative change $E(\alpha)$ between successive control iterates is shown in Figure 3.

![Figure 2](image1.png)

![Figure 3](image2.png)
Several limitations to our approach suggest compelling theoretical research directions. First, the necessary and sufficient conditions for controllability of general ensemble systems remains an open problem, and therefore it cannot always be determined a priori for which bilinear systems and state transfers in the space \( \mathcal{H}_K \) ensemble controls exist. In addition, the convergence properties of the fixed-point iteration described in Section III are unknown. In our future work, we plan to consider the contractive properties of the integral operator equations to derive theorems characterizing the convergence properties of our algorithm, and also determine whether the fixed-point is in fact a minimum norm solution to the ensemble control problem when it exists.

V. CONCLUSIONS

We have introduced a computationally efficient iterative fixed-point method for synthesizing ensemble controls for bilinear systems. By basing our approach on successive linearizations and control iterates synthesized using the singular value decomposition (SVD), we have created a consistent, optimization-free algorithm that leverages the efficiency of a widely-used numerical routine. We have demonstrated its effectiveness for designing controls that accomplish state transfers for certain classes of bilinear ensemble systems, and have conducted multiple simulations to illustrate the sensitivity of the method with respect to the chosen discretization parameters and time horizon. This work provides a novel numerical method for designing excitations for the manipulation of quantum spin systems. Further development of this technique will accelerate the quickly broadening scope of ensemble control theory by contributing powerful new tools for solving forefront problems in numerous fields from neuroscience to quantum computing. Extension of our approach to a universal algorithm for ensemble control of general nonlinear systems, as well as systems involving constraints or stochastic components, will be valuable to applications in fields including chemistry, robotics, and medicine.

REFERENCES