On Nash equilibria in duopolistic power markets subject to make-whole uplift

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Abstract—We consider a duopolistic power market in which firms bid their offer functions while abiding by capacity constraints. Minimal cost allocation decisions and marginal cost prices are determined through the solution of two optimization problems, the first a mixed-integer linear program (the unit commitment problem) and the second a related linear program (the economic dispatch problem) that has constraints derived from the solution of the commitment problem. Traditional marginal-cost pricing mechanisms have relied on using the dual variables associated with the supply-demand constraints from the dispatch problem. Unfortunately, such mechanisms fail to cover the cost of start-up and energy costs faced by dispatched generators. In existing markets, one attempt at a remedy is by providing every dispatched generator with an ex-post out of market settlement called a make-whole uplift. Given the presence of such an uplift mechanism, this paper seeks to characterize the nature of equilibria in duopolistic settings.

The two main conclusions of our analysis are: 1) Pure Nash equilibria are identified when one firm can supply the entire demand; 2) Otherwise, it is shown that mixed-strategy Nash equilibria exist, and these equilibria can be characterized by two coupled differential equations. These results provide a framework for further analysis of electricity markets, and in particular the true value of make-whole uplift payments in a strategic setting.

I. INTRODUCTION

Currently, most organized electricity markets in North America allow for multi-part offers that include marginal costs and start-up costs [1]–[6]. These offers are considered as part of the determination of production schedules. A salient characteristic of start-up costs is that these costs depend only on whether a generator is on or off (commitment) and is independent of the generating level. In some markets, payoff functions completely characterize the benefit of a strategy, given specified adversarial decisions. However, in power markets, this payoff is a consequence of an intermediate allocation and pricing decision [7]. Such a decision is usually derived from solving two sets of optimization problems. Of these, the first is a mixed-integer linear program that prescribes which generators should be on during dispatch. Next, given these on/off decisions, the precise dispatch levels of the committed generators is obtained by solving a linear program (called the economic dispatch problem). Additionally, the Lagrange multiplier associated with the nodal supply-demand constraint in the dispatch problem provides the locational marginal or the marginal-cost price at that node.

Start-up costs are commitment-dependent and are ignored when computing these prices. As a consequence, the payments collected through the auction may be insufficient to compensate the generators. To overcome this problem, uplift payment mechanisms have been introduced, through which additional side payments are made to the generators in recognition of the costs incurred due to commitment decisions.

Strategic behavior in electricity markets with only marginal-cost offers has been extensively studied. A comprehensive survey on the computational methods to calculate optimal offering strategy can be found in [8]. Ignoring the commitment decisions, Fabra et al. investigate the pure-strategy and mixed-strategy equilibria in a duopoly under uniform-price and pay-as-bid auctions, assuming the capacity is perfectly divisible and the cost functions are linear [9]. The application of the pay-as-bid auction is also studied through an experimental method in [10]. Supply function equilibrium (SFE) [11] approaches have also been applied and allow more realistic considerations of market rules. Green and Newbery [12] apply an SFE model to the England and Wales market and obtain consistent results with the empirical data. Players with heterogeneous cost functions and capacity constraints are studied in [13], which also studies the stability of the SFEs. The optimal offering strategy with uncertain demand and competitor’s behavior is studied in [14], [15]. Although the SFE approach provides more sophisticated models, to the best of our knowledge, none of the existing literature considers commitment decisions within a strategic regime.

This paper makes two sets of contributions in the context of duopolistic power markets subject to marginal-cost pricing with make-whole uplift:

(i) In the low-demand regime in which either firm can meet the entire demand, the existence of pure-strategy Nash equilibria is established, and their form is characterized.

(ii) In the high demand regime, both generators must be dispatched to meet demand. In this case it is shown that a pure-strategy Nash equilibrium cannot exist. However, the existence of mixed-strategy Nash equilibria is proven.

The model presented in this paper could be viewed as an extension of the uniform-price duopoly model studied in [9] by including the start-up costs and the associated uplift payment mechanisms. However, the impacts of start-up costs...
are profound. As Fabra et al. show, without the consideration of start-up costs, pure-strategy equilibria always exists under marginal-cost pricing, which is not the case in our model.

The remainder of this paper contains a discussion of other sections and is organized as follows. In Section II, we present a mathematical model for a duopolistic power market under two distinct pricing and uplift mechanisms. We devote Section III to characterizing prices, uplift payments and payoff functions. In Section IV, we characterize the form of equilibria that emerge under the two pricing mechanisms and in low and high demand regimes. The paper concludes with some remarks and final thoughts in Section V.

II. Model

Consider a duopoly in which two generators with capacities $K_1$ and $K_2$ compete to serve an inelastic demand $d$. It is assumed that capacity is perfectly divisible and, for purposes of simplicity, $K_1 = K_2 = K$. Generator $i$ could be either offline, denoted by $u_i = 0$, or online, as specified by $u_i = 1$. In the instance of the latter, the $i$th generator can produce $k_i$ MWh during a given hour, where $k_i \in [0, K]$. Note the generator can provide positive electricity output only if it is online. Suppose generator $i$’s cost of generating $k_i$ MWh is given by an affine function $s_i + c_i k_i$, where $s_i$ and $c_i$ are positive parameters that denote the start-up cost and the marginal cost, respectively. Without loss of generality, we assume that $s_1 \leq s_2$.

A. Firm strategies

In the duopolistic regime, the action or strategy of each generator is given by a single marginal cost offer $p_i \in [0, \rho_{mc}^\text{max}]$ for its entire capacity, which does not necessarily reflect its true cost, $i \in \{1, 2\}$. Note that $\rho_{mc}^\text{max}$ denotes the price cap which is assumed to be greater than either $c_1$ or $c_2$. We assume that both firms are rational profit-maximizing firms and have access to their payoffs as well as that of their rivals.

B. Allocation mechanism

We consider an electricity market organized as a sealed-bid auction. The auction allocation and payment rules specify which bids are accepted, fully or partially, and at what price the transaction is settled. This allocation is usually determined by solving the unit commitment problem and the economic dispatch problem. Furthermore, we assume that the auctioneer determines the allocation to minimize the total “as-bid” cost based on not only the marginal cost offers, but also the start-up costs.

Definition 1 (Allocation Model). The auctioneer determines allocation by solving the mixed-integer linear program:

$$\begin{align*}
\min_{u_1, k_1, u_2, k_2} & \quad u_1 (s_1 + p_1 k_1) + u_2 (s_2 + p_2 k_2) \\
\text{subject to} & \quad u_1 k_1 + u_2 k_2 = d, \\
& \quad u_i \in \{0, 1\}, \quad i = 1, 2, \\
& \quad k_i \in [0, K], \quad i = 1, 2.
\end{align*}$$

In effect, this uplift ensures the successful firms to get at least their offered prices. Given the allocation and payment rules, we define the player’s problem under marginal-cost pricing and make-whole uplift as follow.

Definition 4 (Player’s Problem Under Marginal-Cost Pricing). Suppose generator $i$ is given by a single marginal cost offer $p_i$, where $p_i \in [0, \rho_{mc}^\text{max}]$ and $\rho_{mc}$ is determined by solving the unit commitment problem and the economic dispatch problem.

Such a model is prototypical of currently operational day-ahead markets in North America. In practice, other factors, such as a more complicated cost structure and physical/operational constraints, may result in discontinuities and lumpiness in cost. These impacts are similar to those arising from the presence of start-up costs in our model, into the capacity markets and generation expansion problem, in which the efforts to obtain permits and the huge initial investment to build new generators can be viewed as the “start-up” cost to participate in electricity markets.

It is possible that problem (1) yields multiple solutions. We adopt a tie-breaking rule that favors the allocations with lower actual cost; if a tie still persists, we dispatch player 1 first. Such rules differ from the commonly used pro-rata tie-breaking rules employed elsewhere, because with positive start-up cost, a convex combination of multiple solutions, in general, is not a solution. We adopt such rules solely for identifying the existence of pure-strategy equilibria. When studying the mixed-strategy equilibria, the probability of a tie is zero and the tie-breaking rule has no impacts on market outcomes.

Suppose the solution to (1) is denoted by $\{u_1^*, k_1^*, u_2^*, k_2^*\}$, based on which a payment scheme is considered and comprises of a pricing and an uplift rule.

C. Payment rule

We consider a payment scheme based on marginal-cost pricing with a make-whole uplift. This scheme is currently adopted in most day-ahead markets in North America.

Definition 2 (Marginal-Cost Price). The marginal-cost price is determined by the optimal dual variable $\rho_{mc}$ associated with the supply-demand balance constraint in the following linear programming problem:

$$\begin{align*}
\min_{k_1, k_2} & \quad u_1^* (s_1 + p_1 k_1) + u_2^* (s_2 + p_2 k_2) \\
\text{subject to} & \quad u_1^* k_1 + u_2^* k_2 = d, \\
& \quad k_i \in [0, K], \quad i = 1, 2.
\end{align*}$$

If a pricing rule leads to multiple solutions, we select the minimal one. Note, in the above linear program, $u_1^* s_1$ and $u_2^* s_2$ are constants, and have no impact on the problem. Therefore, marginal-cost pricing may be insufficient in reimbursing the start-up cost. In such cases, a “make-whole” uplift payment is introduced to compensate the generators.

Definition 3 (Make-whole Uplift). The make-whole uplift payment $w_i$ to generator $i$ is defined as $w_i$, where

$$w_i \triangleq \max \{0, u_i^* (s_i + p_i k_i^* - \rho_{mc} k_i^*)\}.$$
and Make-Whole Uplift).

\[
\max_{p_i \in [0, p_{\text{max}}]} \left[ \rho_{mc} u_i^* k_i^* + w_i - u_i^* (s_i + p_i k_i^*) \right]
\]

subject to \( \{u_i^*, k_i^*, u_2^*, k_2^*\} \) solves (1), \( \rho_{mc} \) solves (2).

The resulting pure-strategy Nash equilibrium is given by the tuple of offers \( \{p_i^*\}_{i=1}^2 \), where

\( p_i^* \) solves \( G_i(p_{-i}) \), \( i = 1, 2 \),

where \( p_{-i} \triangleq \{ p_j \}_{j \neq i} \).

### III. Analysis of Payoff Functions

This section is devoted to the analysis of allocation and payment with respect to the players’ offers, based on which the players’ pay-off is analyzed. As introduced before, the allocation outcome is a solution to an MILP. For a stylized market as our model, the allocation is relatively straightforward: Due to the positive start-up costs and tie-breaking rules, if \( 0 < d \leq K \), or in the low demand case, in the allocation outcomes, exactly one generator is on-line to serve the demand; in the high demand case, with \( K < d \leq 2K \), both generators are on-line, and the one with the lower marginal cost offer operates at full capacity. The resulting marginal-cost price depends on the allocation, and is given by the following lemma.

**Lemma 5.** Given commitment decisions \( u_i^* \) and \( u_2^* \), and offer prices \( p_1 \) and \( p_2 \), the marginal-cost price is given by,

\[
\rho_{mc} = \begin{cases} u_1^* p_1 + u_2^* p_2, & 0 < d \leq K, \\
\max(p_1, p_2), & K < d \leq 2K. \end{cases}
\]

**Proof.** Based on the allocation outcome, under marginal-cost pricing, in the low demand case, the price is set at the marginal cost offer price of the on-line generator, while in the high demand case, the marginal-cost price is \( \max(p_1, p_2) \).

With the allocation and price outcomes in hand, we can easily analyze the resulting uplift payment and profit of each player. Note that, depending on the tie-breaking rules, the profit function may be upper semicontinuous or lower semicontinuous. For simplicity, we do not specify the profit at the point \( p_i = p_j \). The proof of Lemma 6, as well as other skipped proofs, can be found in [16], [17].

**Lemma 6.** Given commitment decisions \( u_i^* \) and \( u_2^* \), and offer prices \( p_1 \) and \( p_2 \), in low demand cases, \( 0 < d \leq K \), the profit of player \( i \) under marginal-cost pricing and make-whole uplift, \( \pi_{mc-mw}^{i} \), is

\[
\pi_{mc-mw}^{i} = u_i^* (p_i - c_i) d;
\]

In high demand cases, \( K < d \leq 2K \), the profit of player \( i \) under such a payment mechanism is

\[
\pi_{mc-mw}^{i} = \begin{cases} \left( p_j - c_i \right) K - s_i, & p_i \leq p_j - \frac{s_i}{K}, \\
\left( p_i - c_i \right) K, & p_j - \frac{s_i}{K} < p_i < p_j, \\
\left( p_i - c_i \right) (d - K), & p_i > p_j. \end{cases}
\]

**IV. Analysis of Equilibria**

We devote this section to the analysis of Nash equilibria arising from the duopolistic interactions. It is observed that the structure of the equilibrium offers varies significantly when demand is beyond a certain threshold. In Section IV-A, we examine the nature of equilibria in a low demand regime while in Section IV-B, we focus on a high demand regime.

**A. Low Demand**

In a low demand regime, competition forces players to offer close to actual costs. When the demand may be satisfied by either player, the equilibrium can be captured by the following proposition.

**Proposition 7.** Suppose \( 0 < d \leq K \). Then the following hold:

(a) Suppose \( s_i + c_i d < s_j + c_j d \). Marginal-cost pricing with make-whole uplift mechanism admits a unique pure-strategy Nash equilibrium given by: \( p_i^* = c_j + \frac{s_j - s_i}{d}, p_j^* = c_j \).

(b) Suppose \( s_i + c_i d = s_j + c_j d \). Then, \( p_i^* = c_i, p_j^* = c_j \), and player 1 is dispatched.

**Proof.** Case (a): When \( 0 < d \leq K \), one generator is on with a possibly positive profit while the other is off with zero profit. Consequently, the best response of each generator is to undercut its rival’s offer in terms of total as-bid cost including the start-up cost, until under-cutting is unprofitable (or the offer is less than true marginal cost). Consider the marginal cost pricing mechanism with make-whole uplift. Suppose generator \( i \) is dispatched, then by (4), the marginal cost price is set to be \( p_i \). Consequently, generator \( i \) gets an energy-based payment of \( p_i d \) with a make-whole uplift payment of \( s_i \) leading to a profit of \( (p_i - c_i) d \) while generator \( j \) obtains zero profit. Consequently, the best response\(^1\) of generator \( j \) is to undercut generator \( i \) by submitting an offer \( p_j \) such that \( s_i + p_i d < s_j + p_j d \), until \( p_j < c_j \). This implies that

\(^1\)For generator 1, \( \leq \) is enough

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\( \left( p_j - c_i \right) K - s_i \)

\( \left( p_j - c_i \right) K - s_i \)

\( \left( p_j - c_i \right) (d - K) \)

\( \left( p_j - c_i \right) (d - K) \)

\( \left( p_j - c_i \right) (d - K) \)

\( \left( p_j - c_i \right) (d - K) \)
\[ p_i = \frac{s_i - s_j}{d} + c_j \quad \text{when} \quad p_j = c_j. \]  
In summary, the equilibrium strategy is given by \[ (p_1^*, p_2^*) = \left( c_2 + \frac{s_2 - s_1}{d}, c_2 \right). \]

This is a unique pure-strategy Nash equilibrium and leads to a price Under this equilibrium, \( \rho^{\text{inc}} = c_2 = c_2 + \frac{s_2 - s_1}{d} \) with a total payment to generator 1 of \( s_2 + c_2 d \).

Case (b): As part of a tie-breaking rule, player 1 is dispatched first.

The equilibrium in the low demand case resembles the Bertrand outcome. Indeed, this corresponds well with economic intuition, when with excess capacity, competition drives prices close to marginal cost and in such settings, the pricing rules have little impact on allocation, total payment or profits.

B. High Demand

Next, we consider a high demand case where demand exceeds capacity of either player and observe that players may have far more market power. As a consequence, this may drive the clearing price above the actual cost. It is trivial that when \( d = 2K \), both generators need to supply at full capacity, and hence, offering at \( p^{\max} \) is the pure strategy equilibrium. We now consider the case when \( K < d < 2K \).

By Lemma 6, under marginal-cost pricing and make-whole uplift payment, the profit of player \( i \) is given by the following:

\[
\pi_i(p) = \begin{cases} 
(p_j - c_i)K - s_i, & p_i \leq p_j - \frac{s_i}{K} \\
(p_i - c_i)K, & p_j - \frac{s_i}{K} < p_i < p_j \\
(p_i - c_i)(d - K), & p_i > p_j 
\end{cases}
\]  
(5)

By inspecting the payoff functions, the only candidate strategies for generator \( i \) in response to \( p_j \) are to offer either at \( p_i = p^{\max} \) or at \( p_i = p_j \), and the same argument holds for player \( j \). By inspecting these candidate strategies, we conclude the absence of pure-strategy Nash equilibria.

Proposition 8. Suppose \( d \) satisfies \( K < d < 2K \). Then there is no pure-strategy Nash equilibrium for marginal-cost pricing with make-whole uplift payment mechanism.

As pure-strategy Nash equilibria do not exist, we focus on mixed strategy equilibria, given by a tuple of generalized probability densities \( \{f_1(p), f_2(p)\} \). We use \( \{F_1(p), F_2(p)\} \) to denote the corresponding cumulative distribution function and consider an expected payoff function \( \pi_i \) as follows:

\[
\pi_i(f_1(p), f_2(p)) \triangleq \int_0^{p^{\max}} \pi_i(p_i, f_2(p))f_1(p_i)dp_i,
\]
where \( \pi_i(p_i, f_2(p)) \triangleq \int_0^{p^{\max}} \pi_i(p_i, p_j)f_2(p_j)dp_j. \)

Let \( S_1 \) and \( S_2 \) denote the supports of generator 1 and generator 2 in their respective mixed-strategy densities.\(^2\)

Note that all pure strategies in the support are best responses to the rival’s mixed strategy. In other words, suppose \( \{f_1^*(p), f_2^*(p)\} \) denotes the mixed strategy equilibrium. Then, for \( j = 1 \) and \( 2 \), \( p_j \in S_j \) if and only if \( [18] \)

\[
p_j \in \arg\max_{p_j \in [0, p^{\max}]} \pi_j(p_j, f^*_j(p)).
\]

We begin with an intermediate result that prescribes some properties of the mixed-strategy Nash equilibrium, indeed it exists. Those properties can be proven by contradiction.

Lemma 9. Suppose \( K < d < 2K \) and suppose the mixed strategy equilibrium exists and is denoted by \( \{f_1^*, f_2^*\} \). Then, this equilibrium has the following properties:

(i) Both \( f_1^*(p) \) and \( f_2^*(p) \) have no point masses on the open set \( \{\max(c_1, c_2), p^{\max}\} \);

(ii) \( \max\{p : p \in S_1\} = \max\{p : p \in S_2\} = p^{\max}; \)

(iii) \( \min\{p : p \in S_1\} = \min\{p : p \in S_2\} = \max(c_1, c_2); \)

(iv) \( \forall p' \in S_1 \cap S_2, F_1(p_1 = p' \text { and } p_2 = p') = 0; \) and

(v) \( S_i \) is an interval without holes, i.e., for any two points \( a, b \) such that \( \min S_i < a < b < \max S_i, F_i(b) - F_i(a) > 0. \)

Therefore, \( S_1 = S_2 \) and there are no point masses or holes in the interior of \( S_1 \) or \( S_2 \), implying that the associated cumulative distribution functions, \( F_1^*(p) \) and \( F_2^*(p) \), are differentiable in the interior of their respective supports. Leveraging this fact, we provide the following proposition.

Proposition 10. For any \( p \in (\min(S_1), p^{\max}) \), we have the following:

\[
F_2(p)(d - 2K) + F_2\left(p + \frac{s_1}{K}\right)K - f_2(p)(p - c_1)(2K - d) = 0,
\]
(6)

and

\[
F_1(p)(d - 2K) + F_1\left(p + \frac{s_2}{K}\right)K - f_1(p)(p - c_2)(2K - d) = 0.
\]
(7)

Proof. Suppose player 1 offers \( p \) while player 2 adopts a mixed strategy \( f_2(p) \). Then player 1’s profit is given by

\[
\pi_1(p, f_2(p)) = F_2(p)(p - c_1)(d - K) + [F_2(p + \frac{s_1}{K}) - F_2(p)](p - c_1)K + \int_{p + \frac{s_1}{K}}^{p^{\max}} [(p - c_1)K - s_1]f_2(dp).
\]

In the interior of \( S_1 \), since \( F_2(p) \) is differentiable in \( p \), the payoff function \( \pi_1(p, f_2(p)) \) is also differentiable in \( p \). For all \( p \in S_1 \), the payoff function \( \pi_1(p, f_2(p)) \) is constant since player 1 randomizes over this set. Consequently, we have that

\[
\frac{d\pi_1(p, f_2(p))}{dp} = 0
\]
in the interior of \( S_1 \). Therefore, (6) holds and (7) can be seen to hold in a similar fashion.

Note that (6) and (7) appear to be decoupled, and may be solved independently. However, we will proceed to show...
later in the paper that player strategies are indeed coupled but through boundary conditions.

To establish the existence of the mixed strategy Nash equilibrium, we need to derive cumulative distribution functions $F_1(p)$ and $F_2(p)$ that solve the differential equations given by (6) and (7). It may be recalled that cumulative distribution functions are right-continuous. To simplify the notation, we denote the left-hand limit by $F(a^{-})$, where

$$ F(a^{-}) \triangleq \lim_{x \to a^{-}} F(x). $$

The solutions to differential equations depend on the initial or boundary values. For (6) and (7), because of the constant delay of $-s_i/K$, boundary values in an interval with the length of $s_i/K$ need to be specified. In the context of cumulative distribution functions, we only need to specify $F_i(p^{\max}^{-})$, because we know $F_i(p) = 1, p \geq p^{\max}$. In effect, the point mass at $p^{\max}$ is given by $(1 - F_i(p^{\max}^{-}))$.

To facilitate our discussion and highlight the dependence of $F_i(p)$ on the initial value, we view $F_i(p^{\max}^{-})$ as a parameter $I_i$, where $I_i \triangleq F_i(p^{\max}^{-})$. Given this dependence, we denote the distribution function associated with player $i$ as $F_i(p, I_i)$. Next, we show the continuity of $F_i(p, I_i)$ in $I_i$.

Either (6) or (7) may be written as

$$ f_i(p) = \frac{1}{p - c_j} \left[ \frac{K}{2K - d} F_i \left( \frac{p + s_j}{K} - F_i(p) \right) \right], \quad (8) $$

where $i = 1$ or 2. By recalling that under suitable assumptions, the density function is given by the derivative of the distribution function. Consequently, this relationship can be viewed as a nonlinear system $\dot{x} = g(x, t, \lambda)$, where $\lambda$ denotes a parameter. For such nonlinear systems, a well-known condition for existence and uniqueness is given by Theorem 3.2 of [19]. The continuity of the distribution function $f_i(p, I_i)$ in $I_i$ based on the following proposition.

**Proposition 11.** Suppose $F_i(p, I_i) = 1, p \geq p^{\max}$ and $F_i(p, I_i)$ is continuous in $p$ on $(\min(S_i), p^{\max})$. Then for all $p \in (\min(S_i), p^{\max})$, $F_i(p, I_i)$ exists and is unique and continuous with respect to $I_i$.

Under suitable continuity assumptions of the solution $F_i(p, I_i)$ in terms of the boundary value $I_i$, we are now ready to establish the existence of the mixed-strategy Nash equilibrium.

**Theorem 12.** Suppose for $d \in (K, 2K)$ and for $i = 1, 2$ with $I_i \leq 1$, $F_i(p, I_i) = 1, p \geq p^{\max}$ and $F_i(p, I_i)$ is continuous in $p$ on $(\min(S_i), p^{\max})$. Then, the marginal-cost pricing with a make-whole payment mechanism admits a mixed strategy equilibrium.

**Proof.** We only consider the solutions when $p > c_j$ and define $F_i(c_j) \triangleq F_i(c_j^-)$. Such an interval $[c_j, p^{\max}]$ is sufficient for covering the support of the players’ strategy set because by Lemma 9, $\min(S) \geq \max(c_1, c_2)$.

This proof has two parts: (i) First, we show that there exists a solution to (6) or (7) satisfying the properties of a cumulative distribution function. (ii) Subsequently, we show that such solution is indeed an equilibrium.

(i) Since $d \in (K, 2K)$, then $K/(2K - d) > 1$. Furthermore, since $p > c_j$, $1/(p - c_j) > 0$, by following a procedure similar to that employed in proving Proposition 11 (see [16], [17]), we may conclude if $I_i \in [0, 1]$, $f_i(p) > 0$ by using mathematical induction. This implies that $F_i(p, I_i)$ is increasing in $p$. Let $I_1 = I_2 = F_1(p^{\max}^-) = F_2(p^{\max}^-) = 0$, solve (6) and (7). Then, for all $p < p^{\max}$, $F_i(p, 0) < F_i(p^{\max}^-), 0 = 0, F_2(p, 0) < F_2(p^{\max}^-), 0 = 0$. Let $I_1 = I_2 = F_1(p^{\max}^-) = F_2(p^{\max}^-) = 1$, solve (6) and (7). Note that solvability of the differential equations is guaranteed under the assumed continuity and Lipschitzian requirements. It follows that there are two possibilities:

(a) $F_1(\max(c_1, c_2), 1) \geq 0$ and $F_2(\max(c_1, c_2), 1) \geq 0$.

In this case, choose any generator $i \in \{1, 2\}$ and let $F_i(p, I_i) = F_i(\max(c_1, c_2), 1)$. For the other generator, denoted by $j$, by Proposition 11 and mean-value theorem, since $F_j(\max(c_1, c_2), 0) < 0$ and $F_j(\max(c_1, c_2), 1) > 0$, there exists an $a \in (0, 1)$, such that $F_j(\max(c_1, c_2), a) = 0$. Then we obtain a mixed strategy

$$ \{F_i(p, 1), F_j(p, a) \mid p \in [\max(c_1, c_2), p^{\max}] \} $$

that satisfies the first order necessary condition of the equilibrium.

(b) Either $F_1(\max(c_1, c_2), 1) < 0$ or $F_2(\max(c_1, c_2), 1) < 0$ or both are negative. Due to the continuity of $F_i$ with respect to $p$, there must be a point $p'$ such that $F_i(p', 1) = 0$. If both $F_1(\max(c_1, c_2), 1)$ and $F_2(\max(c_1, c_2), 1)$ are negative, we select $p'$ such that $F_j(p', 1) = 0$ and $F_j(p', 1) \geq 0$. Thus, we obtain a mixed strategy given by

$$ \{F_i(p, 1), F_j(p, 1) \mid p \in [p', p^{\max}] \}, $$

that satisfies the first order necessary condition of the equilibrium. In effect, we have obtained mixed-strategy that are indeed distribution functions.

(ii) To show that the mixed strategy above is indeed an equilibrium, we need to further prove that neither player can profit by unilaterally deviating to another strategy. There are two possible deviations: randomizing over a different support and employing a different distribution on the same support.

By Lemma 9, if one player changes his support, his profit will decrease, implying that there is no incentive for either player to randomize his strategy over a different support. On the other hand, by Proposition 10, all pure strategy in the support against the rival’s mixed strategy leads to constant payoff, implying that a player cannot profit from deviating to a different density over the same support. In effect, neither player can profit from unilateral deviations and therefore, the mixed strategy candidate equilibrium is indeed equilibrium.

\[\square\]

**V. Conclusions**

The contribution of this paper lies in providing a complete analysis of duopolistic models based on electricity markets currently operating in the U.S. Briefly, our contributions pertain to the the nature of equilibria in two distinct regimes: (i) In a low demand regime, a pure-strategy Nash equilibrium is shown to exist; and (ii) In a high-demand regime, where
no firm can independently service demand, we derive differential equations that capture the necessary conditions of the mixed-strategy Nash equilibria. It is shown that these conditions admit solutions that are distribution functions. Furthermore, these distribution functions are seen to be mixed-strategy Nash equilibria. Two extensions to this work can be found in [16], [17] and are summarized as follows:

(i) **Convex hull pricing schemes.** Motivated by the need to reduce or eliminate the out-of-market uplift payment, convex hull pricing has been proposed and will be implemented in the Midwest ISO [20], [21]. Preliminary results in [16], [17] show that in a strategic regime analogous to the setting of this paper, convex-hull pricing is not necessarily superior to marginal-cost pricing in terms of efficiency, consumer’s total payments, or uplift payments.

(ii) **Numerical results.** The equilibria in high demand cases are essentially solutions to boundary value problems. Such problems can be solved numerically and such numerical results provide us with far more insights regarding the performance of the market mechanism.

There is, of course, much more work that must be done to truly understand these pricing mechanisms in real-world settings. The larger question of how to create better markets for electric power is the long-term goal of our research. This topic is difficult, in part, because the optimal allocation of resources may have multiple solutions. For example, coal generators of similar technology are subject to similar costs. From a social planner’s point of view this is a blessing – any of $N$ coal generators will serve the needs of the grid at a particular time. However, the owners of the generators have a very different perspective: they will not accept an arbitrary mechanism to break ties. This fundamental issue must be addressed when attempting to solve the unit commitment problem through market mechanisms.

**REFERENCES**


