On the Convergence of Joint Schemes for Online Computation and Supervised Learning

Hao Jiang  Uday V. Shanbhag

Abstract—Traditionally, the field of deterministic optimization has been devoted to minimization of functions $f(x; \theta^*)$ whose parameters, denoted by $\theta^*$, are known with certainty. Supervised learning theory on the other hand considers the question of employing training data to seek a function from a set of possible functions. Instances of learning algorithms include regression schemes and support vector machines, amongst others. We consider a hybrid problem of computation and learning that arises in online settings, where one may be interested in optimizing $f(x; \theta^*)$ while learning $\theta^*$ through a set of observations. More generally, we consider the solution of parameterized monotone variational inequality problems, which can capture a range of convex optimization problems and convex Nash games. The unknown parameter $\theta^*$ is learned through the noisy observations of a linear function of $\theta^*$, denoted by $\ell(x; \theta^*)$. This paper provides convergence statements for joint schemes when observations are corrupted by noise in regimes where the associated variational inequality problem may be either strongly monotone or merely monotone. The proposed schemes are shown to produce iterates that converge in mean to their true counterparts. Numerical results derived from the application of these techniques to convex optimization problems and nonlinear Nash-Cournot games is shown to be promising.

I. INTRODUCTION

Optimization [1], [2] is a mature field that is primarily concerned with the problem of minimizing a function over a prescribed set and has immense application, ranging from process control, engineering mechanics, and production planning and operations. Key subfields of optimization include problem classes where the function and the set is convex (convex programming [3]), a subset of the variables take on integer values (integer programming [4]), or when the function or the set is parameterized by random variables, requiring the use of an appropriate expectation or risk-based formulation (stochastic programming [5]). Yet, in much of these settings, problem parameters are available in some form. For instance, in process control problems, a range of parameters may be unavailable but can be observed indirectly. Optimizing such systems often requires a stage-wise approach wherein such parameters are learnt and subsequently, one solves the associated optimization problem; in effect, the process of optimization and learning are distinct.

In fact, the question of learning such parameters can be resolved a priori by a variety of supervised learning techniques, that include linear and nonlinear regression, support vector machines, neural networks, amongst others [6].

Unfortunately, in many settings with imperfect information, the observations required for learning a parameter cannot be accessed a priori. For instance, consider a $N$−player Nash-Cournot game [7] where firms compete in quantity. Market prices are prescribed by a price function $p(X; a^*, b^*) \triangleq a^* - b^*X$ where $X \triangleq \sum_{i=1}^N x_i$, $x_i$ denotes the $i$th firm’s quantity decision and $a^*$ and $b^*$ represent the true intercept and slope of the affine price function.

This paper is motivated by the need to optimize systems or compete in markets where parameters need to be simultaneously learnt. An instance of such a scheme in the context of Nash-Cournot games is when every player constructs an estimate of $a^*$ and $b^*$ by observing $p(X_k; a^*, b^*)$. Our goal would then be to develop schemes whereby players observe prices and update their strategies, with the intent of learning $a^*$, $b^*$ and $x^*$, the equilibrium strategy.

In an interest to capture both convex Nash games and convex optimization problems with imperfect information, we employ the variational inequality problem [8]. The variational inequality (VI) is a useful tool in solving optimization problems, noncooperative games, equilibrium problems, amongst others. Given a mapping $F : K \rightarrow \mathbb{R}^n$, where $F$ is a single-valued mapping and $K$ is a closed and convex set, then a variational inequality problem, denoted by VI$(K, F)$, requires an $x^* \in K$ satisfying

$$\langle y - x^* \rangle^T F(x^*) \geq 0, \text{ for all } y \in K. \quad (1)$$

However, in practice, the original problems, be they optimization problems, equilibrium problems, or even Nash games, may well be corrupted by imperfections in the information. One avenue for capturing such imperfections is by considering a parameterized variational inequality problem $\text{VI}(K, F(\cdot; \theta^*))$ with a mapping $F(x; \theta^*)$, parameterized by an unknown vector $\theta^*$. In this paper, we assume that the parameter $\theta^*$ can be learnt by observing a linear function of $\theta^*$ denoted by $\ell(x; \theta^*)$ which may or may not be corrupted by noise. The function $\ell$ may be intimately related to the mapping $F$, as is the case when $\ell$ is derived from considering the residuals between observed prices $p(X_k; \theta_k)$ and predicted prices $p(X; \theta_k)$. Alternately, it could be derived from distinctly different observation process. Succinctly, the goal of this paper lies in developing convergent schemes for computing $x^*$ and learning $\theta^*$ through the use of the observations $\ell(x; \theta^*)$.

In the past, adaptive control has been inspired by similar questions in the context of control theory where control laws are tuned while recursively learning parameters [9]. In operations research literature, Cooper et al. [10] examines...
how incorrect beliefs regarding customer behavior lead to higher fares and systematically lead to lower revenues over time, and represents an instance where firms optimize and learn (incorrectly) simultaneously. More recently, the same authors consider pricing and learning schemes in revenue management problems [11]. Exceptions to this include recent work by Jiang, Shanbhag and Meyn [12] where the authors examine joint schemes for learning (misspecified) price function parameters and equilibrium computation in noise-corrupted regimes. However, little is available by way of general convergence statements for joint computation and learning in the presence of noise-corrupted observations. Motivated by this obvious gap, this paper makes the following contributions. When observations are corrupted by noise, we develop schemes that are equipped with convergence guarantees in expectation. In these schemes, learning is carried out by solving a suitably defined least-squares problem. The schemes are applied towards two sets of problems. Of these, the first is a convex optimization problem with unknown parameters while the second is a Nash-Cournot game where the parameters of the price functions are unavailable, but may be learnt over time.

The rest of this paper is organized as follows. In Section II, we consider two sets of source problems that motivate our general problem of interest. In Section III, we focus on the setting where the observations \( \ell(x; \theta^*) \) are noise-corrupted. In this section, strongly monotone maps \( F \) and their merely monotone counterparts regimes are considered for developing convergent schemes. We apply our proposed techniques to the computation of Nash-Cournot equilibria in imperfect information regimes in Section IV. The paper concludes with a brief set of remarks in Section V.

Throughout the paper, we use \( \|x\| \) to denote the Euclidean norm of a vector \( x \), i.e., \( \|x\| = \sqrt{x^T x} \). We use \( \Pi_K \) to denote the Euclidean projection operator onto a set \( K \), i.e., \( \Pi_K(x) = \text{argmin}_{y \in K} \|x - y\| \). A mapping \( H : \mathbb{R}^n \to \mathbb{R}^n \) is said to be monotone on \( K \) if \( (H(x) - H(y))^T (x - y) \geq 0 \) for all \( x, y \in K \) and is strongly monotone on \( K \) if there exists a constant \( c > 0 \) such that \( (H(x) - H(y))^T (x - y) \geq c \|x - y\|^2 \), for all \( x, y \in K \).

II. SOURCE PROBLEMS AND MOTIVATION

In this section, we motivate our problem of interest by considering two classes of problems, the first an optimization problem and the second, a Nash game. In each instance, we articulate the problem and specify the nature of the observations. Given both sets of problems, we employ a variational framework to construct a general problem under which both the optimization and game-theoretic problem may be cast.

A. Source problems

a) Optimization problems with unknown parameters:
Consider an optimization problem in which \( h_i : \mathbb{R}^n \to \mathbb{R} \) is a convex function in \( x \) for \( i = 1, \ldots, m \), \( K_x \subseteq \mathbb{R}^n \) is a closed and convex set, and \( \theta_i \) is a nonnegative parameter for \( i = 1, \ldots, m \). Then, the optimization problem of interest is given by

\[
\min_{x} \sum_{i=1}^{m} \theta_i h_i(x),
\]

s.t. \( x \in K_x \).

Let \( f(x; \theta) = \sum_{i=1}^{m} \theta_i h_i(x) \), where \( \theta = (\theta_1, \ldots, \theta_m)^T \). Since \( \theta_i \) is nonnegative for all \( i \), and \( h_i(x) \) is convex in \( x \) for all \( i \), the objective function \( \sum_{i=1}^{m} \theta_i h_i(x) \) is convex in \( x \). Thus, problem (2) is equivalent to the following variational inequality problem that requires an \( x \in K_x \) such that

\[
(y - x)^T F(x; \theta) \geq 0, \quad \forall y \in K_x,
\]

where

\[
F(x; \theta) = \sum_{i=1}^{m} \theta_i \nabla_x h_i(x).
\]

The key challenge lies in the lack of information regarding the vector \( \theta \). The value of \( \theta \) may be learnt indirectly by observing a linear function of \( \theta \), denote by \( \ell(x; \omega) \), defined as

\[
\ell(x; \theta^*) = \sum_{i=1}^{m} \theta_i^* g_i(x) + \xi(x),
\]

where \( g_i : \mathbb{R}^n \to \mathbb{R} \) for \( i = 1, \ldots, m \) and \( \xi : \Omega \to \mathbb{R} \), the noise in the observation process, denotes a random variable and \( (\Omega, \mathcal{F}, \mathbb{P}) \) denotes the probability space. A special case of \( \ell(x; \theta) \) is when \( g_i = h_i \) for \( i = 1, \ldots, m \), implying that the objective function of (2) is observable. Our goal lies in constructing sequences \( \{x^k\} \) and \( \{\theta^k\} \) such that \( x^k \) converges to \( x^* \) and \( \theta^k \) converges to \( \theta^* \in \Theta \) where \( x^* \) denotes a solution of (2), given \( \theta^* \), and \( \Theta \) is a closed and convex set in \( \mathbb{R}^m \).

b) Nash-Cournot games with unknown price functions:
Next, we consider an \( N \)-player Nash-Cournot game in which the \( i \)th player solves the following parameterized optimization problem:

\[
\min_{x_i} f_i(x_i; \theta) \triangleq \left[ c_i(x_i) - x_i \left( \theta_0 - \sum_{i=1}^{m} \theta_i x^i \right) \right],
\]

s.t. \( x_i \in X_i \),

where the cost function \( c_i : \mathbb{R}^n \to \mathbb{R} \) is convex for \( i = 1, \ldots, N \), and \( X_i \subseteq \mathbb{R}^n \) is a closed and convex set for \( i = 1, \ldots, N \). Let \( x = (x_1, \ldots, x_N)^T, \theta = (\theta_0, \theta_1, \ldots, \theta_m)^T \) and \( K_x = \prod_{i=1}^{N} X_i \). Then, an equilibrium of this game is given by the following variational inequality. Find \( x \in K_x \), s.t.

\[
(y - x)^T F(x; \theta) \geq 0, \quad \forall y \in K_x,
\]

where

\[
F(x; \theta) = (\nabla_x f_i(x; \theta))^T \big|_{i=1}^{N},
\]
and
\[
\nabla_x f_j(x; \theta) = c_j(x_j) - \left( \theta_0 - \sum_{i=1}^m \theta_i x^{\alpha_i} \right) + x_j \sum_{i=1}^m \theta_i \alpha_i x^{\alpha_i-1},
\]
for \( j = 1, \ldots, n \). Again, it is assumed that every player can observe prices, possibly corrupted by noise, denoted by
\[
\ell(x; \theta^*) \triangleq \theta_0^* - \sum_{i=1}^m \theta_i^* x^{\alpha_i} + \xi(\omega) = g(x)^T \theta + \xi(\omega),
\]
which is a linear function of \( \theta \). Similar to the optimization setting, our goal lies in developing schemes that generate sequences \( \{x^k\} \) and \( \{\theta^k\} \) such that \( x^k \) converges to \( x^* \) and \( \theta^k \) converges to \( \theta^* \in \Theta \), where \( x^* \) is a Nash-Cournot equilibrium of the game given \( \theta^* \) and \( \Theta \) is a closed and convex set in \( \mathbb{R}^m \).

**B. A general variational inequality problem framework**

The solutions of convex optimization problems and constrained Nash games can be compactly captured by variational inequality problems. Inspired by the source problems introduced in Section II-A, we define a parameterized variational inequality problem as follows. Given \( \theta^* \in K_0 \), a solution \( x^* \in K_x \) to VI(\( K_x, F(\cdot; \theta^*) \)) satisfies
\[
(x - x^*)^T F(x^*; \theta^*) \geq 0, \quad \forall x \in K_x,
\]
where \( K_x \subseteq \mathbb{R}^n \) and \( K_0 \subseteq \mathbb{R}^m \) are closed and convex sets. The parameter \( \theta \) can be learnt by observing a linear function of \( \theta^* \) denoted by \( \ell(x; \theta^*) \), defined as
\[
\ell(x; \theta^*) \triangleq \sum_{i=1}^m \theta_i^* g_i(x) + \xi(\omega). \quad (7)
\]
Throughout the remainder of this paper, we employ \( \mathcal{F}_k \) to denote the \( \sigma \)-field on the kth iteration generated by \( x_0, \theta_0 \) and the sequence \( \omega_j \) for \( j = 1, \ldots, k \). More precisely,
\[
\mathcal{F}_k \triangleq \{x_0, \theta_0, \omega_1, \ldots, \omega_k\}.
\]

Succinctly, the goals of this paper lie in developing convergent schemes for computing \( x^* \) and learning \( \theta^* \) through the use of the observations \( \ell(x; \theta^*) \). In the remainder of the paper, for purposes of maintaining a general outlook, we focus on solving VI(\( K, F(\cdot; \theta) \)) while utilizing the observation process. In section III, we assume that noise process is a mean-zero, independent and identically distributed, process. Finally, we note that \( \ell(x; \theta) \) may have no obvious relation to \( F(x; \theta) \), the mapping constructed from the optimization or the game-theoretic problem. Although the source problems in Section II-A provide special cases where \( \ell(x; \theta) \) is indeed related to \( F(x; \theta) \), we consider a general process where \( \ell(x; \theta) \) is available and satisfies some specified conditions. (e.g. conditions in A1).

**III. NOISE-CORRUPTED OBSERVATIONS**

In this section, we assume that there is some noise in the observations. Suppose \( \ell(x; \theta^*) = \sum_{i=1}^m \theta_i^* g_i(x) + \xi(\omega) \), where \( \xi \) is a random variable with mean 0. We can only see \( \ell(x; \theta^*) \) instead of \( \sum_{i=1}^m \theta_i^* g_i(x) \) directly. When there are \( k \) observations, we have \( y^j = \ell(x^j; \theta^*) = \sum_{i=1}^m \theta_i^* g_i(x^j) + \xi^j \) for \( j = 1, \ldots, k \), where \( \xi^j \) are i.i.d. Consider the least squares estimator \( \theta^k \) of \( \theta^* \), that is given by the solution to the following problem:
\[
\min_{\theta \in K_0} \frac{1}{k} \sum_{j=1}^k (y^j - \sum_{i=1}^m \theta_i g_i(x^j))^2. \quad (8)
\]
Given the presence of noise, we define two forms of convergence that will assume relevance.

**Definition 1:** Let \( \{X_k\} \) be a sequence of random variables on the probability space \((\Omega, \mathcal{F}, \mathbb{P})\), and \( X \) is a random variable on the same probability space. Then, the following hold:
1) \( X_k \) converges in mean to \( X \) or \( X_k \xrightarrow{L_1} X \) if
\[
\lim_{k \to \infty} \mathbb{E}(\|X_k - X\|) = 0;
\]
2) \( X_k \) converges in probability to \( X \) or \( X_k \xrightarrow{P} X \) if for all \( \epsilon > 0 \)
\[
\lim_{k \to \infty} \mathbb{P}(\|X_k - X\| \geq \epsilon) = 0.
\]

The remainder of this section is partitioned into two subsections. In Section III-A, we consider a regime where \( F(x; \theta) \) is strongly monotone in \( X \) uniformly in \( \theta \). This condition is weakened in Section III-B where a merely monotone map is considered.

**A. Strongly monotone mapping**

In this subsection, we consider strongly monotone maps and make the following assumptions on both the decision variables and the parameters.

**Assumption 1 (A1):** Suppose the following hold:
1) For every \( \theta \in \Theta \), \( F(x; \theta) \) is both strongly monotone and Lipschitz continuous in \( x \) with constants \( \mu_x \) and \( L_x \), respectively.  
2) \( F(x^*; \theta) \) is Lipschitz continuous in \( \theta \) with constant \( L_\theta \).

**Assumption 2 (A2):** Let \( \{\gamma_{k,x}\} \) and \( \{\gamma_{k,\theta}\} \) be chosen such that:
1) \( \gamma_{k,x} > 0 \) for all \( k \), \( \sum_{k=1}^\infty \gamma_{k,x} = \infty \), and \( \sum_{k=1}^\infty (\gamma_{k,x})^2 < \infty \);  
2) \( \gamma_{k,\theta} > 0 \) for all \( k \), \( \sum_{k=1}^\infty \gamma_{k,\theta} = \infty \), and \( \sum_{k=1}^\infty (\gamma_{k,\theta})^2 < \infty \).

Our algorithm to learn \( x^* \) and \( \theta^* \) is as follows.

**Algorithm 1:**
\[
x^{k+1} = \Pi_{K_x} (x^k - \gamma_{k,x} (F(x^k; \theta^k)),
\]
where \( \theta^k \) is the solution to (8) and is a function of the sequence of noise terms \( \theta_0, \omega_1, \ldots, \omega_k \).

We begin by providing a contraction result which will be subsequently employed in developing our convergence statements.
Lemma 1: Let $H : K \to \mathbb{R}^n$ be a mapping that is strongly monotone over $K$ with constant $\mu$, and Lipschitz continuous over $K$ with constant $L$. Then, for any $\gamma > 0$, we have

$$\Pi_K(x - \gamma H(x)) - \Pi_K(y - \gamma H(y)) \leq q\|x - y\|.$$ (9)

where $q = \sqrt{1 - 2\mu \gamma + \gamma^2 L^2}$.

Proof: By employing the non-expansivity of the Euclidean projector, the left-hand side of (9) may be expressed as follows:

$$\|\Pi_K(x - \gamma H(x)) - \Pi_K(y - \gamma H(y))\|^2 \\
\leq \|(x - \gamma H(x)) - (y - \gamma H(y))\|^2 \\
= \|(x - y) - \gamma (H(x) - H(y))\|^2 \\
= \|x - y\|^2 - 2\gamma (x - y)^T (H(x) - H(y)) \\
+ \gamma^2 \|H(x) - H(y)\|^2.$$

By invoking the strong monotonicity and Lipschitz continuity of $H$ over $K$, we have that

$$\|x - y\|^2 - 2\gamma (x - y)^T (H(x) - H(y)) \\
+ \gamma^2 \|H(x) - H(y)\|^2 \leq (1 - 2\mu \gamma + \gamma^2 L^2)\|x - y\|^2,$$

implying (9).

Our first result provides a convergence statement for the sequence of iterates produced by Algorithm 1 and can be stated as follows.

Theorem 2: Suppose (A1) and (A2) hold. Let $\{x^k, \theta^k\}$ be computed via Algorithm 1. Then, $x^k \xrightarrow{L} x^*$ and $\theta^k \xrightarrow{P} \theta^*$ as $k \to \infty$.

Proof: By the nonexpansivity of the Euclidean projector and the triangle inequality, $\|x_{k+1} - x^*\|$ may be bounded as follows:

$$\|x_{k+1} - x^*\| \\
= \|\Pi_K(x_k - \gamma_{k,x} F(x_k; \theta^k) - \Pi_K(x^* - \gamma_{k,x} F(x^*; \theta^*))\| \\
\leq \|(x_k - x^*) - \gamma_{k,x} (F(x_k; \theta^k) - F(x^*; \theta^*))\| \\
\leq \|(x_k - x^*) - \gamma_{k,x} (F(x_k; \theta^k) - F(x^*; \theta^*))\| \\
+ \gamma_{k,x} \|F(x^*; \theta^k) - F(x^*; \theta^*)\|.$$

By Lemma 1 and the Lipschitz continuity of $F(x^*; \theta)$ in $\theta$ (A1), the right-hand side of (10) can be further bounded by

$$q_{k,x} \|x_k - x^*\| + \gamma_{k,x} \|\theta^k - \theta^*\|,$$

where $q_{k,x} = \sqrt{1 - 2\mu \gamma_{k,x} + (L_k)^2 \gamma_{k,x}^2}$. Recall that the sequences $\{x_k\}$ and $\{\theta_k\}$ are random sequences derived from the sequence of events captured by $\mathcal{F}_k$. Then, by taking expectations, we have that

$$E[\|x_{k+1} - x^*\|] \leq q_{k,x} E[\|x_k - x^*\|] + \gamma_{k,x} E[\|\theta^k - \theta^*\|].$$

Since the sequence $\{\theta^k\}$ is bounded, it is uniformly integrable. Since $\theta^k$ converges to $\theta^*$ in probability, we have that

$$E[\|\theta^k - \theta^*\|] \to 0,$$

It remains to show that $E[\|x_{k+1} - x^*\|] \to 0$ as $k \to \infty$. This holds if (i) $\sum_{k=1}^{\infty} (1 - q_{k,x}) = \infty$ and

$$\lim_{k \to \infty} \frac{\gamma_{k,x} L_k E[\|\theta^k - \theta^*\|]}{1 - q_{k,x}} = 0.$$

By (A2), we have that $\sum_{k=1}^{\infty} (1 - q_{k,x}) = \infty$. Furthermore,

$$\lim_{k \to \infty} \frac{\gamma_{k,x} L_k E[\|\theta^k - \theta^*\|]}{1 - q_{k,x}} = \frac{\gamma_{k,x} (1 + q_{k,x}) L_k E[\|\theta^k - \theta^*\|]}{2 \mu \gamma_{k,x} (L_k)^2 (\gamma_{k,x})^2} = \frac{1 + q_{k,x}}{2 \mu \gamma_{k,x} (L_k)^2 (\gamma_{k,x})^2} = 0,$$

since $q_{k,x} \to 1$ and $\gamma_{k,x} \to 0$ as $k \to \infty$. Therefore, $E[\|x_{k+1} - x^*\|]$ converges to zero.

B. Monotone mapping

In this section, we weaken the rather stringent requirement of strong monotonicity of the map by using a Tikhonov regularization. The following assumptions will be made on both the decision variable and parameter.

Assumption 3 (A3): Suppose the following hold:

1) $F(x; \theta^*)$ is monotone in $x$, and Lipschitz continuous in $x$ with constant $L_0$ uniformly in $\theta$.

2) For every $x$, $F(x; \theta)$ is Lipschitz continuous in $\theta$ with constant $L_0$.

We make the following assumptions on the steplengths and regularization parameter of the algorithm.

Assumption 4 (A4): Let $\{\gamma_{k,x}\}$, $\{\gamma_{k,\theta}\}$ and $\{\epsilon_k\}$ be chosen such that:

1) $\gamma_{k,x} > 0 \forall k, \epsilon_k > 0 \forall k, \sum_{k=1}^{\infty} \gamma_{k,x} \epsilon_k = \infty, \gamma_{k+1,x} \leq \gamma_{k,x} \epsilon_k \leq \epsilon_k, \sum_{k=1}^{\infty} (\gamma_{k,x})^2 < \infty, \sum_{k=1}^{\infty} (\gamma_{k,x} \epsilon_k) < \infty, \gamma_{k,x} \epsilon_k \to 0$, and $\epsilon_k - \epsilon_k \to 0$.

2) $\gamma_{k,\theta} > 0 \forall k, \sum_{k=1}^{\infty} \gamma_{k,\theta} = \infty, \sum_{k=1}^{\infty} (\gamma_{k,\theta}) < \infty$.

We consider a regularized variant of algorithm 1 to compute $x^*$ and learn $\theta^*$ and is given by the following.

Algorithm 2:

$$x_{k+1} = \Pi_K(x_k - \gamma_{k,x} F(x_k; \theta^k) + \epsilon_k x_k),$$

where $\theta^k$ is the solution to (8).

Before providing a convergence result for Algorithm 2, we introduce the following result.

Lemma 3: Let $H : K \to \mathbb{R}^n$ be a mapping that is monotone over $K$, and Lipschitz continuous over $K$ with constant $L$. Then, for any $\gamma > 0$ and $\epsilon > 0$, we have

$$\|x - y\| - \gamma (H(x) - H(y)) - \epsilon \gamma (x - y) \leq q \|x - y\|,$$

where $q = \sqrt{1 - 2\gamma \epsilon + \gamma^2 (L^2 + \epsilon^2)}$.

Proof: See proof of Theorem 2 in [13].

Lemma 4: Let $H : K \to \mathbb{R}^n$ be a mapping that is monotone over $K$. Given $\epsilon_k > 0$, let $y_k$ be a solution to $\text{VI}(H + \epsilon_k I, K)$. Then,

$$\|y_k - y_{k-1}\| \leq M \epsilon_k \epsilon_k - \epsilon_k,$$

where $M = \|x^*\|$ and $x^*$ is a solution to $\text{VI}(H, K)$.

Proof: See Lemma 3 in [13].
Lemma 5: Let $M \geq 0$ and $L_x > 0$. Suppose (A4) holds. Then, we have
\[
\sum_{k=1}^{\infty} (1 - q_{k,x}) = \infty \quad \text{and} \quad \lim_{k \to \infty} \frac{q_{k,x}}{M \epsilon_{k-1} - \epsilon_k} = 0,
\]
where $q_{k,x} = \sqrt{1 - 2 \gamma_{k,x} \epsilon_k + (\gamma_{k,x})^2((L_x)^2 + (\epsilon_k)^2)}$.

Proof: See proof of Theorem 2 in [13].

The convergence result for Algorithm 2 can be stated as follows.

Theorem 6: Suppose (A3) and (A4) hold. Let $\{x^k, \theta^k\}$ be computed via Algorithm 2. Then, $x^k \overset{L^1}{\rightarrow} x^*$ and $\theta^k \overset{P}{\rightarrow} \theta^*$ as $k \to \infty$.

Proof: Suppose $y^k$ is a solution to the following fixed-point problem
\[
y^k = \Pi_{\mathcal{K}_s}(y^k - \gamma_{k,x}(F(y^k; \theta^*) + \epsilon_k y^k)).
\]
Then, we have by the triangle inequality that $\|x^{k+1} - x^*\|$ may be bounded as per
\[
\|x^{k+1} - x^*\| \leq \|x^{k+1} - y^k\| + \|y^k - x^*\|. \quad (13)
\]
Term 2 converges to zero by the convergence statement of Tikhonov regularization methods [8]. By using the non-expansivity of the Euclidean projector and triangle inequality, term 1 can be bounded by
\[
\|x^{k+1} - y^k\| = \|\Pi_{\mathcal{K}_s}(x^k - \gamma_{k,x}(F(x^k; \theta^k) + \epsilon_k x^k)) - \Pi_{\mathcal{K}_s}(y^k - \gamma_{k,x}(F(y^k; \theta^*) + \epsilon_k y^k))\|
\leq \|(x^k - y^k) - \gamma_{k,x}(F(x^k; \theta^k) - F(y^k; \theta^*))\| - \epsilon_k \gamma_{k,x} \|x^k - y^k\| \leq \|(x^k - y^k) - \gamma_{k,x}(F(x^k; \theta^k) - F(y^k; \theta^*))\| - \epsilon_k \gamma_{k,x} \|x^k - y^k\| + \gamma_{k,x} \|F(x^k; \theta^k) - F(x^k; \theta^*)\|.
\]
By Lemma 3 and 4 and (A3), the right hand side of (14) can be further bounded by
\[
q_{k,x} \|x^k - y^k\| + \gamma_{k,x} L_0 \|\theta^k - \theta^*\|
\leq q_{k,x} \|x^k - y^{k-1}\| + q_{k,x} M \epsilon_{k-1} - \epsilon_k + \gamma_{k,x} L_0 \|\theta^k - \theta^*\| \quad (15)
\]
where $q_{k,x} = \sqrt{1 - 2 \gamma_{k,x} \epsilon_k + (\gamma_{k,x})^2((L_x)^2 + (\epsilon_k)^2)}$ and $M = \|x^*\|$. Again, we may recall that the sequences $\{x_k\}$ and $\{\theta_k\}$ are random sequences derived from the sequence of events captured by $\mathcal{F}_k$. By combining (14) and (15) and taking expectations, we have that
\[
\mathbb{E}[\|x^{k+1} - y^k\|] \leq q_{k,x} \mathbb{E}[\|x^k - y^{k-1}\|] + q_{k,x} M \epsilon_{k-1} - \epsilon_k + \gamma_{k,x} L_0 \mathbb{E}[\|\theta^k - \theta^*\|].
\]
If the sequence $\{\theta_k\}$ is bounded, it is uniformly integrable. Furthermore, since $\theta^k$ converges to $\theta^*$ in probability, it follows that $\mathbb{E}[\|\theta^k - \theta^*\|] \to 0$.

By Lemma 5, we have
\[
\sum_{k=1}^{\infty} (1 - q_{k,x}) = \infty \quad \text{and} \quad \lim_{k \to \infty} \frac{q_{k,x}}{M \epsilon_{k-1} - \epsilon_k} = 0.
\]
Furthermore, we have that
\[
\lim_{k \to \infty} \frac{\gamma_{k,x} L_0 \mathbb{E}[\|\theta^k - \theta^*\|]}{(1 - q_{k,x})} = \lim_{k \to \infty} \frac{\gamma_{k,x}(1 + q_{k,x}) L_0 \mathbb{E}[\|\theta^k - \theta^*\|]}{(2 \gamma_{k,x} \epsilon_k - \gamma_{k,x}^2 L_x^2 - \epsilon_k^2)} = \lim_{k \to \infty} \frac{\gamma_{k,x}(1 + q_{k,x}) L_0 \mathbb{E}[\|\theta^k - \theta^*\|]}{(2 \gamma_{k,x} \epsilon_k - \gamma_{k,x}^2 L_x^2 - \epsilon_k^2)} = 0,
\]
a consequence of observing that $\mathbb{E}[\|\theta^k - \theta^*\|] \to 0$ as $k \to \infty$, $\gamma_{k,x} \epsilon_k$ is bounded for all $k$. Therefore, from (13), we obtain that $\mathbb{E}[\|y^k - x^*\|] \to 0$ as $k \to \infty$.

IV. Numerical results

In this section, we illustrate our algorithms through several examples based on the source problems introduced in Section II-A. We examine the schemes the regime where observations are indeed corrupted by noise.

A. Sample problems

We consider both convex optimization problems (2) and Nash-Cournot games (4) in this section.

1) Convex optimization problems: Suppose the objective function in (2) is given by $\sum_{i=1}^{3} \theta_i h_i(x)$, where
\[
(h_1(x), h_2(x), h_3(x)) = (e^{x_1-a} + e^{x_2-b-3}, e^{-(x_1-a-3) + e^{-(x_2-b)}}, (x_3 - c)^2),
\]
and $\theta^* = (3, 3e^{-3}, 5)^T$. Then, the optimal solution to (2) is $x^* = (a, b, c)^T$.

2) Nash-Cournot games: Suppose in (4) that $f_i(x; \theta^*) = c_i x_i - x_i (a^* - b^* x^*)$, where $a = 1.5$ and $\theta^* = (a^*, b^*)^T = (100, 10)^T$. Suppose we do not know $\theta^*$, but we can observe the price $\ell(x; \theta^*) = a^* - b^* x^*.$

B. Imperfect observations

We now consider the case when $\xi(\omega)$ is a mean-zero random variable in our observation (7) and as earlier, convex optimization problems and Nash-Cournot games are examined. We find that the least-squares learning techniques coupled with gradient schemes are seen to be capable of contending with such problems.
1) Optimization problems: Suppose the optimization problem (2) is as defined in Section IV-A. As earlier, \( \theta^* \) is unavailable but may be learnt by observing \( \ell(x; \theta^*) = g(x)^T \theta^* + \xi \), where \( g(x) = h(x) \) and \( \xi \) is a normal random variable with mean zero and standard deviation 5. Then, we can use Algorithm 1 with diminishing steplengths to learn both \( x^* \) and \( \theta^* \). Let \( x^k_i \in [0.1, 20] \) for all \( i, k \), and \( \gamma_{k,x} = 1/k \). After 10000 iterations, we have the following result for different values of \( a, b, c \)

<table>
<thead>
<tr>
<th>( a )</th>
<th>( b )</th>
<th>( c )</th>
<th>( \frac{|x^* - x^k|}{1 + |x^k|} )</th>
<th>( \frac{|\theta^* - \theta^k|}{1 + |\theta^k|} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>2</td>
<td>3</td>
<td>2.2 \times 10^{-2}</td>
<td>4.3 \times 10^{-3}</td>
</tr>
<tr>
<td>2</td>
<td>3</td>
<td>1</td>
<td>3.0 \times 10^{-2}</td>
<td>3.3 \times 10^{-3}</td>
</tr>
<tr>
<td>4</td>
<td>6</td>
<td>8</td>
<td>1.3 \times 10^{-2}</td>
<td>4.6 \times 10^{-3}</td>
</tr>
<tr>
<td>8</td>
<td>4</td>
<td>6</td>
<td>1.4 \times 10^{-2}</td>
<td>7.1 \times 10^{-3}</td>
</tr>
</tbody>
</table>

2) Nash-Cournot games: As in the previous subsection, we consider the Nash-Cournot games defined in Section IV-A. Again, \( \theta^* \) is unavailable but may be learnt through observing a noise-corrupted variant of the price \( \ell(x; \theta^*) = a^* - b^* X^a + \xi \), where \( \xi \) is a normal random variable with mean zero and standard deviation 5. Then, we may use Algorithm 1 to learn both \( x^* \) and \( \theta^* \). Let \( x^k_i \in [0.1, 20] \) for all \( i, k \), and \( \gamma_{k,x} = 1/k \). After 10000 iterations, we have the following result for different dimensions \( n \) of \( x^* \)

<table>
<thead>
<tr>
<th>( n )</th>
<th>( \frac{|x^* - x^k|}{1 + |x^k|} )</th>
<th>( \frac{|\theta^* - \theta^k|}{1 + |\theta^k|} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>4</td>
<td>9.7 \times 10^{-5}</td>
<td>6.4 \times 10^{-5}</td>
</tr>
<tr>
<td>5</td>
<td>8.3 \times 10^{-3}</td>
<td>5.2 \times 10^{-3}</td>
</tr>
<tr>
<td>6</td>
<td>3.7 \times 10^{-3}</td>
<td>2.3 \times 10^{-3}</td>
</tr>
<tr>
<td>7</td>
<td>1.5 \times 10^{-3}</td>
<td>9.3 \times 10^{-4}</td>
</tr>
</tbody>
</table>

V. Concluding remarks

Motivated by online computational problems where parameters may be unavailable, we consider the development of joint schemes for computing a solution of the variational problem in question while learning the unknown parameter. More specifically, we consider a parameterized monotone variational inequality that can capture the convex optimization problems, convex Nash games and certain economic and traffic equilibrium problems. Unlike in standard offline supervised learning settings, observations appear over the span of time, in response to changing decisions. Often, the observations may be corrupted by noise. Schemes are proposed in such regimes and lead to probabilistic convergence guarantees. To contend with merely monotone settings, we provide regularized variants that admit similar convergence properties.

We believe that the avenue of computation and learning presents new and interesting challenges from several distinct sources. Of these, one direction of interest is that arising from more general learning problems that, while convex, may be complicated by nonsmoothness, singular Hessians and rapid growth in size. Contending with such problems through smoothing, regularization and decomposition remains a goal of future work.

REFERENCES