Generalization of Proportional Adaptation Law for $\mathcal{L}_1$ Adaptive Controller

Justin Vanness, Evgeny Kharisov and Naira Hovakimyan

Abstract—The paper presents a generalized framework for an $\mathcal{L}_1$ adaptive state-feedback controller with proportional adaptation gain. Using the decoupling property for estimation and control loops of the $\mathcal{L}_1$ adaptive controller, we show that this generalized architecture has the potential of unifying several nonlinear control methods in a single framework. In all cases, the proposed controller, similar to other $\mathcal{L}_1$ controllers, offers systematic tuning of robustness and performance bounds, by increasing the adaptation gain and careful selection of the lowpass filter. Simulations verify the theoretical findings.

I. INTRODUCTION

We review several nonlinear control methods in this paper, including model reference adaptive control (MRAC) [1]–[3], sliding mode control [4], [5], and funnel control [6]–[8], and present a variant of $\mathcal{L}_1$ control architecture that shares certain similarities with these methods. One of the main benefits of the sliding mode control is its robustness with respect to parametric uncertainty. On the other hand, several drawbacks are pointed out in the literature. In particular, it involves large control authority and can often lead to chattering [9]. In [10] it is shown that in some cases chattering can lead to closed-loop system instability. There are multiple solutions that aim to address the above drawbacks using different approaches [11]. A variant of $\mathcal{L}_1$ controller with switching adaptation laws, which addresses the above issues by means of the methods of the $\mathcal{L}_1$ adaptive control theory, was presented in [12].

The funnel control was designed to address significant parametric uncertainty in the presence of measurement noise without explicit parametric estimation [6]. It is applicable to a class of minimum-phase systems of relative degree one with known high-frequency gain [6], [13]. It is also important to notice that funnel control is based on high-gain design, which in some cases may affect the time-delay margin of the system.

MRAC was introduced to control systems in the presence of large parametric uncertainty. The main difference of MRAC approach from robust control is that it has the potential of learning and utilizing a posteriori information about the system, which is obtained during its operation. On the contrary, the robust controllers solely rely on a priori available information [14]. However, performance of the closed-loop MRAC systems strongly depends on the performance of the parameter estimation algorithm of the MRAC, as it is tightly coupled with control. The possibility of learning in MRAC, as well as for other adaptive control methods, depends on “information richness” of the system signals, which cannot be guaranteed a priori. As a consequence, despite the stability proofs [15], [16], performance and robustness results are limited [17].

$\mathcal{L}_1$ adaptive control theory [18] initially was developed as an extension of indirect MRAC, which overcomes the above limitations of MRAC. Namely, it ensures robustness of the closed-loop system in the presence of fast estimation rates [19]; that is fast adaptation in $\mathcal{L}_1$ adaptive control does not lead to high-gain control as it does for MRAC type controllers [20]. By enabling fast adaptation, the $\mathcal{L}_1$ adaptive controller can achieve guaranteed and predictable performance. In particular, for all $\mathcal{L}_1$ adaptive architectures, the transient performance of the closed-loop adaptive system is quantified both for the system output and the input by uniform performance bounds with respect to an $\mathcal{L}_1$ reference system, which incorporates a lowpass filter. The performance bounds can be arbitrarily improved by increasing the adaptation rate. Some recent findings in [21] establish connections between the $\mathcal{L}_1$ adaptive control theory and the internal model control, which are further exploited in [22] towards applying methods from robust control to the filter design of the $\mathcal{L}_1$ adaptive controller. For applications of the $\mathcal{L}_1$ adaptive control theory we refer the reader to [23] and references therein.

The key property of the $\mathcal{L}_1$ adaptive controller is that it decouples the estimation process from control [20]. Since the tradeoff between robustness and performance of the $\mathcal{L}_1$ adaptive controller is resolved by the choice of the lowpass filter in the control law, the decoupling of the estimation allows using different types of adaptation laws and modifications of the state predictor without affecting robustness of the closed-loop system. We exploit this feature in this paper and generalize the $\mathcal{L}_1$ adaptive control architecture with proportional adaptation law. We show that this generalized architecture has a potential of unifying the switching laws and funnel control structure along with several other methods in a single framework, which ensures robustness and performance properties of the above mentioned controllers.

The generalized $\mathcal{L}_1$ adaptive control architecture presented in this paper is applicable to a class of nonlinear state-feedback systems with unmatched uncertainties of the form

$$\dot{x}(t) = A_m x(t) + b_m \omega u(t) + f(t, x),$$

$$y(t) = c^\top x(t), \quad x(0) = x_0,$$

where $x(t) \in \mathbb{R}^n$ and $y(t) \in \mathbb{R}$ are the system state vector and the output respectively; $u(t) \in \mathbb{R}$ is the control signal; $b_m, c \in \mathbb{R}^n$ are known constant vectors; $f(t, x) : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ is a Lipschitz function in $x$ and uniformly bounded in time, which represents the system uncertainty; $A_m \in \mathbb{R}^{n \times n}$

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J. Vanness, E. Kharisov and N. Hovakimyan are with UIUC, Urbana, IL 61801, e-mail: {jvanness2, evgeny, n.hovakimyan}@illinois.edu
is a known Hurwitz matrix specifying the desired poles of the closed-loop dynamics; $\omega \in \mathbb{R}$ is an unknown constant with known sign. However, for the sake of proofs due to page limitations, in this paper we consider a simplified problem formulation only for parametric matched uncertainty. The general result and the proofs can be found in journal version of this paper.

The paper is organized as follows: Section II gives the problem formulation. Section III defines the $L_1$ adaptive control architecture with generalized proportional adaptation law. The stability and the transient performance bounds are derived in Section IV. Three specific examples of control laws that fall under the generalized framework are presented in Section V. Simulation results are given in Section VI, and Section VII concludes the paper.

II. PROBLEM FORMULATION

Consider the system given by

$$\dot{x}(t) = A_m x(t) + b_m (\omega u(t) + \theta^T(t)x(t) + \sigma(t)),$$

$$y(t) = c^T x(t), \quad x(0) = x_0,$$

where $x(t) \in \mathbb{R}^n$ and $y(t) \in \mathbb{R}$ are the system state vector and the output respectively; $u(t) \in \mathbb{R}$ is the control signal; $b_m, c \in \mathbb{R}^n$ are known constant vectors; $A_m$ is a known Hurwitz $n \times n$ matrix specifying the desired poles of the closed-loop dynamics; $\omega \in \mathbb{R}$ is an unknown constant with known sign; $\theta(t) \in \mathbb{R}^n$ is a vector of time-varying unknown parameters; and $\sigma(t) \in \mathbb{R}$ models input disturbances.

Assumption 1. Let $\omega \in \Omega \triangleq [\omega_l, \omega_u], \theta(t) \in \Theta, |\sigma(t)| \leq \Delta, \forall t \geq 0$, where $0 < \omega_l < \omega_u$ are given known upper and lower bounds on $\omega$, $\Theta$ is a known convex compact set, and $\Delta \in \mathbb{R}^+$ is a known (conservative) bound of $\sigma(t)$.

The control objective is to design a full-state-feedback adaptive controller, which ensures that $y(t)$ tracks a given bounded piecewise-continuous reference signal $r(t) \in \mathbb{R}$ with a quantifiable performance bound. The performance specifications are given by the following ideal system:

$$\dot{x}_{id}(t) = A_m x_{id}(t) + b_m k_g r(t),$$

$$y_{id}(t) = c^T x_{id}(t), \quad x_{id}(0) = x_0,$$

where $k_g \triangleq -1/(c^T A_m^{-1} b_m)$.

III. $L_1$ ADAPTIVE CONTROL ARCHITECTURE

The $L_1$ adaptive controller presented in this paper, similar to all other $L_1$ architectures, is comprised of a state predictor, adaptive law, and control law, arranged as shown in Figure 1.

We consider the following state predictor:

$$\dot{\hat{x}}(t) = A_m \hat{x}(t) + b_m \omega_0 u(t) + \hat{\eta}(t), \quad \hat{x}(0) = x_0,$$

$$\dot{\hat{y}}(t) = c^T \hat{x}(t),$$

which has a similar structure as (1), except $\omega$ is replaced by its best available guess $\omega_0 \in \Omega$, and the unknown estimates of $\omega, \theta(t)$, and $\sigma(t)$ are grouped together to form the estimated parameter $\hat{\eta}(t) \in \mathbb{R}^n$, which in general appears in an unmatched fashion.

The adaptive process is governed by the following generalized adaptation law:

$$\dot{\hat{\eta}}(t) = -\Gamma(t) \hat{x}(t),$$

where $\hat{x}(t) \triangleq \hat{x}(t) - x(t)$, and $\Gamma(t)$ is diagonal $n \times n$ piecewise continuous matrix function with the only requirement being that $\lambda_{\min}(\Gamma(t)) > \Gamma_{\min}$ for all $t \geq 0$ for some $\Gamma_{\min} \in \mathbb{R}^+$. This loose condition on the adaptive gain provides a generalized framework for the analysis of stability and performance, under which several methods for generating an adaptive gain can be considered.

The $L_1$ adaptive control law is generated as the output of the following feedback system:

$$u(s) = -k D(s) (\omega_0 u(s) + \hat{\eta}(s) - k_g r(s)),$$

$$\dot{\hat{\eta}}(s) = (P_m b_m)^T (s I - A_m)^{-1} \hat{\eta}(s),$$

$$H_m(s) \triangleq (P_m b_m)^T (s I - A_m)^{-1} b_m,$$

where $r(s)$ and $\hat{\eta}(s)$ are the Laplace transforms of $r(t)$ and $\hat{\eta}(t)$ respectively; and $k > 0$ and $D(s)$ are a feedback gain and a strictly proper transfer function respectively, which lead to a strictly proper stable transfer function

$$C(s) \triangleq \frac{\omega k D(s)}{1 + \omega k D(s)}, \quad \forall \omega \in \Omega,$$

with DC gain $C(0) = 1$, and $P = P^T > 0$ is the solution of the algebraic Lyapunov equation $A_m^T P + P A_m = -Q$ for arbitrary $Q = Q^T > 0$.

Remark 1. Notice that an arbitrary vector $c_0 \in \mathbb{R}^n$, which leads to minimum phase $c_0^T (s I - A_m)^{-1} b_m$, can be used instead of $P b_m$ in both (5) and (6).

The $L_1$ adaptive controller is defined via (2), (3) and (4), subject to the following $L_1$-norm condition:

$$\|G(s)\|_{L_1} L < 1,$$

where

$$L \triangleq \max_{\theta \in \Theta} \|\theta\|_1, \quad H(s) \triangleq (s I - A_m)^{-1} b_m,$$

$$G(s) \triangleq H(s)(1 - C(s)).$$

IV. ANALYSIS OF THE $L_1$ ADAPTIVE CONTROLLER

A. $L_1$ Reference System

Similar to all $L_1$ adaptive controllers, we consider the following $L_1$ reference system

$$\dot{x}_{ref}(t) = A_m x_{ref}(t) + b_m (\omega u_{ref}(t) + \psi_{ref}(t)),$$

$$y_{ref}(t) = c^T x_{ref}(t), \quad x_{ref}(0) = x_0,$$

where

$$\psi_{ref}(t) \triangleq \theta^T(t) x_{ref}(t) + \sigma(t),$$

and the reference control law is given by

$$u_{ref}(s) = -\frac{C(s)}{\omega} (\psi_{ref}(s) - k_g r(s)).$$

The stability of the closed-loop system in (8)-(9) is ensured by the following lemma.
The proof of Lemma 1 is given in [18].

**Remark 2.** The $L_1$ reference system assumes partial compensation of uncertainties within the bandwidth of the control channel. It therefore depends on the unknown system parameters and disturbances and cannot be implemented directly. The $L_1$ reference system is used solely for the purpose of analysis.

**B. Error Dynamics**

The system dynamics in (1) and the state predictor in (2) lead to the following prediction-error dynamics:

$$
\hat{x}(t) = A_m\hat{x}(t) - b_m\eta(t) + \eta(t), \quad \hat{x}(0) = 0,
$$

where $\hat{x}(s)$ is defined as

$$
\hat{x}(s) = \frac{(P_m^b)^T(s\omega - A_m)^{-1}(sI - A_m)^{-1}b_m}{(P_m^b)^T(sI - A_m)^{-1}b_m} \eta(s) - \eta(s).
$$

**Lemma 2.** Given the system in (1) and the $L_1$ adaptive controller defined via (2), (3), and (4), if $\|x_r\|_{L_\infty} \leq \rho$ and $\|u_r\|_{L_\infty} \leq \rho_u$ for some $\rho, \rho_u \in \mathbb{R}^+$ and time $\tau > 0$, then we have:

$$
\|\hat{x}_r\|_{L_\infty} \leq \gamma_0 \leq \gamma_1 \leq \gamma_2,
$$

where $\Delta_\eta$ and $\nu$ are defined as

$$
\Delta_\eta \equiv (\omega_u - \omega)\rho_u + L_\rho + \Delta,
$$
$$
\nu \equiv \inf_{t \geq 0} \lambda_{\min}(\Gamma(t)).
$$

The proof of Lemma 2 is given in the appendix.

**C. Performance Bounds on Closed-loop Adaptive System**

The error bounds between the system’s states and the $L_1$ reference states and the system’s input and the $L_1$ reference input are given in the next theorem.

**Theorem 1.** Given the system in (1) and the $L_1$ adaptive controller defined via (2), (3), and (4), subject to the $L_1$-norm condition in (7), if we have $\|x_0\|_{L_\infty} \leq \rho_0$, then

$$
\|x\|_{L_\infty} \leq \rho, \quad \|u\|_{L_\infty} \leq \rho_u, \quad \|\hat{x}\|_{L_\infty} \leq \gamma_0,
$$

$$
\|x_{ref} - x\|_{L_\infty} \leq \gamma_1, \quad \|u_{ref} - u\|_{L_\infty} \leq \gamma_2,
$$

where $\rho$ and $\rho_u$ are defined as

$$
\rho \triangleq \rho_1 + \gamma_1, \quad \rho_u \triangleq \rho_u + \gamma_2,
$$

and

$$
\rho_r \triangleq \frac{\|H(s)C(s)k_g\|_{L_1}}{1 - \|G(s)\|_{L_1}},
$$
$$
\rho_{ur} \triangleq \frac{\|L(s)C(s)\|_{L_1}}{1 - \|G(s)\|_{L_1}} + \frac{\|G(s)\|_{L_1}}{1 - \|G(s)\|_{L_1}} \|r\|_{L_\infty}
$$

with $\beta$ and $\tilde{\gamma}_0$ being some arbitrary (small) positive constants, such that $\gamma_0 \leq \tilde{\gamma}_0$, and $x_{in}(s) \triangleq (sI - A_m)^{-1}x_0$.

The proof of Theorem 1 is given in [24].

**Remark 3.** We notice that the structure of the performance bounds (12)-(13) is identical to the performance bounds of the $L_1$ adaptive controller given in Section 2.2.1 of [18]. This similarity is a consequence of the decoupling of the estimation loop from the control loop, which is achieved by the $L_1$ adaptive control architectures [20].

**Remark 4.** Notice that the bounds for $\hat{x}$, $\|x_{ref} - x\|_{L_\infty}$, and $\|u_{ref} - u\|_{L_\infty}$ can be made arbitrarily small by increasing the value of $\nu$ in (11). Thus, one can ensure a desired bound by increasing the minimum eigenvalue of the gain matrix $\Gamma(t)$. 3217
V. ADAPTATION Gain GENERATION METHODS

In this section we examine three specific methods for adaptive gain generation that fall under the generalized $L_1$-adaptive controller with proportional feedback presented in the paper.

A. Switching Adaptation Law

We begin by considering the switching-adaptation law presented in [12], which is given by

$$
\tilde{\eta}(t) = -G_{s\text{gn}}(dz \left[(Pb_m)^T \tilde{x}(t)\right]),
$$

where $dz[x]$ is the dead zone function, the size of which is set arbitrarily small. The adaptive architecture presented in [12] has the same performance bounds as the standard $L_1$ adaptive control architectures. The above adaptation law can be presented as a time-varying adaptive gain considered in this paper by approximating it as

$$
\Gamma_{i,i}(t) = \epsilon + \frac{[(Pb)^T_i]}{\epsilon + \|\tilde{x}_i(t)\|},
$$

where epsilon is an arbitrarily small positive constant used to avoid singularity and $\cdot_i$ denotes the $i^{th}$ component of the vector while $i,i$ denotes the $i^{th}$ component of the gain matrix.

In the subsequent sections we now consider examples of adaptive gain generation techniques, which fit the architecture presented in this paper, but have not been considered for $L_1$ adaptive controller thus far.

B. Funnel Adaptation Law

One specific application of time-varying proportional gain is the funnel control. In funnel control, the time-varying gain is defined as follows

$$
\Gamma(t) = \frac{1}{\mathcal{F}(t) - \|\tilde{x}(t)\|_p},
$$

where $p$ can be any vector norm, $\tilde{x}(t) \in \mathbb{R}^n$ is the prediction error, and $\mathcal{F}(t) \in \mathbb{R}$ is a continuous function of time such that $\|\tilde{x}(0)\| < \mathcal{F}(0)$, and $\min_{t \geq 0} \mathcal{F}(t) \geq F_\infty > 0$ and represents a time-varying funnel, in which the error remains, as shown in Figure 2. For example, the Exponential Funnel Boundary [13] is given by

$$
\mathcal{F}_{exp}(t) = F_0 \exp\left(-\frac{t}{T}\right) + F_\infty,
$$

where $T$ is the time constant and $F_0$ and $F_\infty$ are positive constants such that $\mathcal{F}(0) = F_0 + F_\infty$ and $\lim_{t \to \infty} \mathcal{F}_{exp}(t) = F_\infty$.

Notice that $\|\tilde{x}(t)\|$ must be less than $\mathcal{F}(0)$ in order for the initial gain to be positive, which corresponds to the requirement that the error must be initialized within the funnel. Then, as the value of the error approaches the funnel boundary, the denominator in (14) decreases, resulting in an increased gain, and forces the error towards zero. Infinite gain occurs as the error approaches the boundary. Therefore, funnel control can exhibit very large adaptation gains which can lead to robustness issues, such as small time delay margins.

The key difference of our design from conventional funnel control [13] is that the adaptive gain in the $L_1$ architecture considers the error between the actual and the state predictor instead of the error between the output and the reference signal. Due to the filtering structure that decouples robustness and adaptation in $L_1$ theory [23], the high gain resulting from the funnel control acts to improve performance without decreasing robustness. The tradeoff between robustness and performance instead manifests itself in the choice of the lowpass filter.

Remark 5. We notice that there may exist particular operational conditions, for which the funnel control or the sliding mode control may outperform their $L_1$ adaptive control versions. For example, in [25] we have shown how for particular flight conditions MRAC outperforms the $L_1$ controller in the time-delay marginal assessment. The main benefit of the $L_1$ controller, as explained in [25] and various other papers as well, is in its guaranteed and uniform performance across the operational envelope for various changes in system dynamics in terms of uncertainties. This paper shows that the same uniformity extends to other controllers as well and is not limited to MRAC per se.

C. Variable Dependent Adaptation Law

Finally, we present an adaptive gain generation method, dependent on a vector $\psi(t) \in \mathbb{R}^q$, to be selected as a tuning variable, where $q$ can be any number. The gain adjustment equation is given as

$$
\Gamma(t) = \left(\frac{G_L + \frac{G_U}{1 + \alpha\|\psi(t)\|_p}}{1}\right)^\|, 
$$

with $G_L,G_U, \alpha \in \mathbb{R}^+$ and $p$ represents any vector norm. The adaptation gain has a lower bound $G_L$ and an upper bound $G_U + G_L$, while the rate of variation is dependent on the design parameter $\alpha$. Thus, as the norm of the vector $\psi(t)$ increases, the adaptation gain decreases. Note that the lower bound can be set to ensure given performance bounds. We next suggest two variables that can be used to vary the adaptation gain.

The function $\psi(t)$ can be selected based on the CPU rate. The main limiting factor to fast adaptation is the available CPU, since $L_1$ architectures require high CPU demand. Varying the adaptation gain according to CPU demand can be used to regulate the CPU usage, and may lead to improved performance, since in many cases the adaptation gain is selected based on conservative CPU demand estimations.
We illustrate some of the results presented in this section in the simulations below.

VI. SIMULATIONS

Consider the dynamics of a single-link robot arm rotating on a vertical plane:

\[ I \ddot{q}(t) = u(t) + \frac{Mgl \cos(q(t))}{2} + \dot{\sigma}(t) + F_1(t)q(t) + F(t)\dot{q}(t), \]

where \( q(t) \) and \( \dot{q}(t) \) are the angular position and velocity, respectively, \( u(t) \) is the input torque, \( I \) is the unknown moment of inertia, \( M \) is the unknown mass, \( l \) is the unknown length, \( F \) is an unknown time-varying friction coefficient, \( F_1(t) \) is the position-dependent external torque coefficient, and \( \dot{\sigma}(t) \) is an unknown uniformly bounded disturbance. The equations of motion can be cast into the form of (1) where \( \omega = 1/I \) is the unknown control effectiveness, and

\[
A_m = \begin{bmatrix} 0 & 1 \\ -1 & -1.4 \end{bmatrix}, \quad b_m = \begin{bmatrix} 0 \\ 1 \end{bmatrix},
\]

\[
\theta(t) = \begin{bmatrix} 1 + \frac{F_1(t)}{I}, & 1.4 + \frac{F(t)}{I} \end{bmatrix}^\top,
\]

\[
\sigma(t) = \frac{Mgl \cos(x_1(t))}{2I} + \frac{\dot{\sigma}(t)}{I}.
\]

For the simulations, consider the following uncertainty:

\[
\omega = 1,
\]

\[
\theta(t) = [2 + \cos(\pi t), \ 2 + 0.3\sin(\pi t) + 0.2\cos(\pi t)]^\top,
\]

\[
\sigma(t) = \sin\left(\frac{\pi t}{2}\right).
\]

The compact sets can be conservatively set to \( \Omega = \{0.5, 1, 1.8\} \), \( \Theta = \partial = [\vartheta_1, \vartheta_2]^\top \in \mathbb{R}^2 : \vartheta_i \in [-5, 5], i = 1, 2, \) and \( \Delta = 10. \)

We implement the \( L_1 \) adaptive controller according to (2), (3), and (4), subject to the \( L_1 \)-norm condition in (7). Letting \( D(s) = 1/s \), we have

\[
G(s) = \frac{s}{s + \omega k} H(s),
\]

\[
H(s) = \begin{bmatrix} 1 & s \\ s^2 + 1.4s + 1 & s^2 + 1.4s + 1 \end{bmatrix}^\top.
\]

It is straightforward to verify numerically that for \( \omega k > 30 \), one has \( \|G(s)\|_{L_1} \leq 1 \). Since \( \omega > 0.5 \), we set \( k = 60. \)

The adaptive gain for funnel control algorithm in (14) uses the infinity norm. The exponential funnel boundary from (15) is used with \( F_0 = 0.3, F_{\infty} = 0.05, \) and \( T = 2. \)

The plant states are initialized as \( x_0 = [0.1, 0.2]^\top \), while the state predictor is initialized with \( \dot{x}_0 = [0, 0]^\top \). A small initialization error is presented so that the initial error does not start at the origin. Figure 3 shows the state response and the control input. There is an initial error between the output of the system and the reference system, which eventually converges as the funnel narrows. The adaptive gain varies in order to keep the error within the funnel as shown in Figure 4.

Due to page limits, in this paper we provide only simulations for funnel adaptation laws. The simulation results for \( L_1 \) adaptive controller with switching adaptation laws for the same system can be found in [12]. We notice that simulations in [12] show similar uniform performance results, as achieved in Figure 3.

VII. CONCLUSIONS

A generalized framework for an \( L_1 \) adaptive state-feedback controller with proportional adaptation gain was presented. We showed that this generalized architecture has a potential of unifying the switching control and funnel control structure along with several other methods in a single framework. The proposed architecture leads to uniform performance bounds for system output and control signal with guaranteed robustness in the presence of fast adaptation.

REFERENCES


Fig. 3: Performance of proportional $L_1$ adaptive controller using funnel adaptation gain.

Fig. 4: Prediction error transient and adaptation gain history for $L_1$ adaptive controller with funnel adaptation gain.


APPENDIX

Proof of Lemma 2: Consider the Lyapunov function candidate: $V(t) = \frac{1}{2} \dot{x}^T(t) P \dot{x}(t)$. First we prove that $V(t) \leq \frac{1}{2} \lambda_{\max}(P) \left( \frac{\|P b_m \Delta u\|}{\lambda_{\min}(P) \nu} \right)^2$. Since $\dot{x}(0) = x(0)$, we can easily verify that $V(0) = 0$. Taking the time derivative of the Lyapunov function, we obtain

$$V(t) = \frac{1}{2} \left( -\dot{x}^T(t) Q \dot{x}(t) - 2 \dot{x}^T(t) P b_m \eta(t) + 2 \dot{x}^T(t) \Gamma(t) \eta(t) \right).$$

Substituting the adaptation law (3) in for $\eta$ we get

$$\dot{V}(t) = \frac{1}{2} \left( -\dot{x}^T(t) Q \dot{x}(t) - 2 \dot{x}^T(t) P b_m \eta(t) - 2 \dot{x}^T(t) \Gamma(t) \dot{\tilde{x}}(t) \right).$$

which can be upper bounded to achieve the following

$$\dot{V}(t) \leq -\dot{x}^T(t) P \Gamma(t) \dot{\tilde{x}}(t) + \|\dot{\tilde{x}}(t)\| \|P b_m\| \Delta u,$$

where the value $\Delta u$ bounds $\eta(t)$ as follows:

$$\|\eta(t)\|_{L_\infty} \leq (\omega_u - \omega_f) \|u_r\|_{L_\infty} + L \|x_r\|_{L_\infty} + \Delta = \Delta_u.$$ 

Given that $\Gamma(t)$ is diagonal and $P$ is a positive definite and symmetric we obtain

$$\lambda_{\min}(P \Gamma(t)) = \lambda_{\min}(P) \lambda_{\min}(\Gamma(t)) \geq \lambda_{\min}(P) \nu.$$

Then, since $\nu$ and $\lambda_{\min}(P)$ are greater than zero, if

$$\|\dot{\tilde{x}}(t)\| > \|P b_m \Delta u\| \quad \text{then} \quad \dot{V}(t) < 0.$$

If at any time $t_1 > 0$, one has

$$V(t_1) > \frac{1}{2} \lambda_{\max}(P) \left( \frac{\|P b_m \Delta u\|}{\lambda_{\min}(P) \nu} \right)^2,$$

then

$$\frac{1}{2} \lambda_{\max}(P) \left( \frac{\|P b_m \Delta u\|}{\lambda_{\min}(P) \nu} \right)^2 < V(t_1) \leq \frac{1}{2} \lambda_{\max}(P) \|\dot{x}(t)\|^2,$$

which results in $\|\dot{x}(t)\| > \|P b_m \Delta u\| \lambda_{\min}(P) \nu$, causing $\dot{V}(t_1) < 0$. Then it follows that

$$V(t) \leq \frac{1}{2} \lambda_{\max}(P) \left( \frac{\|P b_m \Delta u\|}{\lambda_{\min}(P) \nu} \right)^2,$$

Since $\frac{1}{2} \lambda_{\min}(P) \|\dot{x}(t)\|^2 \leq V(t)$, it follows that

$$\|\dot{x}(t)\| \leq \frac{1}{\nu} \left\| \|P b_m \Delta u\| \lambda_{\min}(P) \right\|^{\frac{1}{3}} \lambda_{\max}(P),$$

and the proof is complete. □