Control of Undirected Four-agent Formations in 3-dimensional Space

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Abstract—We investigate a four-agent tetrahedral formation consisting of mobile autonomous agents in 3-dimensional space. The formation shape is required to be maintained only by the given inter-agent distance constraints. We use the gradient control to maneuver the agents to achieve the desired inter-agent distances. We provide analysis on the global behavior of the agents to explain whether or not the agents converge to the desired equilibrium set. Numerical simulation results are also included.

I. INTRODUCTION

Formation control of mobile autonomous agents has received increased attention in recent years as a part of distributed control systems. Particularly, advances of consensus algorithms contribute to solutions for the distributed formation control problems [1]. In the consensus-based formation control, the desired formation is achieved by controlling the relative displacements between each agent, which results in the concept of displacement-based formation control [2].

Unlike the displacement-based formation control, there have been many attempts to maneuver the formations by controlling the relative distances. For example, Olfati-Saber & Murray provide gradient-based controls obtained from the potential function consisting of distance errors [3]. Depending on the problem posing, formations could be represented by directed or undirected graphs. In the distance-based formation control, the graph rigidity and persistence play important roles in ascertaining the consistency of the formation [4].

By taking the idea in [3], Krick et al. provide more concrete analysis for local asymptotic stability of multi-agent formations which are represented by infinitesimally rigid frameworks for undirected formations [5]. Those approaches used in [3], [5] are categorized into the distance-based formation control [2]. As an extension of the distance-based formations in 2-dimensional space into n-dimensional formations, Oh & Ahn provide local asymptotic stability of rigid formations under the gradient-based controls [6].

Not surprisingly, there exists another control strategy not directly derived from the gradient of a certain potential function. Oh & Ahn interpret the change of inter-agent distances as a virtual dynamics, and then suggest controls that are inversely derived from the virtual controls [7]. They provide global (respectively, local) stability analysis for 3-agent (respectively, n-agent) case.

However, if the formation consists of more than three agents in the plane, most of the works in the previous literature provide only the local stability analysis. For triangular formations of three agents, there are fully analyzed results on the global behavior of the agents [7]–[13]. As an attempt to extend those results to the formations having more than three agents, four-agent formations whose underlying graph is complete are investigated [14], [15] although those results are confined to formations in 2-dimensional space.

The reason why there are many publications on the triangular formation is that the triangular formation in the plane is the simplest rigid (and persistent) formation whose body dimension is not smaller than the embedding dimension. Similarly, four-agent tetrahedral formation is the simplest rigid formation in 3-dimensional space. In that regard, we want to provide an analysis for the global behavior of the tetrahedral formation.

We assume that each agent is modeled as a single integrator. The control law used for each agent is the same as in [5] except for the difference in embedding dimension. Under the given control law, we partition the set of equilibria into desired equilibrium subset and undesired one. Through our analysis, we show that the trajectories of the agents approach the equilibrium subsets. Then, we conclude that the agents do not approach the undesired equilibrium set by further analysis, which means that the desired equilibrium set is attractive. However, the attractiveness of the desired equilibrium set does not guarantee the convergence of the agents. By achieving the exponential decaying rate of the velocity of the agents, the convergence of the agents is verified.

The rest of the paper is organized as follows. We provide mathematical representations and preliminary knowledge to deal with our problem in Section II. Through Section III, we investigate the behavior of the agents under the given control law, and provide the detailed analysis on the equilibrium formations and convergence rate. Numerical simulation results are given in Section IV to support our analysis. Then, we finish the paper with concluding remarks in Section V.

II. PRELIMINARIES

A. Graph Representation of the Formation

To represent the formation in our problem, we use graph representation. Each agent is denoted by a vertex of a graph, and the connection between two agents is denoted by an edge. We assume that we have four autonomous mobile agents in the 3-dimensional Euclidean space. The vertex set is given by \( V = \{1, 2, 3, 4\} \). Each agent is supposed to measure the relative displacements from the
others and control the relative distances. Thus, the edge set is given by \( \mathcal{E} = \{(1, 2), (1, 3), (1, 4), (2, 3), (2, 4), (3, 4)\} \). The formation graph is a pair of the vertex set and the edge set, which is written by

\[ \mathcal{G} = (\mathcal{V}, \mathcal{E}). \]

For all \( i \in \mathcal{V} \), we use \( p_i(t) \) to denote the location of agent \( i \) at time \( t \). By concatenating \( p_i \) for all \( i \in \mathcal{V} \), we write \( p = [p_1^T \cdots p_i^T]^T \in \mathbb{R}^{12} \), and call \( p \) realization of \( \mathcal{G} \). The combination of the graph and corresponding realization is called a framework and written by \( (\mathcal{G}, p) \). Thus, the formation is fully described by the framework.

**B. Motion of the Agents and Control Laws**

We assume that the motion of each agent is governed by

\[ \dot{p}_i = \frac{d}{dt}p_i(t) = u_i \in \mathbb{R}^3, \quad \forall i \in \mathcal{V}, \]

where \( u_i \) is the control input for agent \( i \), and \( u = [u_1^T \cdots u_i^T]^T \in \mathbb{R}^{12} \). The desired formation shape is specified by \( \hat{p} = [\hat{p}_1^T \cdots \hat{p}_i^T]^T \).

For convenience, we define relative position vectors as

\[ z_1 = p_2 - p_1, \quad z_2 = p_3 - p_1, \quad z_3 = p_4 - p_1. \]

Concatenated relative position vector is written by \( z = [z_1^T \ldots z_i^T]^T \in \mathbb{R}^9 \). Additionally, we define a matrix whose columns consist of the relative position vectors as \( Z = [z_1 \ldots z_4]^T \in \mathbb{R}^{3 \times 4} \).

We use \( e_{ij} \) to denote the squared-distance error between agent \( i \) and \( j \) as

\[ e_{ij}(t) = \|p_i(t) - p_j(t)\|^2 - d_{ij}^2, \quad \forall (i, j) \in \mathcal{E}, \]

where \( d_{ij} = \|p_i - p_j\| \), and \( \| \cdot \| \) is the Euclidean norm. To simplify our analysis, we confine the desired formation to a regular tetrahedral formation, i.e., \( d_{ij} = d \) for all \( (i, j) \in \mathcal{E} \).

In terms of achieving the desired formation up to congruence by controlling only the relative distances, our control objective is specified by \( \lim_{t \to \infty} e_{ij} = 0, \forall (i, j) \in \mathcal{E} \).

Define an potential function consisting of the squared-distance errors as

\[ \phi(p) = \frac{1}{4} \sum_{(i, j) \in \mathcal{E}} e_{ij}^2. \]

We use the gradient control proposed in the literature [5], [14], which is given by

\[ \dot{p} = u = -\frac{\partial \phi}{\partial p}^T = -R_G^T e \]

\[ = \begin{bmatrix} e_{12}z_1 + e_{13}z_2 + e_{14}z_3 \\ -e_{12} - e_{23} - e_{24} \end{bmatrix} \]

\[ = \begin{bmatrix} e_{23}z_1 + (-e_{13} - e_{23} - e_{34})z_2 + e_{34}z_4 \\ e_{24}z_1 + e_{34}z_2 + (-e_{14} - e_{24} - e_{34})z_3 \end{bmatrix}, \]

where \( e = [e_{12} \ldots e_{34}]^T \). Note that the control law \( u \) is designed for each agent to use only the relative measurements from the neighboring agents.

Let \( \mathcal{Q}, \mathcal{D}, \mathcal{U} \) and \( \mathcal{C} \) denote subsets defined by

\[ \mathcal{Q} = \{p : \dot{p} = 0\}, \]

\[ \mathcal{D} = \{p : e = 0\}, \]

\[ \mathcal{U} = \mathcal{Q} \setminus \mathcal{D} = \{p : \dot{p} = 0, e \neq 0\}, \]

\[ \mathcal{C} = \{p : \det Z = 0\}. \]

**C. Rigidity and Rigidity Matrix**

The matrix \( R_G \) in (1) is called the rigidity matrix of \( (\mathcal{G}, p) \). A framework \( (\mathcal{G}, p) \) having more than two vertices in the 3-dimensional space is said to be infinitesimally rigid if and only if \( \text{rank } R_G = 3|\mathcal{V}| - 6 \). Roughly speaking, a framework is rigid if there is no continuous deformation preserving the length of the edges. In Fig. 1, for example, if it is not allowed to change the length of any edge \( (i, j) \in \mathcal{E} \), then the framework cannot be distorted. For more information on the rigidity, refer to [16], [17] or [18].

Note that the number of rows of \( R_G \) is equal to \( |\mathcal{E}| \). Thus, \( R_G \) has full row rank if and only if \( (\mathcal{G}, p) \) is infinitesimally rigid in our problem \( (|\mathcal{V}| = 4, |\mathcal{E}| = 6) \).

**III. ANALYSIS**

To observe the behavior of the squared-distance errors, let us take the time derivative of \( \phi \). Then, we have

\[ \dot{\phi} = \frac{\partial \phi}{\partial p}^T = \frac{1}{2} \| \frac{\partial \phi}{\partial p} \| \leq 0. \]

Therefore, \( \phi \) is bounded, which means that \( e_{ij}, \forall (i, j) \in \mathcal{E} \), and \( z_i, \forall i \in \{1, 2, 3\} \), are also bounded. Since \( \phi \) is bounded below and non-increasing, \( \phi \) converges to a constant as \( t \to \infty \). Moreover, \( \phi \) is uniformly continuous. Thus, \( \phi \) converges to zero as \( t \) goes to infinity by Barbalat’s lemma [19, Lemma 8.2].

**Lemma 1:** Under the given control law in (2), the trajectory \( p \) of the agents approaches \( \mathcal{D} \) or \( \mathcal{U} \) as \( t \to \infty \).

**Proof:** Note that \( \phi \) is zero if and only if \( u = 0 \) from (2) and (3). We know that \( \phi \) converges to zero as \( t \) goes to infinity. Therefore, \( p \) approaches \( Q \) as \( t \) goes to infinity. Since \( \mathcal{D} \cup \mathcal{U} = \mathcal{Q} \), \( p \) approaches \( \mathcal{D} \) or \( \mathcal{U} \).
A. Undesired Equilibrium Formations

Since \( z_1, z_2, \) and \( z_3 \) exist in \( \mathbb{R}^3 \), if they are linearly independent, then \( \dot{p} = 0 \) implies that \( e = 0 \) from (2). The contrapositive is given by the following statement.

**Lemma 2:** If \( \mathbf{p} \in \mathcal{U} \), then \( z_1, z_2, \) and \( z_3 \) are linearly dependent.

Lemma 2 means that any formation formed by \( \forall \mathbf{p} \in \mathcal{U} \) should exist on a plane due to the linear dependence of \( z_1, z_2, \) and \( z_3, \) which means that \( \det Z = 0 \). Hence, \( \mathcal{U} \subset \mathcal{C} \).

Now, we want to investigate the shape of undesired equilibrium formations. To find the undesired equilibrium formations, we need to find \( z, \) which will be denoted by \( z^* \), satisfying the following equation with \( e \neq 0 \).

\[
- \mathcal{R}_{G}(z)^T e = 0. \tag{4}
\]

Unfortunately, if a particular \( z^* \) satisfies (4), then every other \( z \) generated from the formation produced by translation and rotation satisfies (4). To show this clearly, let \( \mathbf{t} = [1 \ 1 \ 1]^T \otimes [k_x \ k_y \ k_z]^T \in \mathbb{R}^{12} \) where \( \otimes \) denotes the Kronecker product. Then, \( z^* \) produced by a certain \( \mathbf{p}^* \) is the same as \( z^* \) produced by \( \mathbf{p}^* + \mathbf{t} \). Hence, the translation of the formation does not affect \( z^* \) satisfying (4). Let \( W \in \mathbb{R}^{3 \times 3} \) be a rotation matrix in 3-dimensional space. Let \( z^{*'} \) be a relative position vector produced by \( [I_4 \otimes W]p^* \). Since \( ||z^*_i|| = ||z^{*'}_i|| \) for all \( i \in \mathcal{V} \), rotation by \( W \) does not change \( e \). Also, we have \( [\mathcal{R}_{G}(z^{*'})]^T = [I_4 \otimes W] [\mathcal{R}_{G}(z^*)]^T \) because \( z^{*'} = Wz^*_i \).

From the fact that \( W \) is invertible, \( [I_4 \otimes W] \) is also invertible, so \( [\mathcal{R}_{G}(z^{*'})]^T e = 0 \) if and only if \( [\mathcal{R}_{G}(z^*)]^T e = 0 \).

As a consequence, we only need to find a particular \( z^* \) satisfying (4) up to translation and rotation of the formation to characterize the undesired formations. Let

\[
\begin{align*}
\mathbf{z}_1 &= [z_1 \ y_1 \ z_1]^T, \\
\mathbf{z}_2 &= [z_2 \ y_2 \ z_2]^T, \\
\mathbf{z}_3 &= [z_3 \ y_3 \ z_3]^T.
\end{align*}
\]

We know that every undesired equilibrium formation exists on a plane due to the linear dependence of \( z_1, z_2, \) and \( z_3 \) from Lemma 2. Hence, we assume that \( z_1 = z_2 = z_3 = 0 \) without loss of generality. However, we still have infinitely many solutions satisfying (4) due to the rotation on the \( x-y \) plane. To remove the degree of freedom by the rotation, let one of \( y_1, y_2, \) and \( y_3 \) be zero, say \( y_1 = 0 \), so that \( z_1 \) is parallel to the \( x \)-axis. Thus, we have

\[
\begin{align*}
\mathbf{z}_1 &= [x_1 \ 0 \ 0]^T, \tag{5a} \\
\mathbf{z}_2 &= [x_2 \ y_2 \ 0]^T, \tag{5b} \\
\mathbf{z}_3 &= [x_3 \ y_3 \ 0]^T. \tag{5c}
\end{align*}
\]

**Lemma 3:** Seven different formation shapes in Fig. 2 are all that can be produced by \( \forall \mathbf{p} \in \mathcal{U} \).

**Proof:** In (2), it is true that \( u_1 + u_2 + u_3 + u_4 = 0 \) under the given gradient control law. Hence, if we find \( x_1, \ x_2, \ y_2, \ x_3 \) and \( y_3 \) satisfying

\[
u_1 = 0, \quad u_2 = 0, \quad u_3 = 0, \tag{6}
\]

then they automatically satisfy that \( u_4 = 0 \) and \( \nabla \phi = 0 \). Thus, we are only required to find \( x_1, \ x_2, \ y_2, \ x_3 \) and \( y_3 \), satisfying (6), i.e., \( u_4 = 0 \) is redundant. Note that the equations in (6) are nonlinear simultaneous equations in terms of \( x_1, x_2, y_2, x_3 \) and \( y_3 \). We set \( y_1 = 0 \) to remove the degree of freedom by rotation by fixing the direction of \( z_1 \). However, if \( x_1 = 0 \), for example, then we again face the problem due to rotation, and (6) will generate infinitely many solutions. Thus, we need to solve (6) under some conditions case by case.

1. When \( x_1 = x_2 = x_3 = 0 \): the formations generated by solving (6) are parallel to the \( y \)-axis. Corresponding formation shape is depicted in Fig. 2(d–g).

2. When only one of \( x_1, x_2, \) and \( x_3 \) is nonzero with the others being zero: if \( x_k, k \in \{2, 3\} \), is nonzero, then let \( y_k = 0 \) to remove the degree of freedom by rotation and solve (6). Then we have undesired formation shape depicted in Fig. 2(e).

3. When only one of \( x_1, x_2, \) and \( x_3 \) is zero with the others being nonzero: if \( x_1 = 0 \), then let one of \( y_2 \) and \( y_3 \) be zero to remove the degree of freedom by rotation and solve (6), otherwise just solve (6) without any constraints on \( y_2 \) and \( y_3 \) (see Fig. 2(a,c,d,g)).

4. When non of \( x_1, x_2, \) and \( x_3 \) are zero: since we already set \( y_1 = 0 \), there is no degree of freedom by rotation. Thus, just solve (6) (see Fig. 2(a–c,e,g)).

The number of solutions produced by the cases through from 1) to 4) is finite. For example, the case 4) provides thirty-nine combinations of \( \{x_1, x_2, y_2, x_3, y_3\} \), and one of them is

\[
\left( \sqrt{\frac{2}{5}d}, \sqrt{\frac{3}{10}d}, \sqrt{\frac{3}{10}d}, \sqrt{\frac{3}{10}d}, \sqrt{\frac{3}{10}d} \right),
\]

which corresponds to the shape in Fig. 2(b). Therefore, by checking every solution, we can conclude that the formation
shapes in Fig. 2 are all of the undesired ones at equilibrium.

Remark 1: Note that we only focus on the shape of the formation, but absolute position and the ordering of the agents. In other words, formations with different ordering for $i, j, k, l$ in Fig. 2 are considered to be the same.

B. Repulsiveness of the Undesired Equilibrium Set

Now, we are going to show that $p$ does not approach $\mathcal{U}$ as $t \to \infty$ if $p(0) \notin \mathcal{C}$ by showing that $\det Z$ does not converge to zero and is bounded away from zero as $t \to \infty$. This technique was used by Cao et al. for three-agent directed formations in the plane [10], [11].

By taking the derivative of $\det Z$, we have

$$\frac{d}{dt} \det Z = \det \begin{bmatrix} z_1 & z_2 & z_3 \\ \dot{z}_1 & \dot{z}_2 & \dot{z}_3 \end{bmatrix} + \det \begin{bmatrix} z_1 & z_2 & z_3 \\ \dot{z}_1 & \dot{z}_2 & \dot{z}_3 \end{bmatrix} - 2\sigma \det Z,$$

where $\sigma(t) = \sum_{(i,j) \in \mathcal{E}} c_{ij}$. Hence, we know that

$$\det Z = \exp \left[ -2 \int_{\tau}^{t} \sigma(s) \, ds \right] \det Z_{\tau}, \quad t \geq \tau \geq 0, \quad (7)$$

where $\det Z_{\tau}$ is the determinant of $Z$ at $t = \tau$.

Lemma 4: For all $p \in \mathcal{U}$, it is true that $\sigma < 0$.

Proof: Since there are a finite number of configurations, which are illustrated in Fig. 2, for all $p \in \mathcal{U}$, we know that $\sigma < 0$ for all $p \in \mathcal{U}$ by calculating $\sigma$ case by case.

Note that $\sigma$ changes continuously as $p$ varies. Thus, there exists an open set $\Sigma$ such that

$$\Sigma = \{ p : \sigma < 0 \}, \quad \Sigma \supset \mathcal{U}, \quad \Sigma \cap \mathcal{D} = \emptyset,$$

from Lemma 4. Then, we have the following statement.

Lemma 5: If $p(0) \notin \mathcal{C}$, then $p$ does not approach $\mathcal{D}$ as $t \to \infty$.

Proof: Suppose that $p$ approaches $\mathcal{U}$ as $t \to \infty$ even if $p(0) \notin \mathcal{C}$. Then, it must be true that

$$\lim_{t \to \infty} \det Z = 0, \quad (8)$$

because $\mathcal{U} \subset \mathcal{C}$. Also, there exists a finite time $T_f$ such that $p(T_f) \notin \mathcal{U}$, and $p(t) \in \Sigma$ for all $t \in [T_f, \infty)$. From (7), we have

$$\det Z = \exp \left[ -2 \int_{T_f}^{t} \sigma(s) \, ds \right] \times \exp \left[ -2 \int_{0}^{T_f} \sigma(s) \, ds \right] \det Z_0, \quad t \geq T_f \geq 0.$$

Meanwhile, $\lim_{t \to \infty} \exp \left[ -2 \int_{T_f}^{t} \sigma(s) \, ds \right] \geq 1$ for all $t \geq T_f$ from the fact that $\sigma < 0$ for all $p \in \Sigma$, and $\lim_{t \to \infty} \exp \left[ -2 \int_{0}^{T_f} \sigma(s) \, ds \right] \det Z_0$ is a non-zero finite constant from the fact that $\det Z_0 \neq 0$ if and only if $p(0) \notin \mathcal{C}$. Apparently, it contradicts to (8), which completes the proof.

C. Convergence to the Desired Equilibrium Set

From Lemma 1 and 5, we know that $p$ approaches $\mathcal{D}$ if $p(0) \notin \mathcal{C}$. However, this does not mean that $p$ converges to a point. In the next theorem, we investigate whether $p$ converges to a point or permanently drifts along $\mathcal{D}$.

Theorem 1: If $p(0) \notin \mathcal{C}$, then $p$ exponentially converges to a point in $\mathcal{D}$.

Proof: For a positive number $\rho$, define $\Omega(p)$ as $\{ z : \phi \leq \rho^2/4 \}$. From the fact that $\lim_{||z|| \to \infty} \phi = \infty$, it is true that $\Omega(p)$ is bounded and closed for a fixed $\rho$, i.e., $\Omega(p)$ is compact. Since the desired formation is apart from any undesired formation existing on a plane, there exists $\rho_e > 0$ such that $\text{rank} R_G = 3|\mathcal{V} - 6$, $\forall z \in \Omega(\rho_e)$. Thus, $R_G$ has full row rank, and $R_G R_G^T$ is positive definite for all $z \in \Omega(\rho_e)$. Let $\mu(R_G R_G^T)$ denote the smallest eigenvalue of $R_G R_G^T$. From the fact that $\Omega(\rho_e)$ is compact, define $\lambda$ as $\inf_{z \in \Omega(\rho_e)} \mu(R_G R_G^T)$. Due to the positive definiteness of $R_G R_G^T$, $\lambda$ is strictly positive. Now, from (3), we have

$$\dot{\phi} = -e^T R_G R_G^T e \leq -\lambda e^T e = -4\lambda \phi, \quad \forall z \in \Omega(\rho_e).$$

If $p$ reaches $\Omega(\rho_e)$ in finite time, then $\phi$ converges to zero exponentially fast by the comparison lemma [19, Lemma 3.4], which means that $\sigma_{ij}$ also exponentially converges to zero for all $(i, j) \in \mathcal{E}$. Furthermore, since $z$ is bounded and the right side of (2) consists of $e$ and $z$, we know that $p$ also exponentially converges to zero vector. Therefore, $p$ converges to a point as $t \to \infty$. From Lemma 1 and 5, we know that $\phi$ monotonically converges to zero as time goes to infinity if $p(0) \notin \mathcal{C}$. Thus, it is apparent that there exists a finite time $T_{\rho_e}$ such that $\phi \leq \rho_e^2/4$ for all $t \geq T_{\rho_e}$. Therefore, $p$ reaches $\{ p : z \in \Omega(\rho_e) \}$ in finite time unless $p(0) \in \mathcal{C}$.

IV. Simulations

In this section, we provide some simulation results. Fig. 3 shows the case where $p(0) \notin \mathcal{C}$. Every squared-distance error converges to zero as time goes on, and the agents form a regular tetrahedral formation.

V. Concluding Remarks

A. Conclusion

We have analyzed the four-agent undirected formation evolving in 3-dimensional space. The four agent tetrahedral formation can be a stepping stone to multi-agent formations consisting of more than four agents. We used conventional gradient control law to achieve the desired formation under the single-integrator-modeled dynamics. The control law is fully implemented in a decentralized manner. The subset which the agents approach as time goes on is partitioned by the desired equilibrium set and the undesired equilibrium set. Then, our analysis addresses the condition for the agents to converge to the desired equilibrium set with exponential convergence rate.


\textbf{B. Future Work}

We have mainly focused on the analysis of the desired equilibrium formation. However, we do not provide enough analysis explaining the behavior of the agents converging to the undesired equilibrium formations. Thus, more compact analysis should be carried out to fully understand the systems even when they evolve in the planar formation. In Section II-B, we assume that all the desired relative distances are equal to the same distance $d$. This assumption should be relaxed to extend our results to more general situations. Although, we have only considered four-agent systems, to analyze the formations consisting of arbitrarily many agents with more complex graph topology would be ultimate research direction.

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