Marcum Q-functions and Explicit Feedback Laws for Stabilization of Constant Coefficient $2 \times 2$ Linear Hyperbolic Systems

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Abstract—We find the exact analytical solution to a Goursat PDE system governing the kernels of a backstepping-based boundary control law that stabilizes a constant-coefficient $2 \times 2$ system of first-order hyperbolic linear PDEs. The solution to the Goursat system is related to the solution of a simpler, explicitly solvable Goursat system through a suitable infinite series of powers of partial derivatives which is summed explicitly in terms of special functions, including Bessel functions and the generalized Marcum $Q$-functions of the first order. The Marcum functions are common in certain applications in communications but have not appeared previously in control design problems.

I. INTRODUCTION

The class of $2 \times 2$ systems of first-order hyperbolic linear PDEs has attracted considerable attention due to the many examples of physical systems that can be modelled by the class, such as open channels [1], [2], [3], [4], transmission lines [5], gas flow pipelines [6] or road traffic models [7].

Given the range of applications, many techniques for stabilization of these systems have been proposed in the literature. Such methods include the use of the explicit evolution of the Riemann invariants along the characteristics [8] and the use of control Lyapunov functions [9] (which are also extensible to $n \times n$ systems [10]). Other approaches include [11], [12], [13] (which use a Lyapunov method), [14], [15], [1] (using a Riemann invariants approach), and [16].

Recently, a new design approach based on the backstepping method [17], [18] has been developed [19] for $2 \times 2$ hyperbolic linear systems. The method allows the design of full-state boundary control laws, boundary observers and output-feedback control laws, which guarantee $L^2$ stability of the closed-loop system and convergence of the state estimates. The results have been extended to include the quasilinear case [20] (making the closed-loop system locally exponentially stable in the $H^2$ sense), a disturbance rejection problem [21], and the case of an underactuated hyperbolic system consisting of $n$ rightward-convecting equations coupled with one leftward-convecting equation [22].

In this paper we derive exact analytical expressions for the problem of boundary stabilization for constant-coefficient $2 \times 2$ system of first-order hyperbolic linear PDEs, with actuation at only one of the boundaries.

Having explicit stabilizing feedback-laws for infinite-dimensional systems is rare; it allows a better understanding of the structure of the control law and its dependence with respect the different parameters of the system. Also, having an analytical expression makes implementation simpler and more precise. Most importantly, when the PDE plant parameters are unknown, the only way to design implementable adaptive controllers [23], [24], [25], [26], [27], [28] is when the control gains are available as explicit functions of plant parameters. However, explicit laws are seldom found in the literature, even for 1-D constant-coefficient systems, except in the simplest of cases. The ability to produce explicit control laws for many non-trivial systems has been the distinguishing quality of the backstepping approach (see [17] for examples). However, explicit controllers have heretofore not been available for first-order hyperbolic linear $2 \times 2$ systems. For these systems, the class of constant-coefficient parameters gives a very wide range of plants, with 7 distinct parameters that can have any value, with only the speeds of propagation being restricted to be positive. This class of hyperbolic systems contains linearizations of quasi-linear hyperbolic system around constant equilibria.

The control kernels obtained in this paper are given in terms of modified Bessel functions of the first kind (frequently seen in explicit controllers designed by using backstepping), and also in terms of the generalized Marcum $Q$-function of first order. This function was developed by Marcum for radar analysis [29] and arises in performance analysis of partially coherent, differentially coherent, and noncoherent communications [30], [31] and also in statistics [32], but to the best of the authors’ knowledge it is the first time that it appears in control theory outside specific applications in communications.

To derive the explicit solutions, we start from the backstepping design of [19]. The kernels used in the feedback law are found by solving a well-posed $2 \times 2$ system of first-order hyperbolic linear PDEs in a triangular domain (known as the kernel equations). When the plant model has constant coefficients, the resulting kernel equations have a very specific structure which can be exploited to obtain an explicit solution in terms of special functions. The procedure to find analytical solutions is as follows. First, we apply scaling to reduce the number of constant parameters in the kernel equations from seven to two. The resulting reduced equation is transformed (by using a series approach) to an infinite set of equations which are simpler and can be explicitly solved recursively. Finding the sum of the series solution and reverting the scaling transformations, we finally obtain explicit expressions for the kernels. The appearance of the generalized Marcum $Q$-function, which did not show up previously in explicit expressions of backstepping controllers, is mainly due to a coupling of the two PDEs in the $2 \times 2$ hyperbolic plant.

The paper is organized as follows. In Section II we state our main result—the exact analytical expressions for
these feedback laws. The proof of this result is detailed in Section III. Next we present an example in Section IV. We finish in Section V with some concluding remarks. We also include an appendix with some technical lemmas.

II. STABILIZATION OF CONSTANT COEFFICIENT 2 × 2 LINEAR HYPERBOLIC SYSTEMS

Consider the following system
\[
\begin{align*}
    u_t &= -\epsilon_1 u_x + c_1 u + c_2 v, \\
    v_t &= c_2 v_x + c_3 u + c_4 v, \\
\end{align*}
\]
evolving in \( x \in [0, 1] \), \( t > 0 \), with boundary conditions
\[
u(0, t) = g v(0, t), \quad v(1, t) = U(t),
\]
where \( U(t) \) is the actuation variable. The initial conditions, denoted as \( u_0 \) and \( v_0 \), are assumed to belong to \( L^2([0, 1]) \).

In (1)–(2), \( \epsilon_1, \epsilon_2, c_1, c_2, c_3 \) and \( c_4 \). When the coefficients of (1)–(3) are such that the system is open-loop unstable, it is necessary to design a feedback law for \( U(t) \) that results in a stable closed-loop system.

System (1)–(3) is the most general possible heterodirectional \( 2 \times 2 \) linear hyperbolic system (without including integral or boundary terms). In this context, “heterodirectional” means that the two state variable \( u(x,t) \) and \( v(x,t) \) evolve in opposite spatial directions (with speeds \( \epsilon_1 \) and \( \epsilon_2 \), respectively) as time moves forward. For this reason, the boundary conditions (3)–(3) are at opposite boundaries.

In Section I a number of procedures to design feedback laws for \( 2 \times 2 \) linear hyperbolic systems have been reviewed. However, only backstepping is able to deal with (1)–(3) for arbitrary values of the coefficients. Thus, following [19], we apply the backstepping method, which allows to find a stabilizing linear full-state feedback law as follows
\[
U(t) = \int_0^1 k_u(\xi)u(\xi,t)d\xi + \int_0^1 k_v(\xi)v(\xi,t)d\xi,
\]
where \( k_u(\xi) \) and \( k_v(\xi) \) are the control kernels, which are found by solving an auxiliary set of partial differential equations. We next state the main result of this paper, which gives explicit formulae for \( k_u(\xi) \) and \( k_v(\xi) \) that stabilize the closed-loop system.

**Theorem 1:** Consider the system (1)–(3) with initial conditions \( u_0 \) and \( v_0 \) and control law (4), where the control kernels of \( k_u(\xi) \) and \( k_v(\xi) \) are explicitly given for \( q = 0 \) by
\[
\begin{align*}
    k_u(\xi) &= c_3 H(\xi) \left( I_0 \left[ \frac{2 \epsilon}{\epsilon_1 + \epsilon_2} \mu(\xi) \right] + \eta^2(\xi) I_2 \left[ \frac{2 \epsilon}{\epsilon_1 + \epsilon_2} \mu(\xi) \right] \right), \\
    k_v(\xi) &= \bar{c} \left( 1 + \frac{\epsilon_2}{\epsilon_1} \right) H(\xi) \frac{\xi}{\mu(\xi)} I_1 \left[ \frac{2 \epsilon}{\epsilon_1 + \epsilon_2} \mu(\xi) \right],
\end{align*}
\]
and for \( q \neq 0 \) by
\[
\begin{align*}
    k_u(\xi) &= c_3 \epsilon q \left( \frac{\epsilon}{q^{\epsilon_1}} \right) H(\xi) \left[ F_0(\xi) + F_1(\xi) + \eta(\xi) F_1(\xi) \right], \\
    k_v(\xi) &= H(\xi) \left[ F_0(\xi) + F_1(\xi) + \frac{1}{\eta(\xi)} F_1(\xi) \right],
\end{align*}
\]
where
\[
\begin{align*}
    &H(\xi) = \frac{1}{\epsilon_1 + \epsilon_2} \exp \left[ \frac{(c_1 - c_4)(1 - \xi)}{\epsilon_1 + \epsilon_2} \right], \\
    &\eta(\xi) = \sqrt{\frac{1 - \xi}{1 + \frac{\epsilon_2}{\epsilon_1} \xi}}, \\
    &\mu(\xi) = \sqrt{\frac{1 - \xi}{1 + \frac{\epsilon_2}{\epsilon_1} \xi}}, \\
    &F_0(\xi) = \frac{c_2}{\eta} I_0 \left[ \frac{2 \epsilon}{\epsilon_1 + \epsilon_2} \mu(\xi) \right], \\
    &F_1(\xi) = \frac{c_3}{\epsilon_1 + \epsilon_2} \mu(\xi), \\
    &F_{11}(\xi) = \left( \frac{c_3}{\epsilon_1} - \frac{c_2}{\eta} \right) \Pi \left[ \frac{q c_2 (1 - \xi)}{\epsilon_2 (\epsilon_1 + \epsilon_2)} \frac{c_2 (1 + \frac{\epsilon_2}{\epsilon_1} \xi)}{q (\epsilon_1 + \epsilon_2)} \right],
\end{align*}
\]
with \( I_n \) denoting the modified Bessel function of the first kind (of order \( n \)), and the function \( \Pi(x, y) \) being given by
\[
\Pi(x, y) = e^{x+y} Q_1 \left( \sqrt{2x}, \sqrt{2y} \right)
\]
where \( Q_1 \) the generalized Marcum Q-function of first order [29], which is given by
\[
Q_1 \left( \sqrt{2x}, \sqrt{2y} \right) = 1 - ye^{-x} \int_0^1 e^{-sy} I_0 \left( 2\sqrt{xy} \right) ds.
\]
Then, for any value of the coefficients \( c_1, c_2, c_3, c_4, q, \) and under the assumption that \( u_0, v_0 \in L^2(0,1) \) and \( c_1, c_2 > 0 \), the equilibrium \( u \equiv v \equiv 0 \) is exponentially stable in the \( L^2 \) sense. Moreover, the equilibrium is reached in finite time \( t = \frac{1}{\epsilon_1} + \frac{1}{\epsilon_2} \).

The proof of the stability part of Theorem 1 is based on [19] and it is given in detail in Section III.

The formulae (5), (6) for \( q = 0 \) are found by letting \( q \to 0 \) in (7), (8) and by applying Lemma A6, thus finding
\[
F_{11}(\xi) = -F_0(\xi) - \eta(\xi) F_1(\xi) + q c_2 \left( 1 + \frac{\epsilon_2}{\epsilon_1} \xi \right) I_0 \left[ \frac{2 \epsilon}{\epsilon_1 + \epsilon_2} \mu(\xi) \right] + \eta^2(\xi) I_2 \left[ \frac{2 \epsilon}{\epsilon_1 + \epsilon_2} \mu(\xi) \right] + O(q^2),
\]
from which (5), (6) follow.

The same kernels of Theorem 1 can be used to locally stabilize (in terms of the \( H^2 \) norm) equilibria of quasi-linear \( 2 \times 2 \) systems if the resulting linearization around the equilibrium has constant coefficients (see [20] for additional details). In particular, this situation happens when stabilizing constant equilibrium profiles of the nonlinear plant (since linearization results in a constant coefficient system). In Section IV we give a quasi-linear example.

The result stated in Theorem 1 requires full-state knowledge. In [19] a dual method to design an anti-coltocated boundary observer is presented (which was extended to the quasilinear case in [34]). The observer only needs measurements of \( u(0, t) \) and uses a copy of the plant driven by output injection to obtain estimates of the state. The design procedure requires solving a set of partial differential
equations to obtain the output injection gain kernels. In the constant-coefficient case, it can be shown that the observer kernel equations can be directly transformed into the same set of PDEs that produce the control kernels (7)–(8). Thus, one finds explicit expressions for the observer kernels which are structurally very similar to (7)–(8). Combining this explicit observer with the explicit full-state controller of Theorem 1 one can obtain a stabilizing explicit output-feedback controller which only needs measurements of $u(0,t)$.

III. PROOF OF THEOREM 1

To prove Theorem 1 we introduce three steps. First, we follow the backstepping method as outlined in [19], arriving at a control law that requires the solution of a set of kernel PDEs. Next, we simplify and reduce the equations as much as possible arriving at a set of reduced equations that contain a minimal set of parameters. Then, we solve the reduced equations by posing a series solution which allows us to formulate the problem as an infinite chain of PDEs whose solution can be formally stated as derivatives of an initial function (itself the solution of a simpler PDE, explicitly solvable). Finally, we also apply some technical lemmas (contained in the Appendix) to express the result in terms of the generalized Marcum $Q$-function of first order.

A. Backstepping stabilization laws for $2 \times 2$ linear hyperbolic systems

Following [19], to apply the backstepping method, system (1)–(3) needs first to be put in the proper (anti-diagonal) form, which requires eliminating the $c_1$ term in (1) and the $c_4$ term in (2). For that, define new variables $y$ and $z$ by exponentially scaling $u$ and $v$ as follows

$$y(x,t) = e^{-c_1/\epsilon_1}x u(x,t), \quad z(x,t) = e^{c_4/\epsilon_2}x v(x,t).$$

(19)

Computing the derivatives of these new variables and using the boundary conditions (3)–(3) to find boundary conditions for $y$ and $z$, we reach the system expressed in $y$-$z$ variables

$$y_t = -\epsilon_1 y_x + c_2 e^{\frac{c_4}{\epsilon_2}}x z(x,t),$$

$$z_t = c_2 z_x + c_3 e^{\frac{c_4}{\epsilon_2}}x y(x,t),$$

$$y(0,t) = qz(0,t), \quad z(1,t) = V(t),$$

(20)–(22)

where we have defined $V(t) = e^{c_4/\epsilon_2}U(t)$.

The backstepping method can be now be applied to (20)–(22), and following [19], the closed-loop system is exponentially stable if we set

$$V(t) = \int_0^1 K^{vu}(1,\xi) y(\xi,t) d\xi + \int_0^1 K^{vv}(1,\xi) z(\xi,t) d\xi,$$

(23)

where $K^{vu}(x,\xi)$ and $K^{vv}(x,\xi)$ are the solution to the following kernel equations

$$\epsilon_2 K_{xu}^{vu} - c_1 K_{xu}^{vu} = c_3 e^{\frac{c_4}{\epsilon_2}}x K^{vu},$$

$$\epsilon_2 K_{xv}^{vu} + c_2 K_{xv}^{vu} = c_3 e^{\frac{c_4}{\epsilon_2}}x K^{vu},$$

(24)–(25)

evolving in the triangular domain $T = \{(x,\xi) : 0 \leq \xi \leq x \leq 1\}$, with boundary conditions

$$K^{vu}(x,x) = -\frac{c_3 e^{\frac{c_4}{\epsilon_2}}x}{\epsilon_1 + \epsilon_2},$$

$$K^{vv}(x,0) = \frac{\epsilon_1}{\epsilon_2} K^{vu}(x,0).$$

(26)–(27)

The system (24)–(27) is guaranteed by [19] to have a unique solution. Once the solution is computed, we recover the feedback law for $U(t)$ in terms of the original variables $u$ and $v$ just by using (19), reaching (4) with

$$k_u(\xi) = K^{vu}(1,\xi)e^{-\frac{c_4}{\epsilon_2}\xi - \frac{c_4}{\epsilon_2}}, \quad k_v(\xi) = K^{vv}(1,\xi)e^{-\frac{c_4}{\epsilon_2}(\xi-1)}.$$

B. Reducing the kernel equations

Next we simplify the system of PDEs (24)–(27). In the first place, looking at the structure of the equations, we find that it is not a constant coefficient system. However it is possible to transform it to a constant coefficient case by defining new kernel variables; as before, this is carried out by using an exponential scaling, as follows

$$G^{vu}(x,\xi) = e^{-\frac{c_4}{\epsilon_2}(\frac{\xi}{1+\epsilon_2})} K^{vu}(x,\xi),$$

$$G^{vv}(x,\xi) = e^{-\frac{c_4}{\epsilon_2}(\frac{\xi}{1+\epsilon_2})} K^{vv}(x,\xi).$$

(28)–(29)

Using (24)–(27) we arrive at

$$\epsilon_2 G_{xu}^{vu}(x,\xi) - c_1 G_{xu}^{vu}(x,\xi) = c_3 G^{vu}(x,\xi),$$

$$\epsilon_2 G_{xv}^{vu}(x,\xi) + c_2 G_{xv}^{vu}(x,\xi) = c_3 G^{vu}(x,\xi),$$

$$G^{vu}(x,x) = -\frac{c_3}{\epsilon_1 + \epsilon_2},$$

$$G^{vu}(x,0) = \frac{\epsilon_1}{\epsilon_2} G^{vu}(x,0).$$

(30)–(33)

Defining now $c = \sqrt{\frac{\epsilon_1}{\epsilon_2}}\epsilon_1$, $G^{vu} = \sqrt{\frac{\epsilon_1}{\epsilon_2}} G^{vu}$, $\epsilon = \frac{\epsilon_1}{\epsilon_2}$, $\hat{q} = \sqrt{\frac{\epsilon_1}{\epsilon_2}}\epsilon_1$, $\hat{x} = c x$, $\hat{\xi} = \epsilon_1 \xi$, $G^{vu} = -\sqrt{\frac{\epsilon_1}{\epsilon_2}} G^{vu} + \epsilon_1$, and $G^{vu} = -G^{vu} + \epsilon_1$, we further simplify the equations, arriving at

$$\hat{G}_{xu}^{vu}(\hat{x},\hat{\xi}) + \epsilon_1 \hat{G}_{xu}^{vu}(\hat{x},\hat{\xi}) = \hat{G}^{vu}(\hat{x},\hat{\xi}),$$

$$\hat{G}_{xv}^{vu}(\hat{x},\hat{\xi}) - \epsilon_1 \hat{G}_{xv}^{vu}(\hat{x},\hat{\xi}) = \hat{G}^{vu}(\hat{x},\hat{\xi}),$$

$$\hat{G}^{vu}(\hat{x},0) = 1,$$

$$\hat{G}^{vu}(\hat{x},\hat{\xi}) = \hat{q} \hat{G}^{vu}(\hat{x},0),$$

(34)–(37)

which only depends on two parameters, $\epsilon$ and $\hat{q}$. We call this the reduced kernel equations.

C. Solving the reduced kernel equations

To solve (34)–(37), we pose a solution as a series in the parameter $\hat{q}$, in the following way

$$\hat{G}^{vu} = \sum_{n=0}^{\infty} \hat{q}^n A_n(x,\xi), \quad \hat{G}^{vu} = \sum_{n=0}^{\infty} \hat{q}^n B_n(x,\xi).$$

(38)

The convergence of this series will be seen later. Then, the equations verified by the coefficients $A_n$ and $B_n$ are:

$$(\partial_x - \epsilon \partial_\xi) A_n(x,\xi) = B_n(x,\xi),$$

$$A_n(x,\xi) = A_n(x,\xi),$$

(39)–(40)
We need to solve this system of equations for all values of \( n \). To do so, we decouple the system by finding the equation verified by \( B_n(x,0) \). For that we need the following Lemma, whose proof is omitted.

**Lemma 1:** Denote \( b_0(x) = B_0(x,x) \). Then \( b_0(x,x) = x \), \( b_1(x,x) = 1 \), and \( b_n(x,x) = 0 \) for \( n > 1 \).

Combining (39) and (40) and Lemma 1 we reach the following set of partial differential equations

\[
(\partial_x + \partial_\xi)(\partial_x - c\partial_\xi)B_n(x,\xi) = B_n(x,\xi),
\]

with boundary conditions

\[
B_n(x,0) = 0, \quad B_n(x,\xi) = A_{n-1}(x,0), \quad \forall n > 0.
\]

The plan to solve this set of differential equations is the following.

1) First solve for \( B_0(x,\xi) \) (since it is autonomous).
2) Obtain \( A_0(x,\xi) \) from (40).
3) Iterate this procedure for \( n \geq 1 \), first solving for \( B_n \) by using \( A_{n-1}(x,0) \) in (45) and then obtaining \( A_n(x,\xi) \) from (40).

The next result (that can be proved by induction) shows this plan can be carried out explicitly.

**Lemma 2:** Call \( \Phi(x,\xi) \) the (smooth) function verifying

\[
(\partial_x + \partial_\xi)(\partial_x - c\partial_\xi)\Phi(x,\xi) = \Phi(x,\xi),
\]

\[
\Phi(x,x) = x, \quad \Phi(x,0) = 0.
\]

Then, defining \( A_n \) and \( B_n \) as

\[
B_n(x,x) = (\partial_x + \partial_\xi)^n\Phi(x,\xi),
\]

\[
A_n = (\partial_x + \partial_\xi)^n\Phi(x,\xi),
\]

the set of equation (39)–(42) is verified for all \( n \geq 0 \).

Substituting the result of Lemma 2 in (38) we get

\[
\hat{G}^{uv}(x,\xi) = \sum_{n=0}^{\infty} \hat{\eta}^n(\partial_x + \partial_\xi)^n\Phi(x,\xi).
\]

The kernel \( \hat{G}^{uv} \) is then obtained from (34) by differentiation.

**D. Obtaining explicit formulae for the solution of the reduced kernel equations**

Using Lemma A4 to solve the \( \Phi \) equation, we get

\[
\Phi(x,\xi) = (1 + \epsilon) \xi - \frac{2}{1 + \epsilon} \sqrt{(x - \xi)(\epsilon x + \xi)} \frac{2}{1 + \epsilon} \sqrt{(x - \xi)(\epsilon x + \xi)}
\]

Expanding \( \Phi(x,\xi) \) as a power series by using Lemma A5

\[
\Phi = \xi \sum_{k=0}^{\infty} \frac{1}{1 + \epsilon} (\frac{1}{k!}) k(\epsilon x + \xi)^{k+1}.
\]

and writing \( \xi = \frac{\epsilon + x - \xi}{1 + \epsilon} \), we get

\[
\Phi = \sum_{k=0}^{\infty} \left[ \frac{1}{(k+1)!} \frac{1}{k!(1 + \epsilon) \epsilon x + \xi} \right]^{k+1}.
\]

To explicitly write (50), we need \( (\partial_x + \partial_\xi)^n\Phi(x,\xi) \), which is easier to compute in (53). The following holds.

**Lemma 3:** For all \( n > 0 \)

\[
(\partial_x + \partial_\xi)^n\Phi = \sum_{k=0}^{\infty} \frac{1}{(k+n-1)!} \frac{1}{k!} \frac{1}{(1 + \epsilon) \epsilon x + \xi}^{k+1} \left( 1 - \epsilon \frac{1}{(k+n)(k+n+1)} \right)
\]

We skip the proof (by induction) due to lack of space.

Applying Lemma 3 in (50) we get absolute convergence of the series, and we obtain for \( \hat{G}^{uv} \), after some manipulation,

\[
\hat{G}^{uv} = \sum_{k=0}^{\infty} \frac{1}{(k+n-1)!} \frac{1}{k!(1 + \epsilon) \epsilon x + \xi}^{k+1} \left( 1 - \epsilon \frac{1}{(k+n)(k+n+1)} \right)
\]

and using Lemma A5 to express the power series as modified Bessel functions, we finally get

\[
\hat{G}^{uv} = \sqrt{\frac{\epsilon x + \xi}{x - \xi}} I_1 \left( \frac{2}{1 + \epsilon} \sqrt{(x - \xi)(\epsilon x + \xi)} \right)
\]

where we have defined

\[
\Pi(x,y) = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{x^n y^{n+m}}{n!(n+m)!}.
\]

which can also be written in terms of the the generalized Marcum Q-function of first order, as shown in Lemma A6. Next, using (34) and differentiating \( \hat{G}^{uv} \) we obtain \( \hat{G}^{vu} \). Then, undoing the various scaling transformations that were made to arrive at the reduced equation, we get the \( K^{uv} \) and \( R^{uv} \) backstepping kernels in term of the original coefficients (we skip the expressions due to lack of space). Finally, applying (28) to compute the control kernels, an defining the intermediate functions (9)–(15) to simplify the resulting expression, we verify (7)–(8). Finally we apply [19] to finally arrive at the result of Theorem 1.

**IV. EXAMPLE: STABILIZATION OF A CONSTANT EQUILIBRIUM PROFILE FOR A NONLINEAR PLANT**

Consider a system with quadratic nonlinearities

\[
u_t + u_x = \frac{1}{2}(u - v^2),
\]

\[
v_t - v_x = \frac{1}{2}(v - u^2),
\]

\[
u(0,t) = v(0,t), v(1,t) = U(t).
\]
This example is inspired by models of thermoacoustic combustion instabilities in elongated combustion chambers with momentum-dependent heat release [35], [36].

If \( U(t) = 1 \), there is an equilibrium at \( u \equiv v \equiv 1 \), and the objective is to stabilize the system around this equilibrium. Define error variables \( \tilde{u}(x, t) = u(x, t) - 1 \) and \( \tilde{v}(x, t) = v(x, t) - 1 \), and \( \tilde{U}(t) = U(t) - 1 \). The error system is

\[
\begin{align*}
\tilde{u}_t + \tilde{u}_x &= \frac{c}{2}((\tilde{u} + 1)) - (\tilde{v} + 1)^2, \\
\tilde{v}_t - \tilde{v}_x &= \frac{c}{2}((\tilde{v} + 1)) - (\tilde{u} + 1)^2, \\
\tilde{u}(0, t) &= \tilde{v}(0, t), \quad \tilde{v}(1, t) = \tilde{U}(t).
\end{align*}
\]

Linearizing system (61)–(62) around the origin, we obtain

\[
\begin{align*}
\tilde{u}_t + \tilde{u}_x &= \frac{c}{2}(-2\tilde{v}), \\
\tilde{v}_t - \tilde{v}_x &= \frac{c}{2}(-2\tilde{u}),
\end{align*}
\]

which is a constant coefficient system; following the notation of Section II, we get \( c_1 = c/2, c_2 = -c, c_3 = -c, c_4 = c/2, \epsilon_1 = \epsilon_2 = 1, q = 1 \).

To see the potential unstability, take a time and a space derivative in (64) and subtract the resulting expressions. Then

\[
\tilde{u}_{tt} - \tilde{u}_{xx} = c\tilde{u}_t + \frac{3c^2}{4}\tilde{u},
\]

which is a wave equation with in-domain antidamping (for positive values of \( c \)) and anti-stiffness, which is well-known to yield instability [37]. Numerical simulations of the open-loop nonlinear system (Fig. 2) show the system becoming unstable for large enough \( c \).

The explicit solution for the control kernels is, in this case,

\[
\begin{align*}
k_u &= \frac{c}{2} \left\{ I_0 \left[c\sqrt{1 - \xi^2}\right] - \sqrt{\frac{1 - \xi}{1 + \xi}} I_1 \left[c\sqrt{1 - \xi^2}\right] \right\}, \\
k_v &= \frac{c}{2} \left\{ I_0 \left[c\sqrt{1 - \xi^2}\right] - \sqrt{\frac{1 + \xi}{1 - \xi}} I_1 \left[c\sqrt{1 - \xi^2}\right] \right\}.
\end{align*}
\]

In Fig. 1 we plot the control kernels for different values of \( c \). Notice the exponential increase as \( c \) grows (the apparent peaks are due to the logarithmic scale).

To find the numerical solution of the open-loop and closed-loop system we use the HPRD solver for Matlab [38]. We use the value \( c = 6 \) and find that the open-loop system is unstable as shown in Fig. 2. The control law makes the origin asymptotically stable as shown in Fig. 3.

V. CONCLUSIONS

In this work we have derived an explicit control law to solve the problem of boundary stabilization for constant-coefficient \( 2 \times 2 \) system of first-order hyperbolic linear PDEs. The control law is found by applying the backstepping method and then solving the resulting control kernel equations, which has required the development of a method that expresses their solution as the sum of solutions of an infinite set of explicitly solvable equations.

The resulting explicit expressions contain not only modified Bessel functions of the first kind (frequently seen in explicit controllers previously found by using backstepping) but also generalized Marcum Q-functions of first order. This is the first time that this special function appears in control theory outside specific applications in communications. It is expected that this function might appear in other explicit controllers designed by the backstepping method.

REFERENCES

(71)

Next we present a lemma that allows us to express power series as modified Bessel functions

**Lemma A5:** The following identity holds.

\[
\Phi(x,\xi) = (1 + \epsilon)\xi - \sum_{k=0}^{\infty} \frac{1}{1 + \epsilon} \frac{2^{2k+n}}{(k+n)!} \left( x - \xi \right)^{k+n} \frac{(\epsilon x + \xi)^{k+n}}{k!}.
\]

Finally, we give a result to express a double series as the generalized Marcum Q-function of order 1 [33].

**Lemma A6:** The function \( \Pi(x, y) \) defined as

\[
\Pi(x, y) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{y^n p^{n+m}}{n!(n+m)!}
\]

(73)

can be written as

\[
\Pi(x, y) = e^{x+y} Q_1(2\sqrt{2x}, 2\sqrt{2y})
\]

(74)

and also it can alternatively be written as

\[
\Pi(x, y) = e^{x+y} - \int_0^y e^{(1-s)y} I_0(2\sqrt{xy}) \, ds.
\]

(75)

Additionally, \( \Pi(\delta x, \frac{y}{\delta}) \) can be approximated for small \( \delta \) as

\[
\Pi(\delta x, \frac{y}{\delta}) = I_0(2\sqrt{xy}) + \frac{\delta^2 x}{y} I_2(2\sqrt{xy}) + O(\delta^3).
\]

(76)