LMI searches for discrete–time Zames–Falb multipliers

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Abstract— Recent works have proposed different searches for Zames–Falb multipliers in the continuous-time domain. Although Zames–Falb multipliers are also defined for the discrete-time domain, there is no equivalent literature for their search. In this paper, we propose the discrete-time counterparts of two recently reported algorithms for the search of a Zames–Falb multiplier. The two searches find causal and anticausal multipliers respectively. Several examples show the effectiveness of the method. The examples illustrate the necessity of conducting both causal and anticausal searches to ensure competitiveness over other methods for establishing the stability of loops with slope-restricted nonlinearities.

I. INTRODUCTION

The stability of a feedback interconnection between a linear time-invariant system $G$ and any nonlinearity $\Phi$ within the class of nonlinearities $\Phi$ is referred to as the Lur’e problem (see Section 1.3 in [3] for a history of this problem). As the stability is obtained for the whole class of nonlinearities, the adjective “absolute” or “robust” is added. In the classical solution of this problem frequency-domain conditions on the linear system are determined by the class of nonlinearities. The inclusion of a multiplier was proposed to reduce the conservativeness of the approach. The stability problem is translated into the search of a multiplier $M$ which belongs to the class of multipliers associated with the class of nonlinearities $\Phi$, where $G$ and $M$ satisfy some frequency conditions.

In the case of memoryless slope-restricted and odd nonlinearities, the class of Zames-Falb multipliers [32] is the widest class in the literature [5]. The search for a Zames–Falb multiplier suitable for a linear biproper plant $G$ has attracted some recent attention [27], [26], [25], [8], [6], [7]; see also references therein. None of these provides a search over the whole class of Zames–Falb multipliers since this is a challenging problem. If the multiplier is given as $(1-H)$ then the $\mathcal{L}_1$-norm of the LTI system $H$ must be strictly smaller than 1 while the frequency-domain condition

$$\text{Re}\{((1-H(j\omega))G(j\omega)) > 0, \quad (1)$$

must be satisfied for all frequencies. In some of the searches a conservative bound of the $\mathcal{L}_1$-norm is used in order to synthesise the multiplier in a convex manner; in other searches the multiplier is parametrised in such a way that the $\mathcal{L}_1$-norm can be straightforwardly computed but the use of the KYP lemma for the frequency domain condition is sacrificed. For further discussion see [7].

The possibility of using the Zames–Falb multiplier for studying the stability and robustness properties of input-constrained model predictive control (MPC) [16] provides an inherent motivation for discrete-time analysis, since MPC is naturally formulated in discrete time. More generally, the absolute stability problem of discrete-time Lur’e systems with slope–restricted nonlinearities has also attracted recent attention [2], [1], [14]. These all take a Lyapunov function approach; as an advantage they generate easy-to-check LMI conditions. However one might expect that improved results could be obtained via a multiplier approach, since this provides a more general condition. In fact some of these approaches can be interpreted as a search over a small subclass of Zames–Falb multipliers, see [2] for further details.

In [30], [29], the discrete-time counterparts of the Zames–Falb multipliers [32] are given. The conditions are the natural counterparts to the continuous-time case, where the $\mathcal{L}_1$-norm is replaced by the $\ell_1$-norm and the frequency-domain inequality must be satisfied on the unit circle. In the continuous-time case, the use of improper multipliers has generated “extensions” of the original that have been analysed in [5]. In the discrete-time case, the conditions for the Zames–Falb multipliers are necessary and sufficient to preserve the positivity of the nonlinearity [29]; it follows that the class of Zames–Falb multiplier is the widest class of multipliers that can be used. The result has been extended to MIMO system [23], repeated nonlinearities in [18] and MIMO repeated nonlinearities in [19]. These works are focused on the description of the available multipliers, but no explicit search method is discussed.

This paper proposes two LMI searches, for causal and anticausal discrete-time Zames–Falb multipliers respectively. In the spirit of [27] and using the LMI $\ell_1$-norm characterisation proposed in [22], we develop an LMI condition for a causal multiplier. In the spirit of [7], this LMI condition can also be used to find an anticausal multiplier by inverting the linear plant. As expected, the numerical results show an improvement over other searches available in the literature. However, they also point to some non-trivial differences from the continuous-time case. In particular, for an example of a
second order plant (Ex. 1), the searches only provide a third of the Nyquist value (see Definition 2.4), whereas in the continuous case all the third order plants tested approach the Nyquist value [7], which is the analytical result since the Kalman conjecture holds for first, second, and third order plants in continuous time [4].

The structure of this paper is as follows. Section II presents preliminary results related with the discrete-time Zames–Falb multipliers as well as the LMI result that is used to bound the \( \ell_1 \)-norm properties of the multiplier. In Section III, both searches, causal and anticausal, are given. Section IV presents five examples where the searches have been tested. In each case at least one of the two searches provides very competitive results. Conclusions are given in Section V.

II. NOTATION AND PRELIMINARY RESULTS

Let \( Z \) be the set of integer numbers. Let \( \ell(Z) \) be the space of all real-valued sequences \( f : Z \rightarrow \mathbb{R} \), and let \( \ell_2(Z) \) be the Hilbert space of all square-summable real-valued sequences \( f : Z \rightarrow \mathbb{R} \) with the inner product defined as

\[
\langle f, g \rangle = \sum_{i=-\infty}^{\infty} f(i)g(i),
\]

for \( f, g \in \ell_2(Z) \). Let \( \ell \) be the subspace of \( \ell(Z) \) such that \( f \in \ell \) if \( f(i) = 0 \) for all \( i < 0 \) (the subspace \( \ell_2 \) of \( \ell_2(Z) \) is defined similarly).

A truncation of the sequence \( f \) at \( T \in Z \) is given by \( f_T(i) = f(i), \forall i \leq T \) and \( f_T(i) = 0, \forall i > T \). In addition, \( \ell_1 \) is the space of all absolute summable sequences, and given a sequence \( h : Z \rightarrow \mathbb{R} \) such that \( h \in \ell_1 \), then its \( \ell_1 \)-norm is given by

\[
\|h\|_1 = \sum_{i=-\infty}^{\infty} |h(i)|.
\]

The memoryless nonlinearity \( \phi : \mathbb{R} \rightarrow \mathbb{R} \) with \( \phi(0) = 0 \) is said to be bounded if there exists \( C \) such that \( |\phi(x)| < C|x| \) for all \( x \in \mathbb{R} \) and \( \phi \) is said to be monotone if for any two real numbers \( x_1 \) and \( x_2 \) then

\[
0 \leq \frac{\phi(x_1) - \phi(x_2)}{x_1 - x_2}.
\]

Moreover, \( \phi \) is slope-restricted in the interval \( S[0,k] \), henceforth \( \phi_k \), if

\[
0 \leq \frac{\phi_k(x_1) - \phi_k(x_2)}{x_1 - x_2} \leq k,
\]

for all \( x_1 \neq x_2 \). The nonlinearity \( \phi_k \) is said to be odd if \( N(x) = -N(-x) \) for all \( x \in \mathbb{R} \). Let us define the feedback interconnection of a stable LTI system \( G \) and a slope-restricted nonlinearity \( \phi_k \), represented in Fig. 1 and given by

\[
\begin{align*}
\{v(i) &= f(i) + (Gw)(i), \\
\{w(i) &= -\phi_k(v(i)) + u(i). \}
\end{align*}
\]

It is well-posed if the map \((v,w) \mapsto (u,f)\) has a causal inverse on \( \ell \times \ell \), and this feedback interconnection is \( \ell_2 \)-stable if for any \( f,u \in \ell_2 \), both \( w,v \in \ell_2 \). In addition, \( G(z) \) means the transfer function of the LTI system \( G \). The transfer function with state space realization will be given by \( G(z) = C(zI - A)^{-1}B + D \), in short

\[
G(z) \sim \begin{bmatrix} A & B \\ C & D \end{bmatrix}.
\]

The notation \( RL_{\infty} \) is used for the space of all real rational transfer functions without poles on the unit circle, \( RH_{\infty} \) is used for the space of all real rational transfer functions such that all their poles have absolute value strictly greater than 1, and \( RH_{\infty} \) is used for the space of all real rational transfer functions such that the absolute value of all their poles is strictly lower than \( 1^2 \). With some reasonable abuse of the notation, given a rational transfer function \( H(z) \) bounded at the imaginary axis, \( \|H\|_1 \) means the \( \ell_1 \)-norm of the impulse response \( h \) of \( H(z) \).

Let \( \bar{M} \) denote a linear time invariant operator mapping a time domain input signal to a time domain output signal and let \( M \) denote the corresponding transfer function. We consider that the domain of convergence includes the unit circle, so that the \( \ell_1 \)-norm of the inverse \( z \)-transform of \( M \) is bounded if \( M \in RL_{\infty} \). We say the multiplier \( \bar{M} \) is causal if \( M \in RH_{\infty} \), \( \bar{M} \) is anticausal if \( M \in RH_{\infty} \), and \( \bar{M} \) is noncausal otherwise. See [10] for further discussion on causality and stability. Henceforth, we will use \( M \) for both the operator and its transfer function.

The following theorem provides the absolute stability of system (6) subject to the existence of an appropriate Zames–Falb multiplier. In this paper, we will restrict our attention to slope-restricted and odd nonlinearities.

Theorem 2.1 ([30], [29]): Consider the feedback system in Fig. 1 with \( G \in RH_{\infty} \), and \( \phi_k \) an odd nonlinearity slope-restricted in \( S[0,k] \). Suppose that there exists a Laurent operator \( M : \ell_2(Z) \rightarrow \ell_2(Z) \) whose impulse response is \( m : Z \rightarrow \mathbb{R} \) and satisfies \( \sum_{m=-\infty}^{\infty} |m(i)| < 2m(0) \) and

\[
\text{Re}\left\{M(e^{j\omega})(1 + kG(e^{j\omega}))\right\} > 0 \quad \forall \omega \in [0,2\pi].
\]

Then the feedback interconnection (6) is \( \ell_2 \)-stable.

This theorem characterizes the class of Zames–Falb multipliers.

Definition 2.2: Let \( M \in RL_{\infty} \) be a SISO rational transfer function. Then \( M \) belongs to the class of Zames–Falb multipliers, \( \mathcal{M} \), if \( \|M\|_1 < 2m(0) \).

Given a bounded linear operator \( M : \ell_2(Z) \rightarrow \ell_2(Z) \), the symbol \( M^\sim \) means the \( \ell_2 \)-adjoint of \( M \) mapping \( \ell_2(Z) \) into

\[2^\text{This notation is not always adopted in the literature since the definition of the } z \text{-transform is not uniform in the use of } z \text{ or } z^{-1}. \text{ See [10], [31].}\]
ℓ2(Z). This operator satisfies \( \langle y, Mx \rangle = \langle M^\sim y, x \rangle \) for all \( u \in \ell_2(Z) \) and \( y \in \ell_2(Z) \). As a result, \( M^\sim \) is anticausal if and only if \( M \) is causal [12]. In particular, if the transfer function of the operator \( M \) is given by \( M(z) \), then the transfer function of \( M^\sim \) is \( M(1/z)\top \), where the symbol \( \top \) means transpose. In the time domain, the impulse response is reflected with respect to \( t = 0 \), i.e. given a linear operator \( M \) with an impulse response \( m(i) \) then the impulse response of \( M^\sim \) is \( m(i)\top \). As a result, \( M^\sim \) is anticausal if and only if \( M \) is causal.

Moreover, \( M^\sim \) is anticausal if and only if \( M \) is causal, as a result, \( M^\sim \) is anticausal if and only if \( M \) is causal.

As the class of Zames–Falb multipliers is parametrized by using the \( \ell_1 \)-norm of an LTI system, the following result can be used to ensure that the search of the multiplier is carried out over the class of Zames–Falb multipliers.

Lemma 2.3 ([22]): Consider a dynamical system \( G \) represented by \( x(i + 1) = A x(i) + B u(i), \ y(i) = C x(i) + D u(i) \), and \( x(0) = 0 \). Suppose that there exist \( \mu > 0 \), \( 0 < \lambda < 1 \) and \( P = P\top \) such that

\[
\begin{bmatrix}
A\top PA - \lambda P & A\top PB \\
B\top PA & \mu I
\end{bmatrix} < 0,
\]

\[
\begin{bmatrix}
(\lambda - 1)P + C\top C & C\top D \\
D\top P & (\mu - \gamma^2)I + D\top D
\end{bmatrix} < 0.
\]

Then \( ||G|| < \gamma \). Furthermore, \( A \) has all its eigenvalues in the open unit disk.

Definition 2.4: Given a stable linear plant with state space representation \( (A, B, C, 0) \), the Nyquist value, \( k_N \), is the supremum of the values \( k \) such that the matrix \( A - BKC \) is Schur, i.e. all eigenvalues are within the open unit disk, for all \( K \in [0, k] \).

Finally, the discrete version of the KYP lemma will be used to derive convex conditions from frequency domain inequalities.

Lemma 2.5 ([21]): Given \( A, B, M \) with \( \det(e^{i\omega}I - A) \neq 0 \) for all \( \omega \in [0, 2\pi] \) and the pair \( (A, B) \) controllable, the following two statements are equivalent:

(i) For all \( \omega \in [0, 2\pi] \),

\[
\left( e^{i\omega}(I - A) \right)^{-1}B \right) \top M \left( e^{i\omega}(I - A)^{-1}B \right) \leq 0.
\]

(ii) There is a matrix \( P \in \mathbb{R}^{n \times n} \) such that \( P = P\top \) and

\[
M + \begin{bmatrix}
A\top PA & A\top PB \\
B\top PA & B\top PB
\end{bmatrix} \leq 0.
\]

The corresponding equivalence for strict inequalities holds even if the pair \( (A, B) \) is not controllable.

III. MAIN RESULTS

A. Causal search for discrete-time Zames-Falb multiplier

In the spirit of [27], a search over the class of causal discrete-time Zames–Falb multipliers is presented.

Proposition 3.1: Let

\[
G(z) = \begin{bmatrix}
A_g & B_g \\
C_g & D_g
\end{bmatrix}
\]

where \( A_g \in \mathbb{R}^{n \times n}, B_g \in \mathbb{R}^{n \times 1}, C_g \in \mathbb{R}^{1 \times n} \) and \( D_g \in \mathbb{R}^{1 \times 1} \). Let \( \phi_k \) be an odd nonlinearity slope-restricted in \( \mathcal{S}[0,k] \). Without loss of generality, assume that the feedback interconnection of \( G \) and a linear gain \( k \) is stable. Define \( A_p, B_p, C_p \) and \( D_p \) as follows:

\[
A_p = A_g,
\]
\[
B_p = B_g,
\]
\[
C_p = kC_g,
\]
\[
D_p = 1 + kD_g.
\]

Assume that there exist positive definite symmetric matrices \( S_{11} > 0, P_{11} > 0 \), unstructured matrices \( \tilde{A}, \tilde{B} \) and \( \tilde{C} \) with the same dimension as \( A_n, B_n \) and \( C_n \), respectively, and positive constants \( 0 < \mu < 1 \) and \( 0 < \lambda < 1 \) such that the LMIs (16), (17), and (18) (given on the following page) are satisfied. Then the feedback interconnection (6) is \( \ell_2 \)-stable.

Proof: Let us consider the system \( \tilde{G} = 1 + k\tilde{G} \) where \( \tilde{G}(z) = C_p(zI - A_p)^{-1}B_p + D_p \) and a causal operator \( M \) whose state-space representation is given by

\[
M(z) \sim \begin{bmatrix}
A_u & B_u \\
C_u & D_u
\end{bmatrix}.
\]

The state space representation of \( (z)\tilde{G}(z) \) is given by

\[
M(z)\tilde{G}(z) \sim \begin{bmatrix}
A_p & 0 \\
B_p & B_u D_p
\end{bmatrix} \begin{bmatrix}
A_u & B_u \\
C_u & D_u
\end{bmatrix} = \begin{bmatrix}
A_t & B_t \\
C_t & D_t
\end{bmatrix},
\]

where we have applied Schur complements.

Using the lemma, (22) is equivalent to the frequency condition

\[
\left( e^{i\omega}I - A_t \right)^{-1}B_t \right) \top M \left( e^{i\omega}I - A_t^{-1}B_t \right) \leq 0,
\]

for all \( \omega \in [0, 2\pi] \), which can be rewritten as

\[
\text{Re}\{M(e^{i\omega})\tilde{G}(e^{i\omega})\} > 0 \quad \forall \omega \in [0, 2\pi].
\]

Therefore, the existence of \( P \) satisfying (22) implies the positivity condition of the system \( M(1 + kG) \).

On the other hand, using Lemma 2.3 with the system \( (A_n, B_n, C_n, 0) \), if there exists \( Y \) with \( Y = Y\top \) such that

\[
\begin{bmatrix}
A_n & Y & A_n Y B_n \\
A_n Y B_n & B_n & Y B_n - \mu I
\end{bmatrix} < 0,
\]

\[
\begin{bmatrix}
(\lambda - 1)Y + C_n C_n \mu - I
\end{bmatrix} < 0,
\]

then \( M \) is a causal Zames-Falb multiplier.
In the spirit of [11], [27], the rest of the proof consists in applying a change of variables to show that the feasibility of LMIs (16), (17), and (18) is equivalent to the feasibility of (22), (25) and (26). As a result, the feasibility of (16), (17), and (18) implies the existence of a Zames–Falb multiplier $M \in \mathcal{RH}_m$ such that $M(e^{j\omega})(1 + kG(e^{j\omega})) > 0$ for all $\omega \in [0,2\pi]$. Hence the conditions in Theorem 2.1 are fulfilled and the feedback interconnection (6) is $\ell_2$-stable.

Remark 3.2: Similar to the continuous case, the inequalities (16), (17), and (18) are not LMIs if $\lambda$ is defined as variable. Hence, the use of this result requires a linear search of $\lambda$ between 0 and 1.

Remark 3.3: The change of variable is the same as in the continuous case. Therefore the multiplier can be recovered following [6]
\begin{align*}
A_u &= -(P_1 - S_{11})^{-1}\hat{A}, \quad (27) \\
B_u &= -(P_1 - S_{11})^{-1}\hat{B}, \quad (28) \\
C_u &= \hat{C}, \quad (29)
\end{align*}
where the last equation has been modified to be consistent with our definition of the multiplier.

B. Anticausal multiplier

The anticausal counterpart of the above search can be stated as follows:

Proposition 3.4: Let $G \in \mathcal{RH}_m$ be represented in the state space by $A_g$, $B_g$, $C_g$ and $D_g$ where $A_g \in \mathbb{R}^{n \times n}$, $B_g \in \mathbb{R}^{n \times 1}$, $C_g \in \mathbb{R}^{1 \times n}$ and $D_g \in \mathbb{R}^{1 \times 1}$. Let $\phi_k$ an odd nonlinearity slope-restricted in $S[0,k]$. Without loss of generality, assume that the feedback interconnection of $G$ and a linear gain $k$ is well-posed and stable. Define $A_p$, $B_p$, $C_p$ and $D_p$ as follows:
\begin{align*}
A_p &= A_g - B_g(kD_g + 1)^{-1}kC_g, \quad (30) \\
B_p &= B_g(kD_g + 1)^{-1}, \quad (31) \\
C_p &= (kD_g + 1)^{-1}kC_g, \quad (32) \\
D_p &= (kD_g + 1)^{-1}. \quad (33)
\end{align*}
Assume that there exist positive definite symmetric matrices $S_{11} > 0$, $P_{11} > 0$, unstructured matrices $\hat{A}_u$, $\hat{B}_u$ and $\hat{C}_u$, and positive constants $0 < \mu < 1$ and $0 < \lambda < 1$ such that the LMIs (16), (17), and (18) are satisfied, then the feedback interconnection (6) is $\ell_2$-stable.

Proof: The result is based on phase properties of the adjoint and inverse systems, and it is the same as provided in [7] for the continuous time counterpart, where the inner product between two signals in $\ell_2(Z)$ is defined by (2).

The existence of a causal Zames–Falb multiplier such that $M_c(e^{j\omega})(1 + kG(e^{j\omega}))^{-1} > 0$ for all $\omega \in [0,2\pi]$, ensures the existence of an anticausal Zames–Falb multiplier such that $M_{ac}(e^{j\omega})(1 + kG(e^{j\omega})) > 0$ for all $\omega \in [0,2\pi]$, where $M_{ac} = M_c^{-1}$. Hence the conditions in Theorem 2.1 are fulfilled and the feedback interconnection (6) is $\ell_2$-stable.

Remark 3.5: Once the search has provided the matrices $A_u$, $B_u$, and $C_u$, then the multiplier is given by:
\begin{equation}
M_{ac}(z) = C_u \left(\frac{1}{z}I - A_u\right)^{-1}B_u + 1, \quad (34)
\end{equation}
which can be written as
\begin{equation}
M_{ac}(z) \sim \left[\begin{array}{c|c}
A_u^{-T} & A_u^{-T}C_u^{-1} \\
\hline
B_u^{-1}A_u^{-1} & 1 - B_u^{-1}A_u^{-1}C_u^{-1} \end{array}\right], \quad (35)
\end{equation}
if $A_u$ is non-singular. If $A_u$ is singular, then the result is still valid but the multiplier does not have a forward representation. Note that the region of convergence of this transfer function does not include $z = \infty$ and the term $m_0$ in the inverse z-transform of $M_{ac}(z)$ corresponds with $M_{ac}(0)$, i.e. $(Z^{-1}(M_{ac}))(0) = M_{ac}(0)$.

IV. EXAMPLES

Table I shows the examples that have been considered for testing the criteria. Examples 1 and 5 have been used previously in the literature. As commented, a linear search over $\lambda$ is carried out finding a maximum slope at every value of $\lambda$, see Fig. 2. As a difference with the continuous-time counterpart, this search is very simple since it is restricted to a compact interval. Results of both searches are given in Table IV.

The examples show the necessity of conducting both causal and anticausal searches in order to obtain competitive results against other criteria given in the literature. In particular, we can analyse the differences between Examples 2 and 3. Fig. 3 shows the Bode plots (phase only) of $1 + 0.7687G_2$. 5261
The numerical results appear to show differences between the continuous-time and discrete-time domains. For example, either the causal or the anticausal search approaches the Nyquist value for first, second and third order systems in the continuous-time domain [7]. Example 1 shows that this need not be the case in the discrete-time domain. Future work is needed to provide a better understanding of the links between the continuous-time and discrete-time Zames–Falb multipliers.

To conclude, further research in alternative searches for the discrete time Zames–Falb multipliers is required. Other methods, e.g. [8], [9], which are not restricted by causality considerations, should be extended to the discrete time to understand the conservatism of the method proposed in this paper.

VI. ACKNOWLEDGEMENTS

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REFERENCES

Fig. 3. Phase of $1 + 0.7687G_2$ in blue and solid and $1 + 0.3058G_3$ in red and dashed.

<table>
<thead>
<tr>
<th>Criterion</th>
<th>Ex. 1</th>
<th>Ex. 2</th>
<th>Ex. 3</th>
<th>Ex. 4</th>
<th>Ex. 5</th>
</tr>
</thead>
<tbody>
<tr>
<td>Circle Criterion [24]</td>
<td>0.79333</td>
<td>0.19836</td>
<td>0.13788</td>
<td>1.5316</td>
<td>1.0275</td>
</tr>
<tr>
<td>Tsypkin Criterion [17]</td>
<td>3.8000</td>
<td>0.24268</td>
<td>0.13788</td>
<td>1.691</td>
<td>1.0275</td>
</tr>
<tr>
<td>Park &amp; Kim [20]</td>
<td>3.8000</td>
<td>0.24268</td>
<td>0.14789</td>
<td>1.691</td>
<td>1.7252</td>
</tr>
<tr>
<td>Haddad &amp; Bernstein [15]</td>
<td>2.9374</td>
<td>0.21429</td>
<td>0.13788</td>
<td>1.5562</td>
<td>1.0275</td>
</tr>
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<td>Ahmad et al. (2013) [2]</td>
<td>5.8818</td>
<td>0.32794</td>
<td>0.14789</td>
<td>1.834</td>
<td>2.4474</td>
</tr>
<tr>
<td>Ahmad et al. (2012) [1]</td>
<td>12.4178</td>
<td>0.72614</td>
<td>0.30576</td>
<td>2.5911</td>
<td>2.4474</td>
</tr>
<tr>
<td>Gonzaga et al. (2012) [14]</td>
<td>0.79297</td>
<td>0.13788</td>
<td>0.30267</td>
<td>2.9374</td>
<td>1.0271</td>
</tr>
<tr>
<td>Causal DT Zames-Falb</td>
<td>12.4355</td>
<td>0.7687</td>
<td>0.2341</td>
<td>3.2365</td>
<td>2.4474</td>
</tr>
<tr>
<td>Anti-causal DT Zames-Falb</td>
<td>1.4994</td>
<td>0.4816</td>
<td>0.3058</td>
<td>3.3606</td>
<td>2.4474</td>
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<tr>
<td>Nyquist Value $k_N$</td>
<td>36.10</td>
<td>2.74</td>
<td>0.312</td>
<td>7.90</td>
<td>2.4474</td>
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TABLE II

Sector/slope bounds obtainable using various stability criteria. These results have been obtained using MATLAB LMI solver.

<table>
<thead>
<tr>
<th>Ex.</th>
<th>$\lambda$</th>
<th>$M_1(z)$</th>
<th>$M_2(z)$</th>
<th>$M_3(z)$</th>
<th>$M_4(z)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1 $10^{-4}$</td>
<td>$z^{-2}$ ($-0.3990 + 7.0060 i$)</td>
<td>$z^{-2}$ ($-1.062 + 0.001 i$)</td>
<td>$z^{-2}$ ($-1.717 + 0.255 i$)</td>
<td>$z^{-2}$ ($-0.0164 + 0.0001 i$)</td>
</tr>
<tr>
<td>2</td>
<td>0.0474</td>
<td>$z^{-2}$ ($-0.3176 - 0.0058 i$)</td>
<td>$z^{-2}$ ($-1.6511 - 0.0011 i$)</td>
<td>$z^{-2}$ ($-1.3325 - 0.0011 i$)</td>
<td>$z^{-2}$ ($-0.0000 - 0.0000 i$)</td>
</tr>
<tr>
<td>3</td>
<td>0.0474</td>
<td>$z^{-2}$ ($-1.4136$)</td>
<td>$z^{-2}$ ($-1.4075 + 0.5066 i$)</td>
<td>$z^{-2}$ ($-1.9211$)</td>
<td>$z^{-2}$ ($-2.325$)</td>
</tr>
<tr>
<td>4</td>
<td>0.3158</td>
<td>$z^{-2}$ ($-0.3990 - 0.0014 i$)</td>
<td>$z^{-2}$ ($-0.3298 - 0.0014 i$)</td>
<td>$z^{-2}$ ($-0.4816 - 0.0014 i$)</td>
<td>$z^{-2}$ ($-0.5016 - 0.0014 i$)</td>
</tr>
<tr>
<td>5</td>
<td>0.5</td>
<td>$z^{-2}$ ($-0.3990 + 0.0026 i$)</td>
<td>$z^{-2}$ ($-0.3298 + 0.0026 i$)</td>
<td>$z^{-2}$ ($-0.4816 + 0.0026 i$)</td>
<td>$z^{-2}$ ($-0.5016 + 0.0026 i$)</td>
</tr>
</tbody>
</table>

TABLE III

Reconstructed multipliers. Note that $M_3$ and $M_5$ are anticausal.