When Does Relaxation Reduce the Minimum Cost of an Optimal Control Problem?

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Abstract—Relaxation is a regularization procedure used in optimal control, involving the replacement of velocity sets by their convex hulls, to ensure the existence of a minimizer. It can be an important step in the construction of sub-optimal controls for the original, unrelaxed, optimal control problem (which may not have a minimizer), based on obtaining a minimizer for the relaxed problem and approximating it. In some cases the infimum cost of the unrelaxed problem is strictly greater than the infimum cost over relaxed state trajectories; there is a need to identify such situations because then the above procedure fails. Following on from earlier work by Warga, we explore the relation between, on the one hand, non-coincidence of the minimum cost of the optimal control and its relaxation and, on the other, abnormality of necessary conditions (in the sense that they take a degenerate form in which the cost multiplier tends to zero).

Keywords: Necessary Conditions, Optimal Control, Differential Inclusions.

I. INTRODUCTION

Consider the optimal control problem, in which the dynamic constraint is formulated as a differential inclusion:

\[
(P) \quad \begin{cases} 
\text{Minimize } g(x(0), x(1)) \\
\text{over } x(.) \in W^{1,1} \text{ satisfying} \\
\dot{x}(t) \in F(t, x(t)) \text{ a.e.,} \\
(x(0), x(1)) \in C,
\end{cases}
\]

the data for which comprise a continuous function \( g : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R} \), a closed set \( C \subset \mathbb{R}^n \times \mathbb{R}^n \) and a multifunction \( F(., .) : [0,1] \times \mathbb{R}^n \to \mathbb{R}^n \). \( W^{1,1} \) denotes the space of absolutely continuous \( \mathbb{R}^n \) valued functions on \([0,1] \).

A state trajectory \( x(.) \) is a \( W^{1,1} \) function that satisfies \( \dot{x}(t) \in F(t, x(t)) \), a.e. The state trajectory \( x(.) \) is said to be feasible if \( (x(0), x(1)) \in C \).

A state trajectory \( \bar{x}(.) \) is a minimizer if it achieves the minimum of \( g(x(0), x(1)) \) over all feasible state trajectories \( x(.) \). It is called a \( L^\infty \)-local minimizer if, for some \( \epsilon > 0 \),

\[
g(x(0), x(1)) \geq g(\bar{x}(0), \bar{x}(1))
\]

for all feasible state trajectories \( x(.) \) such that

\[
\| x(.) - \bar{x}(.) \|_{L^\infty} \leq \epsilon.
\]

Now consider the reachable set \( R \) which, here, we take to be the set of all possible endpoint values of state trajectories:

\[
R := \{(x(0), x(1)) | x(.) \text{ is a state trajectory}\}.
\]

The problem \( P \) can be reformulated in terms of \( R \):

\[
\text{Minimize } g(x(0), x(1)) \text{ over } (x(0), x(1)) \in R \cap C.
\]

A standard framework for studying optimal control problems, in which the existence of minimizing feasible trajectories is guaranteed, is to impose a hypothesis regarding convexity of the velocity sets:

\[
(C) \quad F(t, x) \text{ is convex for all } (t, x) \in [0,1] \times \mathbb{R}^n.
\]

For then we can show (under additional, unrestrictive, hypotheses), by means of a weak sequential compactness analysis, that the set \( R \) is closed. It follows that, if also the set \( R \cap C \) is bounded and non-empty and \( g \) is continuous, then \( P \) automatically has a minimizing feasible \( F \)-trajectory, as the minimizer of a continuous function over a non-empty compact set.

Suppose, on the other hand, that the convexity hypothesis is violated. A standard regularization procedure, aimed at enlarging the domain of \( P \) in order to guarantee the existence of minimizers, is to consider the relaxed problem:

\[
(PR) \quad \begin{cases} 
\text{Minimize } g(x(0), x(1)) \\
\text{over } x(.) \in W^{1,1} \text{ satisfying} \\
\dot{x}(t) \in \text{co } F(t, x(t)) \text{ a.e.,} \\
(x(0), x(1)) \in C,
\end{cases}
\]

where \( \text{co} \) denotes the convex hull of \( F(t, x) \). State trajectories and \( L^\infty \)-local minimizers for \( PR \) are referred to as ‘relaxed state trajectories’ and ‘relaxed \( L^\infty \)-local minimizers’, respectively. Since the velocity sets \( F(t, x) \) have been replaced by their convex hulls, the relaxed problem automatically has a minimizer (under additional, mild hypotheses).

The Relaxation Theorem asserts (under appropriate, unrestrictive hypotheses) that a relaxed state trajectory \( x(.) \) can be approximated arbitrarily closely w.r.t. the \( L^\infty \) norm by a state trajectory. It follows that, if we
define the \( R_{\text{relaxed}} \) to be the reachable set for state trajectories associated with \((PR)\), then
\[
R_{\text{relaxed}} = \bar{R}.
\]

Often
\[
\inf\{g(x(0), x(1)) \mid (x(0), x(1)) \in R \cap C\} = \inf\{g(x(0), x(1)) \mid (x(0), x(1)) \in \bar{R} \cap C\}.
\]

It may arise however that
\[
\inf\{g(x(0), x(1)) \mid (x(0), x(1)) \in R \cap C\} > \inf\{g(x(0), x(1)) \mid (x(0), x(1)) \in \bar{R} \cap C\}.
\]

This can be equivalently written
\[
\inf(PR) < \inf(P), \tag{1.2}
\]
where \( \inf(PR) \) and \( \inf(P) \) denote the infimum costs of \((PR)\) and \((P)\) respectively.

The phenomenon is illustrated by Fig. 1, in which the state dimension \( n = 1, C = \{x_0\} \times \mathbb{R} \) and \( g(x_0, x_1) = x_1 \). The infimum cost for \((P)\) (respectively \((PR)\)) is the lowest intersection of the vertical line through \( x_0 \) with \( R \) (respectively \( \bar{R} \)).

It is important to identify the occurrence of an ‘infimum gap’ (1.2) (between the problem \((P)\) and its relaxation \((PR)\)) for several reasons. First we note that the Hamilton-Jacobi equation is the same for \((P)\) and \((PR)\), but in circumstances where there is a unique generalized solution to the Hamilton-Jacobi equation, this solution coincides with the value function of the relaxed problem; the Dynamic Programming approach fails to provide the value function for \((P)\) in this case. Second, the presence of an infimum gap indicates instability of the minimum cost and of the set of optimal processes under perturbations to the endpoint constraint, which adds to the difficulty of numerical solution.

This paper is concerned with the link between the existence of an infimum gap, at least in a local sense, and the degeneracy of the necessary conditions of optimality, expressed in terms of Clarke’s Hamiltonian inclusion. Two consequences of an infimum gap are explored, which differ according to whether we focus attention on a \( L^\infty \)-local minimizer which cannot also be interpreted as a relaxed \( L^\infty \)-local minimizer, or on a relaxed minimizer, whose cost is strictly less than the infimum cost over admissible (non-relaxed) processes.

**Type A relation:** A \( L^\infty \)-local minimizer satisfies necessary conditions of optimality in abnormal form (i.e. with cost multiplier zero) if it is not also a relaxed \( L^\infty \)-local minimizer.

**Type B relation:** A relaxed \( L^\infty \)-local minimizer satisfies necessary conditions of optimality (for the relaxed problem) in abnormal form if its cost is strictly less than the infimum cost over all feasible processes, whose state trajectories are close (in the \( L^\infty \) sense) to that of the relaxed \( L^\infty \)-local minimizer.

Warga pioneered investigations into the link between the existence of an infimum gap and validity of first order necessary conditions in abnormal form. The link has also been studied by Ioffe in [1], [2].

In his monograph [11] Warga proved a Type B relation for optimal control problems, in which the dynamic constraint is expressed as a controlled differential equation and the set of necessary conditions considered is the Pontryagin Maximum Principle. (The expository paper [10] stresses the contrapositive interpretation ‘if there are no relaxed state trajectories satisfying a relaxed version of the Pontryagin Maximum Principle with cost multiplier \( \lambda = 0 \), then there cannot exist minimizers that are not also relaxed minimizers’). In a subsequent paper [12], Warga generalized his earlier Type B results to allow for nonsmooth data, making use of the ‘derivative container’ local approximations [13].

Type A relations have received less attention, surprisingly so since they come closer to addressing the key underlying question: when is a minimizer not a relaxed minimizer?

In this paper we present, to our knowledge, the first Type B relation for optimal control problems in which the dynamic constraint is formulated as a differential inclusion. An earlier Type B relation for optimal control problems in which the dynamic constraint takes the form of a differential equation with control is due to Warga. We also present a Type A relation for such problems, when ‘relaxed local minimizer’ is interpreted in the standard \( L^\infty \) sense, which is new, in either the differential inclusions or controlled differential equations context.
The following notation will be used throughout the paper: for vectors $x \in \mathbb{R}^n$, $|x|$ denotes the Euclidean length. $B$ denotes the closed unit ball in $\mathbb{R}^n$. Given a multifunction $\Gamma(.) : \mathbb{R}^n \rightrightarrows \mathbb{R}^k$, the graph of $\Gamma(.)$, written $\text{Gr}(\Gamma(.))$, is the set $\{(x,v) \in \mathbb{R}^n \times \mathbb{R}^k | v \in \Gamma(x)\}$. Given a set $A \subset \mathbb{R}^n$ and a point $x \in \mathbb{R}^n$, we denote by $d_A(x)$ the Euclidean distance of a point $x \in \mathbb{R}^n$ from $A$:

$$d_A(x) := \inf\{|x - y| | y \in A\}.$$

$W^{1,1}([0,1];\mathbb{R}^n)$ is the space of absolutely continuous $\mathbb{R}^n$-valued function on $[0,1]$. We write $W^{1,1}$ in place of $W^{1,1}([0,1];\mathbb{R}^n)$, etc. when the meaning is clear.

We shall use several constructs of nonsmooth analysis. Given a closed set $D \subset \mathbb{R}^k$ and a point $\bar{x} \in D$, we denote by $N_D(\bar{x})$ the limiting normal cone of $D$ at $\bar{x}$ as defined, for example, in ([9], Def. 4.2.3). Given a lower semicontinuous function $f : \mathbb{R}^k \rightarrow \mathbb{R}$ and a point $\bar{x} \in \text{dom } f := \{x \in \mathbb{R}^k | f(x) < +\infty\}$ The subdifferential of $f$ at $\bar{x}$ as defined, for example, in ([9], Def. 4.3.1). For further relevant information on nonsmooth analysis, we refer to [4], [6], [8] or [9].

**II. CONDITIONS FOR AN INFIMUM GAP**

In this section we state two theorems relating the existence of a gap (in some local sense) between the infimum costs for the optimal control problem $(P)$ and its relaxed counterpart $(PR)$, and the validity of the Hamiltonian inclusion in abnormal form. The following hypotheses, in which $\bar{x}(.)$ is a given absolutely continuous function, will be invoked:

**H1:** $F(.,x)$ is an $L$-measurable multifunction for each $x$ and $F(.,.)$ takes values closed sets.

**H2:** There exist $k(.,.) \subset L^1$ and $\varepsilon > 0$ such that

$$F(t,x) \subset F(t,x') + k(t)(|x - x'|)B \quad \text{and} \quad F(t,x) \in c(t)B$$

for all $x, x' \in \bar{x}(t) + \varepsilon B$, a.e $t \in [0,1]$.

Define the Hamiltonian function $H(.,.,.) : [0,1] \times \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}$

$$H(t,x,p) := \max_{e \in F(t,x)} p \cdot e.$$

The first theorem is a Type A relation:

**Theorem 2.1:** Let $\bar{x}(.)$ be a $L^\infty$-local minimizer. Assume (H1) and (H2) are satisfied and $g(.,.)$ is Lipschitz continuous on a neighborhood of $(\bar{x}(0),\bar{x}(1))$.

(a): Then there exist $p(.) \in W^{1,1}$ and $\lambda \geq 0$ such that

(i) $(p(0),\lambda) \neq (0,0)$,

(ii) $(-\dot{p}(t),\dot{\bar{x}}(t)) \in \partial_{x,p}H(t,\bar{x}(t),p(t))$ a.e. ,

(iii) $(p(0),-p(1)) \in \lambda \partial g(\bar{x}(0),\bar{x}(1))$ $+ N_C(\bar{x}(0),\bar{x}(1))$.

(b): Suppose that, for every $\epsilon > 0$, there exists a feasible relaxed state trajectory $x(.)$ such that

$$g(\bar{x}(0),\bar{x}(1)) > g(x(0),x(1))$$

and $||x(.) - \bar{x}(.)||_{L^\infty} \leq \epsilon$, (i.e. $\bar{x}(.)$ is not also a relaxed $L^\infty$-local minimizer).

Then conditions (i)-(iii) above are satisfied for some choice of multipliers $(p(.),\lambda)$, such that $\lambda = 0$.

**Comments**

(1): Part (a) is Clarke’s well-known Hamiltonian inclusion ([5]). Interest focuses on part (b), which is a Type A relation.

(2): The contrapositive statement of part (b) is a sufficient condition for the absence of an infimum gap (in a local sense): if $\bar{x}(.)$ is a $L^\infty$-local minimizer such that, given any multipliers $(p(.),\lambda)$ satisfying conditions (i) - (iii), we have $\lambda \neq 0$, then $\bar{x}(.)$ is also a relaxed $L^\infty$-local minimizer.

The second theorem includes a Type B relation:

**Theorem 2.2:** Let $\bar{x}(.)$ be a relaxed feasible state trajectory. Assume (H1) and (H2) are satisfied.

(a): Suppose that $g(.,.)$ is Lipschitz continuous on a neighborhood of $(\bar{x}(0),\bar{x}(1))$ and there exists $\epsilon > 0$ such that

$$g(\bar{x}(0),\bar{x}(1)) = \inf\{g(x(0),x(1)) | x(.) \text{ is a feasible (non-relaxed) state trajectory s.t. } ||x(.) - \bar{x}(.)||_{L^\infty} \leq \epsilon\}.$$

Then there exist $p(.) \in W^{1,1}([0,1];\mathbb{R}^n)$ and $\lambda \geq 0$ such that

(i) $(p(0),\lambda) \neq (0,0)$,

(ii) $(-\dot{p}(t),\dot{\bar{x}}(t)) \in \partial_{x,p}H(t,\bar{x}(t),p(t))$ a.e. ,

(iii) $(p(0),-p(1)) \in \lambda \partial g(\bar{x}(0),\bar{x}(1)) + N_C(\bar{x}(0),\bar{x}(1))$.

(b): Suppose that $g(.,.)$ is continuous on a neighborhood of $(\bar{x}(0),\bar{x}(1))$ and, for some $\epsilon > 0$, $g(\bar{x}(0),\bar{x}(1)) < \inf\{g(x(0),x(1)) | x(.) \text{ is a feasible (non-relaxed) state trajectory s.t. } ||x(.) - \bar{x}(.)||_{L^\infty} \leq \epsilon\}$

Then conditions (i)-(iii) above are satisfied for some choice of multipliers $(p(.),\lambda)$, such that $\lambda = 0$.
Comments:
(1): It is well known that if a relaxed feasible state trajectory \( \bar{x}(.) \) achieves the minimum cost over relaxed feasible state trajectories in an \( L^\infty \) neighborhood about \( \bar{x}(.) \), then \( \bar{x}(.) \) satisfies the Hamiltonian inclusion. (This follows from applying the Hamiltonian necessary condition to the relaxed problem.) Part (a) of the theorem tells us a little bit more: it says that the Hamiltonian inclusion is satisfied at the relaxed feasible state trajectory \( \bar{x}(.) \), if the cost of \( \bar{x}(.) \) coincides with the infimum cost for the problem over the smaller set of (non-relaxed) feasible state trajectories in some \( L^\infty \) neighborhood about \( \bar{x}(.) \), an apparently new result for optimal control problems formulated in terms of differential inclusions. This remains the case even if the infimum is not attained over (non-relaxed) feasible state trajectories on some \( L^\infty \) neighborhood about \( \bar{x}(.) \).

(2) Recall that a Type B relation (involving the Hamiltonian inclusion) asserts that, if a relaxed state trajectory \( \bar{x}(.) \) which is a \( L^\infty \)-local minimizer for the relaxed problem has cost lower than the infimum cost over (non-relaxed) state trajectories in an \( L^\infty \) neighborhood about \( \bar{x}(.) \), then \( \bar{x}(.) \) satisfies the Hamiltonian inclusion in abnormal form. Notice that part (b) of theorem is a stronger statement because it says the Hamiltonian inclusion is satisfied in abnormal form, even if \( \bar{x}(.) \) is not an \( L^\infty \) local relaxed minimizer, but merely has cost strictly less than the infimum cost over (non-relaxed) feasible state trajectories in some \( L^\infty \) neighborhood about \( \bar{x}(.) \). Warga’s Type B relations for optimal control problems involving a differential equation with control term also include this refinement.

Sketch of Proofs. The lengthy proofs of the relations reported in this paper will appear in [7]. Here we supply only a brief overview. Thm. 2.2 is first proved. Take a feasible relaxed state trajectory, as in the theorem statement. The relaxation theorem and a variational principle are used to construct a sequence of optimal control problems (involving only the constraints of the original problem, and not the original cost function) and a sequence of state trajectories \( x_i(.) \to \bar{x}(.) \) (in \( L^\infty \)) that are \( L^\infty \)-local minimizers for these problems. Necessary conditions of optimality, in the form of the Hamiltonian inclusion, are applied for each \( i \). These identify each \( x_i(.) \) as an abnormal extremal for a perturbed version of the original control problem. We deduce, in the limit as \( i \to \infty \), that \( \bar{x}(.) \) is an abnormal extremal w.r.t. the Hamiltonian inclusion condition. We mention that a key feature of the proof is the use of Stegall’s theorem as in [5], in the role of variational principle, to construct a sequence of optimal control problems with the properties above. The assertions of Thm. 2.1 follow from Thm. 2.2. If \( \bar{x}(.) \) is a feasible state trajectory as in the statement of Thm. 2.2, then there exists a sequence of relaxed feasible trajectories \( x_i(.) \to \bar{x}(.) \) (in \( L^\infty \)) , each with lower cost than that of \( \bar{x}(.) \). The assertions of Thm. 2.1 are confirmed by applying Thm. 2.2 to each \( x_i(.) \) and passage to the limit.

III. Examples

In this section we provide an example of an optimal control problem which illustrates the assertions of Thms. 2.1 and 2.2. Earlier examples of optimal control problems, in which the infimum costs over admissible state trajectories and over relaxed admissible state trajectories fail to coincide, are to be found, for example, in ([11], p. 246)

\[
\begin{cases}
\text{Minimize } -x_1(1) \\
\quad \text{over } (x(\cdot) = (x_1(\cdot), x_2(\cdot), x_3(\cdot)) \text{ satisfying }} \\
(\dot{x}(t), \dot{x}_2(t), \dot{x}_3(t)) \in \\
\quad \{(0, x_1(t), |x_2(t)|^2) \cup \{0, -x_1(t), |x_2(t)|^2\} \\
x_2(0) = x_3(0) = x_3(1) = 0.
\end{cases}
\]

This is an example of \((P)\), in which \( n = 3, m = 1 \).

\[
F(t, x_1, x_2, x_3) = \{(0, x_1, x_2^2) \cup \{0, -x_1, x_2^2\}\},
\]

\[
g((x_1^0, x_2^0, x_3^0), (x_1^1, x_2^1, x_3^1)) = -x_1^1 \quad \text{and} \quad C = (\mathbb{R} \times \{0\} \times \{0\}) \times (\mathbb{R} \times \mathbb{R} \times \{0\}).
\]

Claim: \( \bar{x}(\cdot) \equiv (0, 0, 0) \) is a minimizer for \((E)\).

To validate the claim, suppose there exists a feasible state trajectory \( x(.) \) with cost lower than that of \( \bar{x}(.) \). Since \( \dot{x}_1(t) = 0 \) and the cost is \(-x_1(1), x_1 \equiv \alpha \) for some \( \alpha > 0 \). But \( \dot{x}_3(t) = |x_2(t)|^2 \geq 0 \) and \( x_3(0) = x_3(1) = 0 \). We deduce from the relation

\[
x_3(1) - x_3(0) = \int_0^1 |x_2(t)|^2 dt
\]

that \( x_2(.) \equiv 0 \). It follows that \( \dot{x}_2(t) \equiv 0 \) a.e. However \( \dot{x}_2(t) \in \{\alpha\} \cup \{-\alpha\} \), a.e. We conclude that \( \dot{x}_2(t) \not\equiv 0 \) a.e.. From this contradiction we deduce that no feasible state trajectory exists with cost less than that of \( \bar{x}(\cdot) \equiv (0, 0, 0) \), as claimed.

Observe that \( \bar{x}(\cdot) \equiv (0, 0, 0) \) is not a relaxed \( L^\infty \)-local minimizer. This is because, for any \( \alpha > 0 \), the arc

\[
x^\alpha(\cdot) \equiv (\alpha, 0, 0)
\]

satisfies the convexified dynamic constraint \( \dot{x}^\alpha(t) \in \text{co} \ F(x^\alpha(t)) \) a.e., and also the endpoint constraints. It is therefore a feasible relaxed state trajectory. But (by adjustment of \( \alpha \)) \( ||x^\alpha(\cdot) - \bar{x}(\cdot)||_{L^\infty} = \alpha \) can be made arbitrarily small. Yet its cost is \(-\alpha \), which is strictly less than that of \( \bar{x}(\cdot) \).
Illustration of Thm. 2.1: We examine the Hamiltonian inclusion condition at $\bar{x}(.) \equiv (0, 0, 0)$. The Hamiltonian is

$$H(x, p) = |p_2x_1| + p_3x_2^2.$$ 

and

$$\partial H(x, p) = \{ (\gamma p_2, 2p_3x_2, 0, 0, \gamma x_1, x_2^2) | \gamma \in [-1, 1] \}.$$ 

Denote by $(p(.), \lambda)$ the (non-trivial) costate/cost multiplier pair. The Hamiltonian inclusion and transversality condition take the following form: there exists a measurable function $\gamma(.) : [0, 1] \rightarrow [-1, 1]$ such that

$$-\dot{p}_1(t) = \gamma(t)p_2(t), \quad -\dot{p}_2(t) = 2p_3(t)x_2(t)$$

$$-\dot{p}_3(t) = 0,$$

$$-\dot{x}_1(t) = 0, \quad -\dot{x}_2(t) = \gamma(t)x_1(t), \quad \dot{x}_3(t) = x_2^2(t)$$

$$p_1(0) = 0, \quad p_1(1) = \lambda, \quad p_2(1) = 0.$$ 

These conditions (in which $\bar{x}(.) \equiv 0$), are satisfied if and only if, for some $k \neq 0$,

$$p_1(.) \equiv 0, \quad p_2(.) \equiv 0, \quad p_3(.) \equiv k \quad \text{and} \quad \lambda = 0. \quad (\text{III.1})$$

(We can choose $\gamma(.)$ to be any measurable function taking values in $[-1, 1]$.) We see that $\bar{x}(.)$ satisfies the Hamiltonian condition with cost multiplier $\lambda = 0$, as predicted by Thm. 2.1. In this example, all possible choices of multipliers are signed scalings (involving multiplication by the non-zero constant $k$) of the multiplier set $(p(.) \equiv (0, 0, 1), \lambda = 0)$.

Illustration of Thm. 2.2: Now fix $\alpha > 0$ and consider the feasible relaxed state trajectory $x^\alpha(.) \equiv (\alpha, 0, 0)$. As we have observed, $x^\alpha(.)$ has cost strictly lower than that of any feasible state trajectory. The Hamiltonian inclusion conditions are the same as before, but now $\bar{x}(.)$ is replaced by $x^\alpha(.)$. The conditions are satisfied if and only if $p(.)$ and $\lambda$ are given by (III.1). (Now, however, we must choose $\gamma(.) \equiv 0$ since $x_1^\alpha$ is a non-zero value.) We have confirmed that $x^\alpha(.) \equiv (\alpha, 0, 0)$ satisfies the Hamiltonian inclusion condition with cost multiplier $\lambda = 0$, as predicted by Thm. 2.2.

IV. OPEN QUESTIONS

Preceding sections of this paper reveal clean, simply stated, relations between the occurrence of an infimum gap and the existence of abnormal extremals, when the underlying necessary condition of optimality is taken to be Clarke’s Hamiltonian inclusion.

There are many open questions regarding the validity of the relation between infimum gaps and abnormal extremals, when abnormality is defined in relation to other types of necessary conditions. Neither a Type A theorem nor a Type B theorem is known when the underlying necessary condition is either the generalized Euler Lagrange condition, or the partially convexified Hamiltonian inclusion. For abnormality defined in the sense of the Pontryagin maximum principle, a Type A theorem is not known (though here it is possible to prove a weak Type A relation in which the condition on the costate arc is replaced by a less precise ‘averaged’ condition). On the other hand, Type B relations date back to the early work of Warga [11], [12]. In what follows, we state a new Type B relation for the nonsmooth Maximum Principle of [3] expressed in terms if the Clarke generalized gradients. Consider the problem

$$\begin{align*}
&P \text{ Minimize } g(x(0), x(1)) \\
&\text{ over } x(.) \in W^{1,1} \\
&\text{ and measurable functions } u(.) \text{ satisfying} \\
&\dot{x}(t) = f(t, x(t), u(t)) \text{ a.e.,} \\
&u(t) \in U(t) \text{ a.e.,} \\
&(x(0), x(1)) \in C,
\end{align*}$$

the data for which comprise: functions $g : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}$, $f : [0, 1] \times \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}^n$, a closed set $C \subset \mathbb{R}^n \times \mathbb{R}^n$ and a multifunction $U(.) : [0, 1] \rightarrow \mathbb{R}^n$.

A process $(x(.), u(.))$ is a pair of functions, of which the first $x(.) : [0, 1] \rightarrow \mathbb{R}^n$ is an absolutely continuous function, and the second $u(.) : [0, 1] \rightarrow \mathbb{R}^m$ is a measurable function satisfying

$$\dot{x}(t) = f(t, x(t), u(t)) \text{ a.e.} \quad \text{and} \quad u(t) \in U(t) \text{ a.e.}$$

A process $(x(.), u(.))$ is said to be feasible if $(x(0), x(1)) \in C$. The first component $x(.)$ of a (feasible) process $(x(.), u(.))$ is called a (feasible) state trajectory and the second a (feasible) control function.

The related relaxed problem is

$$\begin{align*}
&\text{Minimize } g(x(0), x(1)) \\
&\text{ over } x(.) \in W^{1,1} \\
&\text{ and measurable functions } (u_0(.), \ldots, u_n(.)), \\
&\left(\lambda_0(.), \ldots, \lambda_n(.)\right) \text{ satisfying} \\
&\dot{x}(t) = \sum_{j=0}^{n} \lambda_j(t) f(t, x(t), u_j(t)) \text{ a.e.,} \\
&(u_0(t), \ldots, u_n(t)) \in U(t) \prod_{j=0}^{n} U(t) \text{ a.e.,} \\
&(\lambda_0(t), \ldots, \lambda_n(t)) \in \Sigma \text{ a.e.} \\
&(x(0), x(1)) \in C.
\end{align*}$$

Here, $\Sigma$ is the set of simplicial index values in $n$-dimensional space. Minimizers, $L^\infty$-local minimizers, relaxed minimizers, etc. all have their obvious meaning. Define the Hamiltonian function

$$H(t, x, u, p) := p \cdot f(t, x, u).$$

Theorem 4.1: Let $(\bar{x}(.), \{ (\bar{\lambda}^k, \bar{u}^k(.)) \}_{k=0}^\infty)$ be a relaxed feasible process. Assume that

$$f(. , x, u) \text{ is } L\text{-measurable and } f(t, . , .) \text{ is continuous.} \quad U(.) \text{ is a Borel measurable multifunction taking values compact sets.}$$
There exist $\varepsilon > 0$, $k(.) \in L^1$ and $c(.) \in L^1$ s.t.
\[
|f(t, x, u) - f(t, x', u)| \leq k(t)|x - x'|,
|f(t, x, u)| \leq c(t)
\]
for all $x, x' \in \tilde{x}(t) + \varepsilon B$, $u \in U(t)$, a.e. $t \in [0, 1]$.

(a): Assume that $g(.)$ is Lipschitz continuous on a neighborhood of $(\tilde{x}(0), \tilde{x}(1))$. Now suppose that $(\tilde{x}(.), \{\tilde{u}^k(.)\}^n_{k=0})$ is a $L^\infty$-local relaxed minimizer. Then there exist an arc $p(.) \in W^{1,1}([0, 1]; \mathbb{R}^n)$ and $\lambda \geq 0$ such that

(i) $\|p\|_{L^\infty} + \lambda \neq 0$,

(ii) $-\frac{\partial}{\partial t} H(t, \tilde{x}(t), p(t), \tilde{u}^k(t)) + \lambda \partial g(\tilde{x}(0), \tilde{x}(1)) + N_C(\tilde{x}(0), \tilde{x}(1))$,

(iii) $(p(0), -p(1)) \in \lambda \partial g(\tilde{x}(0), \tilde{x}(1)) + N_C(\tilde{x}(0), \tilde{x}(1))$,

(iv) For $k = 0, \ldots, n$

$H(t, \tilde{x}(t), p(t), \tilde{u}^k(t)) \geq H(t, \tilde{x}(t), p(t), u)$

for all $u \in U(t)$, a.e. $t \in [S, T]$.

(b): Suppose $\varepsilon > 0$ and $\delta > 0$ can be chosen such that

\[g(x(0), x(1)) \geq g(\tilde{x}(0), \tilde{x}(1)) + \delta\]

for all feasible processes $(x(., u(.)))$ such that $\|x(.-) - \tilde{x}(.)\|_{L^\infty} \leq \varepsilon$. Then conditions (i)-(iv) above are satisfied for some choice of multipliers $p(.) \in W^{1,1}$ and $\lambda \geq 0$ such that $\lambda = 0$.

The proof of Theorem 4.1 will be given in a forthcoming paper.

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References


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