Synchronization of Delayed Dynamical Networks with Switching Topologies

Tao Liu and Ming Cao

Abstract—To reduce the conservativeness of switching signals introduced by a common average dwell-time condition, this paper uses the mode-dependent average dwell-time method to study the global exponential synchronization problem of a class of dynamical networks with switching topologies as well as time-varying coupling delays. First, we extend the mode-dependent average dwell-time method into the stability analysis of switched linear systems with time-varying delays. Then, we apply the obtained results to studying the synchronization problem of a particular network whose nodes have Lur’e type dynamics. A new delay-dependent sufficient condition is established in terms of linear matrix inequalities (LMIs) that guarantees the solvability of the problem, and a class of synchronizing switching signals, in which each subnetwork has its own average dwell-time scheme, is identified. Finally, a numerical example is given to show the effectiveness of the proposed results.

I. INTRODUCTION

Due to its extensive applications in many different contexts, synchronization of dynamical networks as well as its related topics such as consensus of multi-agent systems has received a great deal of attention in the past decade [1], [2], [3]. Synchronization of networks with different types of topologies has been studied from different points of view, which has achieved fruitful results and produced various synchronization criteria in the literature.

As a general phenomenon in many real-world networks, switching topologies that are often caused by link failures or creations have attracted increasing attention recently. For example, in a communication network of mobile agents where each agent needs to communicate with its neighbors in order to achieve synchronization, existing links may fail and new links between nearby agents may be created while the agents are moving [3]. These switching topologies have great effects on synchronization of such a switched network, and therefore, research emphasis has been placed on uncovering properties with which synchronization of a switched network can be achieved.

For a switched network that none of its subnetwork is synchronizable, if the switching is “fast” enough with a fixed period such that a related static time-averaging network is synchronizable, then local synchronization can be guaranteed [4], [5]. Moreover, if the outer coupling matrices of the network are simultaneously triangularizable, then synchronization can also be achieved via the design of switching within a pre-given collection of subnetworks [6]. In the case that all subnetworks are synchronizable, global asymptotic synchronization under arbitrary switching can be guaranteed if all subnetworks share a common Lyapunov function [6]. However, for the latter case, many networks don’t have a common Lyapunov function and cannot achieve synchronization under arbitrary switching. For these networks, synchronization is still achievable under the so-called “slow” switching identified by the average dwell-time condition [7].

On the other hand, time delay is also a ubiquitous phenomenon in networks [8]. It may degrade the synchronization performance, and may even destroy the synchronizability of the network. In view of this, the average dwell-time method was applied to studying synchronization of switched networks with a constant coupling delay under the assumption that the outer coupling matrices are simultaneously diagonalizable in [9]. This method was further extended to networks with time-varying coupling delays in [10], where the simultaneously diagonalizable condition was removed.

In switched system theory [11], the average dwell-time method [12], [13] has been extensively used to achieve stability of switched systems, where a common average dwell-time condition for each subsystem is used to specify the stabilizing switching signals. This common average dwell-time may introduce conservativeness as different characteristics of each subsystem has not been considered during the identifying process. To reduce the conservativeness, the mode-dependent average dwell-time method was introduced in [14], where different average dwell-time conditions are applied to different subsystems by taking the individual convergence rate of each subsystem into account. However, the obtained results are only applicable to switched systems without time delays, and cannot be applied to those with time delays directly, and hence cannot be used to study synchronization of switched networks with coupling delays, neither. Therefore, in this paper, we will first extend the mode-dependent average dwell-time method into the stability analysis of switched linear systems with time-varying delays. Then, we will utilize the proposed result to investigate global exponential synchronization of a class of switched networks with time-varying coupling delays.

The rest of this paper is organized as follows. In Section II, the switched network model with time-varying delay is introduced. Section III studies the exponential stability of a class of switched linear systems with time-varying delays by using the mode-dependent average dwell-time method. Then, the obtained result is applied to the synchronization analysis of the switched network in Section IV. Section
V takes a network with 10 modified Chua’s circuits as an example to show the effectiveness of the proposed result. Some conclusions are given in Section VI.

II. NETWORK MODEL AND PRELIMINARIES

Consider a class of switched dynamical networks with time-varying coupling delays. The network consists of \( N \) linearly and diffusively coupled identical nodes with the Lur’e type dynamics. The model of the network is given as follows:

\[
\begin{align*}
\dot{x}_i(t) &= A_k x_i(t) + B f(x_i(t)) + \sum_{j=1}^{N} c_{ij}^k \Gamma_{\sigma(t)} x_j(t - \tau(t)) \\
 x_{i0} &= \phi(\theta), \quad \theta \in [-h, 0], \quad i = 1, 2, \ldots, N.
\end{align*}
\]

Here, \( x_i \in \mathbb{R}^n \) is the state variable of node \( i \), \( \tau(t) \geq 0 \) is time-varying delay, and \( h > 0 \) is a known upper bound on \( \tau(t) \). For \( \theta \in [-h, 0) \), \( x_{i0} = x_i(t + \theta) \) is the initial condition of node \( i \), and \( \phi(\theta) \) is a continuously differentiable function. \( A \in \mathbb{R}^{n \times n} \) and \( B \in \mathbb{R}^{n \times n} \) are two constant matrices, and \( f(\cdot) : \mathbb{R}^n \to \mathbb{R}^n \) is a continuous function. \( A \) and \( B \) together define the dynamics of each node. \( \sigma(t) : [0, \infty) \to \mathcal{M} = \{1, 2, \ldots, m\} \) with \( m \geq 2 \) is a piecewise right continuous function representing the switching signal.

The matrices \( \Gamma_{\sigma(t)} \in \mathbb{R}^{n \times n} \) and \( C_{\sigma(t)} = (c_{ij}^{\sigma(t)}) \in \mathbb{R}^{n \times N} \) are the inner coupling matrix and outer coupling matrix of the network, where \( \Gamma_{\sigma(t)} \) represents how two connected nodes interact with each other and \( C_{\sigma(t)} \) denotes the network topology. We refer to the network in (1) with a fixed \( \sigma(t) = k \in \mathcal{M} \) as the \( k \)-th subnetwork of switched network (1), and assume that \( \Gamma_k \) and \( C_k \) only take values from a pre-given finite set. Therefore, network (1) can be considered as the one that is orchestrated by a switching signal \( \sigma(t) \) between \( m \) subnetworks which have different topologies represented by \( C_k \) and \( \Gamma_k \).

In this paper, we are only interested in undirected networks, i.e., for each \( k \in \mathcal{M} \), \( C_k \) is symmetric, and if there is a connection between nodes \( i \) and \( j \), then \( c_{ij}^k = c_{ji}^k > 0 \), otherwise, \( c_{ij}^k = c_{ji}^k = 0 \). The diagonal entries of \( C_k \) satisfy

\[
c_{ii}^k = - \sum_{i=1, i \neq i}^{N} c_{ij}^k = - \sum_{i=1, i \neq i}^{N} c_{ji}^k.
\]

We suppose that each subnetwork in network (1) is connected, i.e., for each \( k \in \mathcal{M} \), the matrix \( C_k \) is irreducible.

The switching signal \( \sigma(t) \) decides when and which subnetwork is activated during the running time of the network. Therefore, we have the switching sequence

\[
\{x_{i0}, (k_0, t_0), \ldots, (k_r, t_r), \ldots \}, \quad k_r \in \mathcal{M}, \quad r = 0, 1, \ldots
\]

which means \( \sigma(t) = k_r \) when \( t \in [t_r, t_{r+1}) \), i.e., the \( k_r \)-th subnetwork is activated when \( t \in [t_r, t_{r+1}) \).

The purpose of the paper is to find conditions that guarantee the global exponential synchronization of the state of network (1) on the synchronization manifold defined below.

**Definition 1 (155):** Define synchronization manifold \( \mathcal{S} = \{ x^T, x_2^T, \ldots, x_N^T \} \in \mathbb{R}^{nN} : x_i = x_j \quad i, j = 1, 2, \ldots, N \}.

Apparently, the synchronization of network (1) is determined by 5 factors: 1) the node dynamics \( A \) and \( B f(\cdot) \), 2) the inner coupling matrices \( \Gamma_k \), 3) the outer coupling matrices \( C_k \), 4) the time-varying coupling delay \( \tau(t) \) and 5) the switching signal \( \sigma(t) \). It turns out to be a hard problem if no restrictions are used to specify particular properties of these factors. Thus, to make the problem tractable, we use the following two assumptions to characterize the time-varying delay \( \tau(t) \) and the node dynamics \( f(\cdot) \).

**Assumption 1 (113):** \( \tau(t) \) is a differentiable function satisfying \( 0 \leq \tau(t) \leq h \) and \( \tau(t) \leq d < 1 \) for some constant \( d \).

**Assumption 2 (116):** There exists a square matrix \( L \in \mathbb{R}^{n \times n} \), such that

\[
(f(x) - f(y))^T(f(x) - f(y)) \leq (x - y)^T L(x - y).
\]

Also, we will only pay attention to a certain class of switching signals that satisfy an average dwell-time condition. In particular, we adopt the concept of mode-dependent average dwell-time whose definition is given as follows.

**Definition 2 (114):** For a switching signal \( \sigma(t) \) and any \( T > t_0 \geq 0 \), let \( N_{\sigma_k}(T, t) \) be the switching numbers that the \( k \)-th subsystem is activated over the interval \([t, T)\) and \( T_k(T, t) \) denote the total running time of the \( k \)-th subsystem over \([t, T)\), \( k \in \mathcal{M} \). We say that \( \sigma(t) \) has a mode-dependent average dwell time \( T_{\sigma_k} \) if there exist \( N_{\sigma_k} > 0 \) such that

\[
N_{\sigma_k}(T, t) \leq N_{\sigma_k} + \frac{T_k(T, t)}{T_{\sigma_k}}, \quad \forall T > t_0 \geq 0.
\]

**Remark 1:** In order to use the individual characteristics of each subnetwork sufficiently and hence may reduce the conservativeness introduced by a common average dwell-time [12], [13], here we assign every subsystem an individual average dwell-time condition (5). Moreover, condition (5) will reduce to the traditional average dwell-time condition proposed in [12] by dropping the subscript \( k \).

To deduce the main results of the paper, we also need the Schur complement lemma which is given below.

**Lemma 1 (117):** The linear matrix inequality

\[
\begin{pmatrix}
Q & S \\
* & R
\end{pmatrix} > 0
\]

with \( Q = Q^T \), \( R = R^T \) and * referring to the transpose of the corresponding block in a symmetric matrix, is equivalent to

\[
R > 0, \quad Q - SR^{-1}S^T > 0.
\]

III. STABILITY OF SWITCHED LINEAR SYSTEMS WITH TIME-VARYING DELAYS

Before studying synchronization of network (1), in this section we extend the mode-dependent average method proposed in [14] to the exponential stability analysis of a delayed switched linear system in the form of

\[
\begin{align*}
\dot{x}(t) &= Ax(t) + \Gamma \sigma(t)x(t - \tau(t)) \\
x_{i0} &= \phi(\theta), \quad \theta \in [-h, 0].
\end{align*}
\]

where \( x \in \mathbb{R}^n \) is the state of the system, \( \tau(t) \), \( x_{i0} \), \( \phi(\cdot) \) and \( \sigma(t) \) are defined the same as in Section II. For each \( k \in \mathcal{M} \), \( A_k \in \mathbb{R}^{n \times n} \) and \( \Gamma_k \in \mathbb{R}^{n \times n} \) are constant matrices.
For switched system (6), we have the following result based on the mode-dependent average dwell-time method.

**Theorem 1:** Suppose Assumption 1 holds. If for given constants \( \alpha_k > 0 \) \( \forall k \in \mathcal{M} \), there exist matrices \( P_k > 0 \), \( Q_k > 0 \), \( Z_k > 0 \), \( X_{11}^k \), \( X_{12}^k \), \( X_{22}^k \), \( Y_k \) and \( T_k \) with appropriate dimensions such that
\[
\Phi_k = \begin{pmatrix}
\Phi_{11}^k & \Phi_{12}^k \\
\Phi_{21}^k & \Phi_{22}^k
\end{pmatrix} h_k Z_k < 0,
\]
then switched system (6) is exponentially stable under switching signal \( \sigma(t) \) satisfying the mode-dependent average dwell-time condition
\[
T_{a_k} > T_{a_k} = \frac{\ln \mu_k}{\alpha_k},
\]
where \( \Phi_{11}^k = A_k^\top P_k + P_k A_k + Y_k + Y_k^\top + Q_k + h_k X_{11}^k + \alpha_k P_k \), \( \Phi_{12}^k = P_k \Gamma_k - Z_k + T_k + h_k X_{12}^k \), \( \Phi_{22}^k = -T_k - T_k - (1 - d)e^{-\alpha_k h_k} Q_k + h_k X_{22}^k \) and \( \mu_k \geq 1 \) satisfying
\[
P_k \leq \mu_k P_l, \quad Q_k \leq \mu_l Q_l, \quad Z_k \leq \mu_l Z_l, \quad \forall k, l \in \mathcal{M}, k \neq l.
\]

**Proof:** Select the following piecewise Lyapunov functional candidate
\[
V_{\sigma(t)}(t) = V_{\sigma(t)} + V_{\sigma(t)} + V_{\sigma(t)},
\]
where
\[
V_{\sigma(t)}(t) = x^\top(t) P(x(t)),
\]
\[
V_{\sigma(t)}(t) = \int_{t-\tau(t)}^t e^{a_{\sigma(t)}}(t) x^\top(\beta) Q_{\sigma(t)} x(\beta) d\beta,
\]
\[
V_{\sigma(t)}(t) = \int_{t-\tau(t)}^t e^{a_{\sigma(t)}}(t) x^\top(\beta) Z_{\sigma(t)} x(\beta) d\beta.
\]
Since, for \( t \in [t_r, t_{r+1}) \), \( r = 0, 1, \ldots \), \( \sigma(t) = \sigma(t_r) \) is a constant, let \( \sigma(t) = \kappa, \forall t \in [t_r, t_{r+1}) \) with a given \( r \) in the sequel for simplicity. Then the derivative of \( V_{\sigma(t)}(t) \) along system (6) is
\[
\dot{V}_{1k} = 2x^\top(t) P_k (A_k x(t) + \Gamma_k x(t - \tau(t))),
\]
\[
\dot{V}_{2k} \leq \alpha_k \int_{t-\tau(t)}^t e^{a_k(\beta-t)} x^\top(\beta) Q_k x(\beta) d\beta + x^\top(t) Q_k x(t) - (1 - d)e^{-\alpha_k h_k} x^\top(t - \tau(t)) Q_k x(t - \tau(t)),
\]
\[
\dot{V}_{3k} \leq \alpha_k \int_{t-\tau(t)}^t e^{a_k(\beta-t)} x^\top(\beta) Z_k x(\beta) d\beta + h_k x^\top(t) Z_k x(t) - \int_{t-\tau(t)}^t e^{-\alpha_k h_k} x^\top(\beta) Z_k x(\beta) d\beta.
\]
\[
2\left( x^\top(t) Y_k + x^\top(t - \tau(t)) T_k \right) \times \left( x(t) - \int_{t-\tau(t)}^t \dot{x}(\beta) d\beta - x(t - \tau(t)) \right) = 0.
\]
Let \( \xi(t) = (x^\top(t), x^\top(t - \tau(t)))\top \), one has
\[
h_k \xi^\top(t) X_k \xi(t) - \int_{t-\tau(t)}^t \xi^\top(\beta) X_k \xi(\beta) d\beta \geq 0.
\]
Thus, from (13)-(17), the following inequality holds.
\[
\dot{V}_{1k} + \alpha_k V_k(t) \leq h_k \xi^\top(t) X_k \xi(t) - \int_{t-\tau(t)}^t \eta^\top(\tau(t), \beta) \eta(\tau(t), \beta) d\beta,
\]
where \( \eta(\tau(t), \beta) = (x^\top(t), x^\top(t - \tau(t)), \dot{x}^\top(\beta))^\top \), and
\[
\Xi = \begin{pmatrix}
\Phi_{11}^k + h_k X_{11} A_k X_k \\
\Phi_{12}^k + h_k X_{12} Z_k \\
\Phi_{22}^k + h_k X_{22} \Gamma_k
\end{pmatrix} \begin{pmatrix}
\Phi_{11}^k + h_k X_{11} Z_k \\
\Phi_{12}^k + h_k X_{12} \\
\Phi_{22}^k + h_k X_{22}
\end{pmatrix}.
\]
Lemma 1 and (8) guarantee that \( \Xi < 0 \), which leads to
\[
\dot{V}_{1k} \leq -\alpha_k V_k(t).
\]
Integrating both sides of inequality (18) from \( t_r \) to \( t \) gives
\[
V_{1k} \leq e^{-\alpha_k(t-t_r)} V_{1k}(t_r).
\]
Thus, for each \( r = 0, 1, \ldots \) and \( t \in [t_r, t_{r+1}) \), one can obtain
\[
\dot{V}_{\sigma(t)}(t) = \dot{V}_{\sigma(t)}(t) \leq e^{-\alpha_{\sigma(t)}(t-t_r)} V_{\sigma(t)}(t_r).
\]
Based on (11) and (12), the following condition holds at each switching time instant \( t_r, r = 1, 2, \ldots \),
\[
\dot{V}_{\sigma(t)}(t_r) \leq \mu_{\sigma(t)}(t) V_{\sigma(t)}(t_r) = \mu_{\sigma(t)}(t) V_{\sigma(t)}(t_r).
\]
Combining (20) with (21) gives
\[
V_{\sigma(t)}(t) \leq \mu_{\sigma(t)}(t) e^{-\alpha_{\sigma(t)}(t-t_r)} V_{\sigma(t)}(t_r).
\]
For the time interval \([t_0, t]\), let \( N_{\sigma(t)} = N_{\sigma(t)}(t_0) \) be the total number that the \( k \)th subsystem has been activated, \( T_k = T_k(t_0) = \sum_{r=1}^{N_{\sigma(t)}} (t_r - t_{r-1}) \) be the total time that the \( k \)th subsystem has been activated and \( N_\sigma = N_\sigma(t_0) = \sum_{k=1}^{m_0} N_{\sigma(t)}(t_0) \) be the total switching times of switched system (6), where \( \{t_r\} \) is a subsequence of \( \{t_r\} \) denoting the consecutive switching on and off time instances of the \( k \)th subsystem. With these definitions and (21), (22) becomes
\[
\dot{V}_{\sigma(t)}(t) \leq \mu_{\sigma(t)}(t) e^{-\alpha_{\sigma(t)}(t-t_r)} e^{-\alpha_{\sigma(t)}(t-t_{r-1})} V_{\sigma(t)}(t_{r-1})
\]
\[
\leq \mu_{\sigma(t)}(t) \cdots \mu_{\sigma(t)}(t) e^{-\alpha_{\sigma(t)}(t-t_r)} \cdots \alpha_{\sigma(t)}(t_{r-1}) V_{\sigma(t)}(t_0)
\]
\[
= \left( \prod_{p=1}^{N_{\sigma(t)}} \mu_p \right) e^{-\sum_{r=1}^{N_{\sigma(t)}} \alpha_{\sigma(t)}(t_r-t_{r-1})} V_{\sigma(t)}(t_0)
\]
\[
= e^{\sum_{r=1}^{N_{\sigma(t)}} (\alpha_{\sigma(t)} - \alpha_{\sigma(t)})} V_{\sigma(t)}(t_0).
\]
The above inequality can be simplified by using (5) and (10)
\[
\dot{V}_{\sigma(t)}(t) \leq e^{-\sum_{r=1}^{N_{\sigma(t)}} \left( \alpha_{\sigma(t)} - \alpha_{\sigma(t)} \right)} V_{\sigma(t)}(t_0)
\]
\[
= \bar{\mu} e^{-\sum_{r=1}^{N_{\sigma(t)}} \alpha_{\sigma(t)}(t_r-t_{r-1})} V_{\sigma(t)}(t_0),
\]
where \( \bar{\mu} = \prod_{p=1}^{N_{\sigma(t)}} \mu_p \). Let \( \alpha_1 = \min_{\sigma(t)} \left( \alpha_k - \frac{\ln \mu_k}{\alpha_k} \right) \), then, (23) leads to
\[
\dot{V}_{\sigma(t)}(t) \leq \bar{\mu} e^{-\sum_{r=1}^{N_{\sigma(t)}} \alpha_{\sigma(t)}(t_r-t_{r-1})} V_{\sigma(t)}(t_0)
\]
\[
= \bar{\mu} e^{-\alpha_{\sigma(t)}(t-t_r)} V_{\sigma(t)}(t_0).
\]
From equation (12), one has

\[ V_\sigma(t) \leq a\|x(t)\|^2, \quad V_\sigma(t_0) \leq b\|x(t_0)\|^2, \quad \text{(25)} \]

where \( \|x(t)\|_{cl} = \sup_{t_0 \leq \theta \leq 0} \{ \|x(t + \theta)\|, \|\dot{x}(t + \theta)\| \}, \)

\[ a = \min_{k \in \mathcal{M}} \{ \lambda_{\min}(P_k) \}, \quad b = \max_{k \in \mathcal{M}} \{ \lambda_{\max}(P_k) \} + h_{\max_{k \in \mathcal{M}}} \{ \lambda_{\max}(Q_k) \} + h^2/2 \max_{k \in \mathcal{M}} \{ \lambda_{\max}(Z_k) \}. \]

Inequalities (24) and (25) lead to

\[ \|x(t)\|^2 \leq \frac{1}{a} V_\sigma(t) \leq \frac{b}{a} \sum_{i=1}^{N} e_i(t_0). \]

Therefore, switched system (6) with time-varying delay is exponentially stable under any switching signal that satisfies the mode-dependent average dwell time condition (10). \( \blacksquare \)

Remark 2: The LMIs (8) and (9) guarantee that each subsystem in switched system (6) is exponentially stable with different convergence rate \( \alpha_k > 0 \). Traditionally, the average dwell-time method [13] chooses a common convergence rate \( \alpha^* = \min_{k \in \mathcal{M}} \{ \alpha_k \} \) and a common \( \mu \) satisfying (11) to compute the so-called average dwell-time \( T_{\alpha^*} = \frac{\ln \mu}{\ln \alpha^*} \). In Theorem 1, we adopt the mode-dependent average dwell-time method proposed in [14], and for each subsystem, we use its own \( \alpha_k \) and \( \mu_k \) which are decided by the individual dynamical characteristics of the subsystem, and derive its own average dwell-time \( T_{\alpha_k} \). Therefore, we may reduce the conservativeness of the obtained result caused by the common parameters \( \alpha^* \), \( \mu \) and \( T_{\alpha^*} \). Obviously, we have \( \mu = \max_{k \in \mathcal{M}} \{ \mu_k \} \), which implies that switching signals proposed in [13] are special cases of those identified in Theorem 1.

IV. SYNCHRONIZATION OF SWITCHED NETWORKS

With the help of Theorem 1, now we can investigate the synchronization problem of network (1). In order to find synchronization conditions for the network, let’s introduce a reference state \( s(t) \) and define the error vector

\[ e_i(t) = x_i(t) - s(t), \quad i = 1, 2, \ldots, N. \]

Here, we select \( s(t) \) as the average state of all the nodes, i.e.,

\[ s(t) = \frac{1}{N} \sum_{i=1}^{N} x_i(t), \quad \text{and the dynamical equation of } s(t) \text{ is} \]

\[ \dot{s}(t) = \frac{1}{N} \sum_{i=1}^{N} \dot{x}_i(t) = :As(t) + B\tilde{f}(s(t)). \quad \text{(26)} \]

with \( \tilde{f}(s(t)) = \frac{1}{N} \sum_{i=1}^{N} f(x_i(t)) \). By the definition of \( e_i(t) \), we can get an error dynamical system for network (1)

\[ \dot{e}_i(t) = :Ae_i(t) + B(f(x_i(t)) - f(s(t))) + B(f(s(t))) \]

\[ - \tilde{f}(s(t))) + \sum_{i=1}^{N} \frac{\mathbf{G}}{c_{ij}} \mathbf{C}_i(t) e_j(t - \tau(t)). \quad \text{(27)} \]

Let \( e(t) = (e_1^T(t), e_2^T(t), \ldots, e_N^T(t))^T \). Then, (27) can be rewritten as

\[ \dot{e}(t) = (I_N \otimes A) e(t) + (C_{\sigma(t)} \otimes \Gamma_{\sigma(t)}) e(t - \tau(t)) \]

\[ + (I_N \otimes B) (F + \tilde{F}), \quad \text{(28)} \]

where \( F = F(e,s,t) = \left( (f(x_i(t)) - f(s(t)))^T, \ldots, (f(x_i(t)) - f(s(t)))^T \right)^T \),

\[ \tilde{F} = \tilde{F}(e,s,t) = \left( (f(s(t)) - \tilde{f}(s(t)))^T, \ldots, (f(s(t)) - \tilde{f}(s(t)))^T \right)^T. \]

Similarly, for each subnetwork \( k, k \in \mathcal{M} \), we can derive its error dynamical system

\[ \dot{e}(t) = \tilde{A} e(t) + \tilde{C}_k e(t - \tau(t)) + \tilde{B} (F + \tilde{F}), \quad \text{(29)} \]

where \( \tilde{A} = I_N \otimes A, \tilde{C}_k = C_k \otimes \Gamma_k \) and \( \tilde{B} = I_N \otimes B \). Here \( I_N \) is the \( N \times N \) identity matrix, and \( \otimes \) denotes the Kronecker product of two matrices.

Now, we conclude that if switched system (28), which is composed of subsystems (29), is globally exponentially stable, then the global exponential synchronization of the original switched network is guaranteed.

Since for each \( k \in \mathcal{M}, C_k = C_k^T \) is a real symmetric matrix, there exists a unitary matrix \( U_k = (u_{k1}, u_{k2}, \ldots, u_{kN}) \in \mathbb{R}^{N \times N} \) with \( u_{ki} = (u_{k1i}, u_{k2i}, \ldots, u_{kNi})^T \in \mathbb{R}^N \) such that \( U_k^T C_k U_k = \Lambda_k \), where \( U_k^T U_k = I_N, \) \( \Lambda_k = \text{diag}(\lambda_{k1}, \lambda_{k2}, \ldots, \lambda_{kN}) \), \( \lambda_{ki} \), \( i = 1, 2, \ldots, N \) are the eigenvalues of \( C_k \). Moreover, the zero row sum condition (2) and the irreducible assumption for each \( C_k \) ensure that \( \lambda_{k1} = 0 \) is a common eigenvalue of all the \( C_k \) with multiplicity 1 and an associated eigenvector \( u = \frac{1}{\sqrt{N}}(1, 1, \ldots, 1)^T \), and all the other eigenvalues are less than 0, i.e., \( 0 = \lambda_{k1} > \lambda_{k2} \geq \cdots \geq \lambda_{kN} \).

Let \( \bar{U}_k = U_k \otimes I_n \). By using the property \( \bar{U}_k \bar{B} = \tilde{B} \bar{U}_k = U_k \otimes B \) and the unitary transform \( y_k(t) = \bar{U}_k^T e(t) = (y_{k1}^T(t), y_{k2}^T(t), \ldots, y_{kN}^T(t))^T \in \mathbb{R}^{Nn} \), we rewrite system (29) as

\[ \dot{y}_k(t) = \tilde{A} y_k(t) + \tilde{C}_k y_k(t - \tau(t)) + \tilde{B} (G_k + \tilde{G}_k) \quad \text{(30)} \]

where \( \tilde{A} = A_k \otimes \Gamma_k, \) \( \tilde{G}_k = G_k S_k, S_k = I_k F \) and \( \tilde{G}_k = G_k S_k \). As \( \tilde{A} \) and \( \tilde{A}_k \) are block diagonal matrices, the system (30) is equivalent to

\[ \dot{y}_k(t) = A_k y_k(t) + \lambda_{k1} \tilde{G}_k y_k(t - \tau(t)) + (G_k + \tilde{G}_k) \quad \text{(31)} \]

where \( G_k = (u_{ki}^T \otimes I_n)F, \tilde{G}_k = (u_{ki}^T \otimes I_n)F, \) \( i = 1, 2, \ldots, N \).

With the particular choice of \( u_{ki} = u_i \), we can verify that

\[ y_k(t) = (u_{ki}^T \otimes I_n) e(t) = \frac{1}{\sqrt{N}} \sum_{i=1}^{N} e_i(t) = 0. \]

Based on (28), (30) and (31), we can get the main result of the paper which is addressed in the following theorem.

Theorem 2: Suppose Assumption 1 and 2 hold. If for given constants \( h > 0, \alpha_k > 0, \forall k \in \mathcal{M}, \) there exist matrices \( P_{ki} > 0, Q_{ki} > 0, Z_{ki} > 0, X_{k1}^{1}, X_{k1}^{2}, X_{k2}^{1}, X_{k2}^{2}, Y_k \) and \( T_k \) with appropriate dimensions such that

\[ X_{ki} = \begin{pmatrix} X_{k1}^{1} & X_{k1}^{2} \\ X_{k2}^{1} & X_{k2}^{2} \end{pmatrix} \geq 0, \quad \text{(32)} \]

\[ \Phi_{ki} = \begin{pmatrix} \Phi_{k1}^{11} & \Phi_{k1}^{12} & P_{k1} & B & h \lambda_{ki}^2 Z_{ki}^T & 0 \\ \Phi_{k2}^{11} & 0 & h \lambda_{ki} Z_{ki}^T & h \lambda_{ki}^2 Z_{ki}^T & -\varepsilon_{ki} & h B \lambda_{ki} Z_{ki}^T \end{pmatrix} < 0, \quad \text{(33)} \]
for all $i = 2, 3, \ldots, N$ and $k \in M$, then network (1) can achieve global exponential synchronization under any switching signal that satisfies the mode-dependent average dwell-time condition

$$T_{ak} > T_{ak}^* = \frac{\ln \mu_k}{\alpha_k},$$

(35)

where $\phi_{kk}^i = P_{kk}A + A^T P_{kk} + Y_{kk}^i + Q_{kk} + hX_{kk}^i + \epsilon \Lambda L + \alpha_k P_{kk}, \quad \phi_{kk}^{12} = \lambda_k P_{kk} \Gamma_k - Y_{kk}^i + T_{kk} + hX_{kk}^i, \quad \phi_{kk}^{22} = -T_{kk} - T_{kk}^* - (1 - d) e^{-\alpha_k h} Q_{kk}, \quad \lambda_k$ are the nonzero eigenvalues of $C_k$, and $\mu_k > 1$ satisfying

$$\tilde{P}_k \leq \mu_k \tilde{P}_k, \quad \tilde{Q}_k \leq \mu_k \tilde{Q}_k, \quad \tilde{Z}_k \leq \mu_k \tilde{Z}_k, \quad \forall k, l \in M$$

with $\tilde{P}_k = \tilde{P}_k \Sigma_k \tilde{P}_k^T, \quad \tilde{Q}_k = \tilde{Q}_k \Sigma_k \tilde{Q}_k^T$ and $\tilde{Z}_k = \tilde{Z}_k \Sigma_k \tilde{Z}_k^T, \quad \tilde{P}_k = \text{diag}\{P_{k1}, P_{k2}, \ldots, P_{kN}\}, \quad \tilde{Q}_k = \text{diag}\{Q_{k1}, Q_{k2}, \ldots, Q_{kN}\}$ and $\tilde{Z}_k = \text{diag}\{Z_{k1}, Z_{k2}, \ldots, Z_{kN}\}$. $P_{kk} \in \mathbb{R}^{n \times n}$ and $Q_{kk} \in \mathbb{R}^{n \times n}$ are two arbitrary positive definite matrices, and $Z_{kk} \in \mathbb{R}^{n \times n}$ is a zero matrix.

Proof: Select the same piecewise Lyapunov function as in (12) with $P_{\sigma(t)} = \tilde{P}_{\sigma(t)}, \quad Q_{\sigma(t)} = \tilde{Q}_{\sigma(t)}$ and $Z_{\sigma(t)} = \tilde{Z}_{\sigma(t)}$.

Suppose $\forall t \in [t_r, t_{r+1}), \quad \sigma(t) = \sigma(t) = k$. With the equivalence of system (29) and (30), one can calculate the derivative of $V_{\sigma(t)}(t)$ along switched error dynamical system (28) and get

$$\bar{V}_{kk}(t) = y_{kk}^T(t)(A^T \tilde{P}_k + \tilde{P}_k A) y_{kk}(t) + 2y_{kk}^T(t) \tilde{P}_k \Lambda_k \times y_{kk}(t - \tau(t)) + G_k,$n

(36)

$$V_{2k} \leq -\alpha V_{2k} + y_{kk}^T(t) \tilde{Q}_k y_{kk}(t) \times y_{kk}(t - \tau(t)) \tilde{Q}_k y_{kk}(t - \tau(t)),$n

(37)

$$V_{3k} \leq -\alpha V_{3k} + h (\tilde{A} y_{kk}(t) + \tilde{\Lambda}_k y_{kk}(t - \tau(t)) + G_k)^T \tilde{Z}_k (\tilde{A} y_{kk}(t) + \tilde{\Lambda}_k y_{kk}(t - \tau(t)) + G_k) - \int_{t-r(t)}^t e^{-\alpha h \tilde{y}_{kk}^T(\beta)} \tilde{Z}_k \tilde{y}_{kk}(\beta) d\beta,$n

(38)

where we use the properties that $y_{kk}^T(t) \tilde{P}_k \tilde{B}_k \tilde{G}_k = 0, \quad y_{kk}^T(t - d(t)) A \tilde{Z}_k \tilde{B}_k \tilde{G}_k = y_{kk}^T(t - d(t)) \tilde{A} \tilde{Z}_k \tilde{B}_k \tilde{G}_k = \tilde{G}_k^T \tilde{B}_k^T \tilde{Z}_k \tilde{B}_k \tilde{G}_k = 0$, which are implied by the fact $y_{kk} = 0, \quad Z_{kk} = 0$ as well as the properties of the unitary matrix $U_k$.

From (4), we have $G_k^T \tilde{G}_k = F^T \tilde{F} \leq e^T(t) \tilde{L}^T \tilde{L} e(t) = y_{kk}^T(t) \tilde{L}^T \tilde{L} y_{kk}(t)$. By the S-Procedure [17], we get

$$V_k \leq V_{1k} + V_{2k} + V_{3k} - e \left( G_k^T \tilde{G}_k - y_{kk}^T(t) \tilde{L}^T \tilde{L} y_{kk}(t) \right).$$

(39)

With the same procedure as in Theorem 1 and $y_{kk} = 0, Z_{kk} = 0$, we have

$$V_{\sigma(t)}(t) + \alpha_{\sigma(t)} V_{\sigma(t)}(t) \leq \sum_{i=2}^{N} \left( \tilde{Z}_{ki}^T \tilde{Z}_{ki} \tilde{E}_{ki} \tilde{E}_{ki}(t) - \int_{t-r(t)}^t \tilde{E}_{ki}^T(t, \beta) \Theta_{ki} \tilde{E}_{ki}(t, \beta) d\beta \right)$$

(40)

for all $t \in [t_r, t_{r+1}), \quad r = 0, 1, \ldots, \quad \bar{E}_{ki}(t) = (y_{kk}^T(t), y_{kk}^T(t - d(t)), G_k^T, \tilde{E}_{ki}(t), \tilde{E}_{ki}(t - d(t)), \tilde{Y}_{ki}(t, \beta)), \quad \tilde{Z}_{ki} =$$

$$\Phi_{ki}^1 + h \lambda_k^T \tilde{Z}_k A \Phi_{ki}^2 + h \lambda_k^T \tilde{Z}_k \Gamma_k \tilde{P}_k B + h \lambda_k^T \tilde{Z}_k B \tilde{h} \lambda_k \Gamma_k \tilde{Z}_k B.$$
By solving LMIs (32)-(34), we can get $\mu_1 = 2.3$ and $\mu_2 = 2.22$, and hence we have $T_{a1} = 0.4165$ and $T_{a2} = 0.6135$.

Therefore, the network with the given parameters can achieve global exponential synchronization under a class of switching signals that satisfy the mode-dependent average dwell-time condition. Fig. 1 shows synchronization errors of the network under a given switching signal satisfying (35) with $N_{01} = 2$ and $N_{02} = 3$ also shown in Fig. 1.

![Synchronization errors and the switching signal of (1)](image)

**VI. CONCLUSION**

This paper has extended the mode-dependent average dwell-time method to the exponential stability analysis of a switched linear system with time-varying delays. A delay-dependent sufficient condition formulated by LMIs has been explored. By considering the individual convergence rate of each subsystem, a class of switching signals, in which each subsystem has its own mode-dependent average dwell-time condition, has been identified. It has been shown that the switched system is exponentially stable under the specified switching signals. The obtained results have been applied to the study of the synchronization problem of a class of dynamical networks with both switching topology and time-varying coupling delays. A corresponding synchronization criterion has been established under which global exponential synchronization of the network can be guaranteed, and a network with coupled Chua’s circuits has been used to show the effectiveness of the obtained results. Furthermore, this paper only considered the case that all the subnetworks are synchronizable, however, in practice, there are networks whose subnetworks are not all synchronizable. For this more general case, how to identify less conservative synchronizing switching signals by using the individual characteristics of each subnetwork, deserves attention in the future work.

**REFERENCES**


