Interpolation and polynomial fitting in the SPD manifold

L. Machado* and F. Silva Leite*

Abstract—Generalizing to Riemannian manifolds classical methods to approximate data (e.g. averaging, interpolation and regularization) has been a theoretical challenge that has also revealed to be computationally very demanding and often unsatisfactory. One particular manifold that shows up in numerous scientific areas that use tensor analysis, including machine learning, medical imaging, and optimization, is the set of symmetric positive definite (SPD) matrices. In this work, we show that when the SPD matrices are endowed with the Log-Euclidean framework, certain optimization problems, such as interpolation and best fitting polynomial problems, can be solved explicitly. This contrasts with what happens in general non-Euclidean spaces. In the Log-Euclidean framework, the SPD manifold has the structure of a commutative Lie group and when equipped with the Log-Euclidean metric it becomes a flat Riemannian manifold. Explicit expressions for polynomial curves in the SPD manifold are therefore obtained easily, and this enables the complete resolution of the proposed problems.

I. INTRODUCTION

The generalization of interpolating or smoothing splines to non-Euclidean spaces has been required in the past few years mainly due to the astounding development of the mechanical and robotic industries. However, the generalization of the classical methods to generate interpolating or smoothing splines on Euclidean spaces ([11]) is not as straightforward as it might be expected. The main drawback encountered is that, in general, no explicit formulas to the analogues of polynomials in curved spaces are known. These generalizations pose many interesting and challenging mathematical problems, and investigation in this area is far from being exhausted. Since the pioneer work [22] on interpolating cubic splines on non-curved spaces, there has been a growing interest in studying these problems. Along the way, different approaches have been used to attempt overcoming the encountered difficulties. Without being exhaustive we mention the variational approach to splines on manifolds [7], [4]; the geometric approach corresponding to the generalization of the De Casteljau algorithm [23], [6]; the analytic approach [9]; and the more recent technique undertaken in [10], based on rolling and unwrapping.

Contrary to what happens in general, we exhibit here explicit solutions for the interpolation problem and for the best fitting polynomial in the set of the symmetric and positive-definite (SPD) matrices, that we denote by $S^+(n)$. This set of matrices, corresponding to an open and convex half-cone of the subspace of the symmetric matrices, $S(n)$, is therefore a smooth manifold of dimension $\frac{n(n+1)}{2}$. SPD matrices have a very rich structure, possess many interesting features and appear in many engineering applications that use tensor analysis. For instance, data taking values in $S^+(3)$ appear in diffusion tensor imaging (DTI), a modality of magnetic resonance imaging (MRI) that allows visualization of the relative mobility of the water molecules in endogenous tissues [25], [12]. Therefore, there has been an increasing demand for a rigorous framework for dealing with different operations on the SPD manifold such as regularization, interpolation, and averaging of SPD matrices data sets. When equipped with the natural Riemannian metric, [24], some of the referred problems cannot be solved explicitly, and even numerically they may require great computational costs. Such is the case of the geometric average of a finite set of SPD matrices [18], [19], [20]. The difficulties with implementation, together with the importance of such features for practical applications, motivated the development of an alternative framework for the SPD manifold, which appeared in [2]. In this framework, the set of SPD matrices becomes a commutative Lie group, when a logarithmic multiplication compatible with its differential structure is defined. In this case, $S^+(n)$ becomes a flat Riemannian manifold when equipped with the natural bi-invariant Riemannian metric compatible with its Lie group structure. This is the key point to give a closed formula for polynomials in $S^+(n)$. Having the explicit formula for polynomials, the interpolation problem and the best fitting polynomial problem in the Lie group $S^+(n)$ are, therefore, completely solved.

The paper is organized as follows. In section II, we revisit the variational formulation for the interpolation problem in a general Riemannian manifold and present the necessary optimality conditions for this optimization problem. In section III, we gather the main results concerning the Log-Euclidean framework in the SPD manifold, present the explicit formulas for the analogues to polynomial curves and finish with the explicit solution for the interpolation problem stated in section II for this particular manifold. The generalization of the classical least squares problem to the SPD manifold, when equipped with the Log-Euclidean metric, is described in section IV. Special attention is given to the geodesic that best fits a given data set of SPD matrices. We finish the paper with some concluding remarks in section V.

L. Machado is with the Department of Mathematics, University of Trás-os-Montes and Alto Douro, 5000-442 Vila Real, and with the Institute of Systems and Robotics - University of Coimbra, 3030-290 Coimbra, Portugal

F. Silva Leite is with the Department of Mathematics, University of Coimbra, 3001-501 Coimbra, and with the Institute of Systems and Robotics - University of Coimbra, 3030-290 Coimbra, Portugal

*Work developed under FCT project PTDC/EEA-CRO/122812/2010.
II. RIEMANNIAN INTERPOLATING SPLINES

In what follows $M$ denotes an $n$-dimensional Riemannian manifold equipped with the Riemannian metric $\langle \cdot , \cdot \rangle$ and $\nabla$ the Levi-Civita connection on $M$, that is, the unique affine connection in $M$ compatible with the metric $\langle \cdot , \cdot \rangle$. If $t \mapsto W(t)$ is a smooth vector field along a curve $t \mapsto \gamma(t)$ and is induced by a smooth vector field $Y$ in $M$, then the covariant derivative of $W$, denoted by $\frac{dW}{dt}$, is given by $\frac{dW}{dt} = \nabla_{\frac{d}{dt}} Y$. We refer to [5], [17] and [14] for more details about Riemannian geometry.

A. Variational problem

Given a set of distinct points $p_0, \ldots, p_N$ in $M$ and a sequence of distinct instants of time $t_0, \ldots, t_N$, forming a partition of the unit time interval $[0, 1]$, that is, satisfying $0 = t_0 < t_1 < \cdots < t_{N-1} < t_N = 1$, find the critical values of the functional

$$J(\gamma) = \frac{1}{2} \int_0^1 \left( \frac{d^m \gamma}{dt^m} , \frac{d^m \gamma}{dt^m} \right) dt,$$

(1)

over the class $\Gamma$ of all $C^{m-1}$ curves $\gamma : [0, 1] \to M$ satisfying $\gamma |_{[t_i, t_{i+1}]}$ is smooth,

$$\gamma(t_i) = p_i, \quad 0 \leq i \leq N,$$

(2)

and, in addition,

$$\frac{D \gamma}{dt}(t_0) = v_{0j} \quad \text{and} \quad \frac{D \gamma}{dt}(t_N) = v_{Nj},$$

(3)

where $v_{0j}$ and $v_{Nj}$ are fixed vectors for $j = 1, \ldots, m - 1$.

B. Necessary optimality conditions

Theorem 2.1: ([4]) A necessary condition for a curve $\gamma \in \Gamma$ to be an extremal for the functional $J$ is that $\gamma$ is of class $C^{2m-2}$ and, for each $i = 0, \ldots, N - 1$ and each subinterval $[t_i, t_{i+1}]$, it satisfies

$$\frac{d^{2m} \gamma}{dt^{2m}} + \sum_{j=2}^{m} (-1)^j R \left( \frac{d^{2m-j} \gamma}{dt^{2m-j}} , \frac{d^{j-1} \gamma}{dt^{j-1}} \right) \frac{d \gamma}{dt} = 0,$$

(4)

where $R$ is the curvature tensor of the Levi-Civita connection in $M$.

This variational approach provides one possible generalization of interpolating splines from Euclidean spaces to more general Riemannian manifolds. Since interpolating splines are the concatenation of polynomials, we adopt the following definition of a geometric polynomial of degree $2m - 1$ on $M$.

Definition 2.1: A smooth curve $\gamma : I \subset \mathbb{R} \to M$ satisfying the $2m$th order differential equation (4) is a geometric polynomial of degree $2m - 1$ on the manifold $M$.

C. Particular case when $m = 2$

When we consider the case when $m = 2$ in the functional $J$ defined in (1), we obtain the particular case of the geometric cubic splines that appear in [22] and in [7]. In this particular case, we pretend to minimize the squared norm of the acceleration but only that component which is tangent to the manifold, that is, $\frac{d^2 \gamma}{dt^2} = \frac{d}{dt} \left( \frac{d \gamma}{dt} \right)$. The geometric cubic splines are therefore characterized by the fourth order differential equation

$$\frac{D^4 \gamma}{dt^4} + R \left( \frac{D^2 \gamma}{dt^2} , \frac{d \gamma}{dt} \right) \frac{d \gamma}{dt} = 0.$$

(5)

For the particular case when the Riemannian manifold is a connected and compact Lie group endowed with a bi-invariant Riemannian metric, the above differential equation reduces to

$$\ddot{V} + [\dot{V}, V] = 0,$$

(6)

where $V$ denotes the velocity vector field along $\gamma$, that is, $V = \frac{d \gamma}{dt}$ (see [7] for details).

Even for the case of the 3-dimensional rotational group, where the differential equation (6) reduces to

$$\dddot{v} + \dot{v} \times v = 0,$$

with $v \in \mathbb{R}^3$, only in a few cases closed-form expressions for the solutions are available, [21].

III. INTERPOLATING SPLINES ON SPD’S WITH RESPECT TO THE LOG-EUCLIDEAN METRIC

In this section, we will present explicit forms for Riemannian splines for the particular case of the manifold of symmetric and positive definite (SPD) matrices when equipped with the Log-Euclidean metric. This is an interesting issue since as we already have mentioned, explicit forms for Riemannian splines are extremely difficult to obtain.

A. Log-Euclidean metric

Let us denote the subspace of all real $n \times n$ symmetric matrices by $S(n)$ and the open convex half cone of the symmetric and positive definite matrices by $S^+(n)$.

In [2], it has been given to $S^+(n)$ the structure of a Lie group. The key point for this procedure is related to the fact that the exponential map, $\exp : S(n) \to S^+(n)$, which assigns to each $S \in S(n)$ the usual matrix exponential $\exp(S) = e^S$, is a diffeomorphism in $S(n)$. This means that the exponential map has a well defined inverse, the logarithm map, $\log : S^+(n) \to S(n)$. The derivatives of the exponential and the logarithm are given in the following propositions.

Proposition 3.1: ([31]) The differential of the exponential map $\exp : S(n) \to S^+(n)$ at a point $S \in S(n)$ acting on $T \in S(n)$ is given by

$$(d\exp)_S(T) = \int_0^1 \exp(tS) T \exp((1-t)S) \ dt.$$
If we take into account the definition of the adjoint operator,
\[ \text{ad} : \mathfrak{gl}(n, \mathbb{R}) \times \mathfrak{gl}(n, \mathbb{R}) \rightarrow \mathfrak{gl}(n, \mathbb{R}) \]
\[ (X, Y) \mapsto [X, Y] = \text{ad}_X(Y) \]
where \([X, Y] = XY - YX\), and if we use the fact that (see [13])
\[ \exp(X)Y\exp(-X) = \exp(\text{ad}_X(Y)), \]
the expression for the differential of the exponential map (7), can be rewritten as
\[ (d\exp)_T = \left. \frac{\exp u - 1}{u} \right|_{u = \text{ad}_S P} (T) \exp(S), \]
where the operator \(\frac{\exp u - 1}{u}\) denotes the sum of the power series
\[ \sum_{m=0}^{\infty} \frac{u^m}{(m+1)!}. \]

Now, the expression for the differential of the logarithm map can be easily derived from (8), applying the chain rule to the identity \(\exp(\log(P)) = P, P \in S^+(n)\).

**Proposition 3.2:** The differential of the logarithm map \(\log : S^+(n) \rightarrow S(n)\) at a point \(P \in S^+(n)\) acting on \(S \in S(n)\) is given by
\[ (d\log)_P(S) = \left. \frac{u}{\exp u - 1} \right|_{u = \text{ad}_{\log} P} (SP^{-1}). \]

Introducing the so-called logarithm multiplication
\[ P_1 \circ P_2 = \exp(\log P_1 + \log P_2), \]
where \(P_1, P_2 \in S^+(n)\), it has been proved in [2] that \((S^+(n), \circ)\) is an Abelian Lie group isomorphic to the vector space \(S(n)\).

In general Lie groups, the existence of bi-invariant metrics is not guaranteed. However, every Abelian Lie group \(G\) can be endowed with a bi-invariant metric. In fact, given any inner product \(\langle \cdot, \cdot \rangle\) in the Lie algebra \(\mathfrak{g}\) the induced bi-invariant metric in \(G\) is given by
\[ \langle u, v \rangle_g = \langle (dL_{g^{-1}})g(u), (dL_{g^{-1}})g(v) \rangle_e, \]
where \(u, v \in T_e G\), \(e\) denotes the identity in \(G\) and \(L_g\) denotes the left multiplication by \(g\) in \(G\).

In the next result we state the most important facts concerning bi-invariant metrics in the Lie group \((S^+(n), \circ)\).

**Theorem 3.1:** ([2]) Let \(\langle \cdot, \cdot \rangle\) be a bi-invariant metric on \(S^+(n)\). Then, the following properties hold.

1) The geodesics are the left (or right) translated versions of one-parameter subgroups. That is, given any \(P \in S^+(n)\), the geodesics through \(P\) are of the form
\[ t \mapsto \exp(\log P + tS), \]
for \(S \in S(n)\).

2) The Riemannian exponential and logarithm are given by
\[ \exp_P(S) = \exp(\log P + (d\log)_P(S)) \]
\[ \log_P(Q) = (d\exp)_P^{-1}(\log Q - \log P), \]
for all \(P, Q \in S^+(n)\) and \(S \in S(n)\).

3) The bi-invariant metric between two tangent vectors \(S_1, S_2 \in S^+(n)\) at a point \(P \in S^+(n)\) is given by
\[ \langle S_1, S_2 \rangle_P = \langle (d\log)_P(S_1), (d\log)_P(S_2) \rangle_P. \]

4) The distance between two SPD matrices \(P, Q \in S^+(n)\) is given by
\[ d(P, Q) = \|\log(P) - \log(Q)\|_F, \]
where \(\|\cdot\|_F\) is the norm induced by the inner product (11).

5) The map \(\exp : S(n) \rightarrow S^+(n)\) is an isometry.

**Definition 3.1:** The bi-invariant metric in the Lie group \(S^+(n)\) defined by (11) is called Log-Euclidean metric.

Since the exponential map \(\exp : S(n) \rightarrow S^+(n)\) is an isometry and \(S(n)\) is a vector space, the Lie group \(S^+(n)\) is a complete, simply connected and flat manifold (the sectional curvature is zero everywhere in \(S^+(n)\)). Therefore, the curvature tensor \(R\) associated to the metric (11) vanishes identically.

If we define for \(P \in S^+(n)\) and \(\lambda \in \mathbb{R}\), the logarithm scalar multiplication by
\[ \lambda \circ P = \exp(\lambda \log P), \]
then \((S^+(n), \circ, \circ)\) has the structure of a vector space with addition \(\circ\) and scalar multiplication \(\circ\).

By construction, \(\exp : S(n) \rightarrow S^+(n)\) is a linear isomorphism meaning that the vector space structure in \(S(n)\) is transferred to \(S^+(n)\) using the \(\exp\) and \(\log\) mappings.

**B. Explicit interpolating splines**

Since the Abelian Lie group \(S^+(n)\) is a flat manifold, it is possible to obtain an explicit solution for the interpolating spline problem formulated in subsection II-A for this particular Riemannian manifold. In fact, since the curvature tensor \(R\) vanishes, a necessary condition for \(\gamma : [0, 1] \rightarrow S^+(n)\) to be a solution for the minimization problem
\[ \min_{\gamma} \int_0^1 \frac{1}{2} \left( \frac{D^m \gamma}{dt^m} - \frac{D^m \gamma}{dt^m} \right)^2 dt, \]
over the class \(\Gamma\) of curves satisfying (2) and (3), is that \(\gamma\) is of class \(C^{2m-2}\) in the interval \([0, 1]\), it is smooth when restricted to each subinterval \(\gamma \mid_{[t_i, t_{i+1}]}\) for \(i = 0, \ldots, N-1\), it satisfies the differential equation
\[ \frac{D^{2m} \gamma}{dt^{2m}} = 0, \forall t \in [t_i, t_{i+1}], \]
and also
\[ \gamma(t_i) = P_i, \quad 0 \leq i \leq N, \]
and
\[ \frac{D^{j} \gamma}{dt^{j}}(t_0) = S_{0j} \quad \text{and} \quad \frac{D^{j} \gamma}{dt^{j}}(t_N) = S_{Nj}, \]
where \(P_i \in S^+(n)\), and \(S_{0j}\) and \(S_{Nj}\) are fixed vectors in \(S(n)\), for \(i = 0, \ldots, N\) and \(j = 1, \ldots, m-1\).
In order to obtain the explicit solution for the above, we will follow closely the approach given in [4] for connected, compact and Abelian Lie groups.

Let \( \{X_1, \ldots, X_l\} \), with \( l = \frac{n(n+1)}{2} \), be a frame field of left invariant and parallel vector fields along \( \gamma \) on \( S^+(n) \) and let \( v_j(t) \), for \( j = 1, \ldots, l \), be the components of \( \frac{d}{dt} (t) \) with respect to \( \{X_1, \ldots, X_l\} \). Then,

\[
\frac{dy_j}{dt}(t) = \sum_{j=1}^l v_j(t) X_j(\gamma(t)),
\]

and

\[
\frac{d^{2m-1}}{dt^{2m-1}} (t) = \sum_{j=1}^l \frac{d^{2m-1}}{dt^{2m-1}} v_j(t) X_j(\gamma(t)).
\]

It can be proved that the solution of the differential equation (13) is given explicitly by

\[
\gamma(t) = \prod_{j=1}^l \exp[f_j(t) X_j],
\]

where \( f_j(t) = \int v_j(t) dt \) and \( \frac{d^{2m-1}}{dt^{2m-1}} (t) = 0 \), \( \forall t \in [t_1, t_{i,+}] \), for \( j = 1, \ldots, l \) and \( i = 0, \ldots, N - 1 \).

Therefore, for each \( j = 1, \ldots, l \), \( f_j(t) \) is an Euclidean polynomial of degree \( 2m-1 \), that can be written explicitly as

\[
f_j(t) = a_0^j + a_1^j t + a_2^j t^2 + \cdots + a_{2m-1}^j t^{2m-1}.
\]

Since the Lie algebra \( S(n) \) is Abelian, the explicit expression for \( \gamma \) is given by

\[
\gamma(t) = \prod_{j=1}^l \exp[f_j(t) X_j] = \prod_{j=1}^l \exp\left[\sum_{k=0}^{2m-1} a_k^j(t) t^k X_j\right]
\]

\[
= \exp\left[\sum_{j=1}^l \sum_{k=0}^{2m-1} a_k^j(t) t^k X_j\right]
\]

\[
= \exp\left[\sum_{k=0}^{2m-1} \left(\sum_{j=1}^l a_k^j(t) X_j\right) t^k\right]
\]

\[
= \exp\left[\sum_{k=0}^{2m-1} S_k t^k\right],
\]

where \( S_k \in S(n) \), for \( k = 0, \ldots, 2m - 1 \).

Definition 3.2: A geometric polynomial of degree \( 2m - 1 \) on the Lie group \( S^+(n, \circ) \), is a smooth curve \( \gamma: I \subset \mathbb{R} \to S^+(n) \) given explicitly by

\[
\gamma(t) = \exp(S_0 + S_1 t + S_2 t^2 + \cdots + S_{2m-1} t^{2m-1}),
\]

where \( S_0, \ldots, S_{2m-1} \in S(n) \).

IV. BEST FITTING POLYNOMIALS IN SPD’S

In this section, we present the generalization of the classical least squares problem to the Riemannian manifold \( S^+(n) \) equipped with the Log-Euclidean metric. For the most part of Riemannian manifolds, the generalization of the classical approach is not straightforward since, in general, no explicit formulas to the analogue of polynomials are known.

To overcome this difficulty, a variational problem depending on a smoothing parameter has been proposed in [15] and in [16], and the corresponding Euler-Lagrange equations have been derived. However, in general, finding solutions becomes a very difficult task.

For the case treated here, of the Lie group \( S^+(n) \), since explicit formulas to the analogue of polynomials are known, the problem of finding a polynomial curve that best fits a given set of points in \( S^+(n) \) can be completely solved.

A. Problem’s statement

Given a collection of distinct points in \( S^+(n) \),

\[
P_0, \ldots, P_N,
\]

and a sequence of instants of time

\[
0 = t_0 < t_1 < \cdots < t_{N-1} < t_N = 1,
\]

find a polynomial curve in \( S^+(n) \), of degree \( m \) \( (m \leq N) \),

\[
t \mapsto \gamma(t) = \exp(S_0 + S_1 t + S_2 t^2 + \cdots + S_m t^m),
\]

that yields the minimum value for the functional

\[
E(\gamma) = \frac{1}{2} \sum_{i=0}^{N} d^2(P_i, \gamma(t_i)),
\]

being \( d \) the distance function defined by (12).

B. Normal equations

In order to find the necessary optimality conditions for the problem stated in the previous subsection, notice that finding the minimum for the functional \( E \), defined by (17), is equivalent to find the minimum for the function

\[
F: S(n) \times \cdots \times S(n) \to \mathbb{R}
\]

\[
(S_0, \ldots, S_m) \mapsto \frac{1}{2} \sum_{i=0}^{N} \|\log P_i - \log(\gamma(t_i))\|^2
\]

Now, in order that \( (S_0, \ldots, S_m) \) is a minimizer of \( F \), one should have

\[
(dF)_{(S_0, \ldots, S_m)}(V_0, \ldots, V_m) = 0,
\]

forall \( V_i \in S(n) \) and \( i = 0, \ldots, m \).

But,

\[
(dF)_{(S_0, \ldots, S_m)}(V_0, \ldots, V_m) =
\]

\[
\sum_{i=0}^{N} \langle \log P_i - \log(\gamma(t_i)), -V_0 - V_1 t_1 - V_2 t_1^2 - \cdots - V_m t_1^m \rangle,
\]

and therefore, condition (18) gives the following system of equations

\[
\sum_{i=0}^{N} \log(\gamma(t_i)) = \sum_{i=0}^{N} \log P_i,
\]

\[
\sum_{i=0}^{N} t_i \log(\gamma(t_i)) = \sum_{i=0}^{N} t_i \log P_i,
\]

\[
\sum_{i=0}^{N} t_i^m \log(\gamma(t_i)) = \sum_{i=0}^{N} t_i^m \log P_i
\]
which are the analogue to the normal equations in the Euclidean case. ly to what happens in the Euclidean space, this system of equations can be solved explicitly. To do that, let us introduce the matrix

\[
Y = \begin{bmatrix}
    1 & t_0 & t_0^2 & \cdots & t_0^n \\
    1 & t_1 & t_1^2 & \cdots & t_1^n \\
    \vdots & \vdots & \vdots & \ddots & \vdots \\
    1 & t_N & t_N^2 & \cdots & t_N^n 
\end{bmatrix},
\]

and the block-column matrices

\[
X = \begin{bmatrix}
    S_0 \\
    S_1 \\
    \vdots \\
    S_m 
\end{bmatrix},
\]

and

\[
P = \begin{bmatrix}
    \log P_0 \\
    \log P_1 \\
    \vdots \\
    \log P_N 
\end{bmatrix}.
\]

Then, the linear system of equations (19) is equivalent to the following equation

\[
(Y^\top Y \otimes I_n)X = (Y \otimes I_n)^\top P,
\]

where \( I_n \) represents the identity matrix of order \( n \) and \( \otimes \) denotes the Kronecker product.

The solution can now be given explicitly by

\[
\gamma(t) = \exp(\sum_{i=0}^{N} t_i \log P_i),
\]

where \( \gamma(t) = \exp(S_0 + S_1 t) \). In this case, \( S_0 \) and \( S_1 \) are given explicitly by

\[
S_0 = \frac{\sum_{i=0}^{N} t_i^2 \log P_i - \sum_{i=0}^{N} t_i \sum_{i=0}^{N} t_i \log P_i}{(N + 1) \sum_{i=0}^{N} t_i^2 - \left( \sum_{i=0}^{N} t_i \right)^2},
\]

and

\[
S_1 = \frac{\sum_{i=0}^{N} t_i \log P_i + \sum_{i=0}^{N} t_i \log P_i}{(N + 1) \sum_{i=0}^{N} t_i^2 - \left( \sum_{i=0}^{N} t_i \right)^2}.
\]

**Proposition 4.1:** The geodesic that best fits the given points (15) at the given instants of time (16) passes through the Log-Euclidean mean of those points, that is, \( P = \exp\left(\frac{1}{N+1} \sum_{i=0}^{N} \log P_i\right) \), at the instant of time precisely equal to the arithmetic mean of the times, that is, \( t = \frac{1}{N+1} \sum_{i=0}^{N} t_i \).

**Proof:** From (20), we note that the first equation

\[
\sum_{i=0}^{N} S_0 + t_i S_1 = \sum_{i=0}^{N} \log P_i,
\]

is equivalent to

\[
S_0 + \left(\frac{1}{N+1} \sum_{i=0}^{N} t_i\right) S_1 = \frac{1}{N+1} \sum_{i=0}^{N} \log P_i,
\]

meaning that \( \gamma\left(\frac{1}{N+1} \sum_{i=0}^{N} t_i\right) = \exp\left(\frac{1}{N+1} \sum_{i=0}^{N} \log P_i\right) \).

For the particular case when \( N = 1 \), that is, when we are given two SPD matrices \( P_0 \) and \( P_1 \), and two instants of time \( t_0 = 0 \) and \( t_1 = 1 \), the geodesic that best fits these data is given explicitly by

\[
\gamma(t) = \exp\left(\frac{1}{2} \log P_0 + \frac{1}{2} \log P_1 - \frac{1}{6} \log P_2 - \log P_0\right),
\]

which is the geodesic that joins \( P_0 \) to \( P_1 \) (figure 1).

When we are given three SPD matrices, \( P_0, P_1 \) and \( P_2 \), and the instants of time \( t_0 = 0 \) and \( t_1 = \frac{1}{2} \) and \( t_2 = 1 \), the geodesic that best fits these data is given explicitly by

\[
\gamma(t) = \exp\left(\frac{1}{2} \log P_0 + \frac{1}{3} \log P_1 - \frac{1}{6} \log P_2 - \log P_0\right).
\]

We finish this section by illustrating the solutions of the geodesic fitting problem for the particular case of the SPD manifold with dimension 6, \( S^+(3) \). In this case, the spectral decomposition of a SPD matrix \( P \) is given by \( P = U D U^\top \), where \( D = \text{diag}(\lambda_1, \lambda_2, \lambda_3) \) is a diagonal matrix and \( U \) is an orthogonal matrix whose columns are the eigenvectors \( v_i \) of \( P \). The positive eigenvalues \( \lambda_1, \lambda_2, \lambda_3 \) and the orthogonal matrix \( U \) provide a parametrization for the elements of \( S^+(3) \) which enables to visualize the SPD matrix \( P \) as an ellipsoid whose principal axes are borne by the eigenvectors \( v_i \) of \( P \) and their half lengths are equal to \( (\lambda_i)^{-\frac{1}{2}} \).

**V. CONCLUSION AND FUTURE WORK**

In this paper we have presented the explicit solutions for the interpolation problem and for the analogue to the classical least squares problem in the set of the symmetric and positive-definite matrices \( S^+(n) \), when it is equipped with the Log-Euclidean metric. This occurs in very special cases since explicit formulas to the analogues of polynomials in general manifolds are extremely hard to obtain. In fact, polynomials on Riemannian manifolds have been defined...
in the literature, [4], as being smooth curves that are the solution of a highly non-linear differential equation (4), involving the curvature tensor. In this particular case, the set $S^+(n)$, with the binary operation (10), is a commutative Lie group, which is isomorphic to the vector space of the symmetric matrices $S(n)$ and is therefore a flat Riemannian manifold. This is the key for obtaining the explicit expression for polynomials in the Lie group $S^+(n)$ and therefore to solve the proposed optimization problems. In section IV, we exhibited the normal equations to the generalization of the classical least squares problem. In this case, we are given a finite set of SPD matrices and times and pretend to obtain the polynomial in $S^+(n)$ that best fits the data. We illustrate the solution of the geodesic fitting problem in figures 1 and 2 for the particular case of $S^+(3)$. With the Log-Euclidean framework, the smoothing geometric splines appearing in [16] can also be completely determined for this particular manifold. These geometric splines turn out to be more general than the interpolating splines since, in that case, it is not required that the curve passes exactly through the given points but rather that it goes reasonably close to them while minimizing a certain cost functional.

REFERENCES


