On using disconnected level sets Lyapunov functions in the context of sampled-data systems

Julien Louis, Marc Jungers, Jamal Daafouz, Member, IEEE

Abstract—The main objective of this paper is to give an interpretation of using non-convex and disconnected level sets Lyapunov functions in the stability analysis of discrete time systems obtained by the discretization of a continuous time Lur’e system. For simplicity reasons, Euler discretization scheme is used to illustrate the features of the proposed method. The main result of this paper shows that it is possible to build, for the original continuous time system, a sequence of bounded connected sets that converges to the origin using this type of Lyapunov functions. To this end, sufficient LMI conditions ensuring the stability of the discrete-time model and an upper bound on the error between the sampled state and the continuous trajectory are used to prove the proposed results. An example will be considered to illustrate this questioning.

I. INTRODUCTION

Lur’e systems denote an interconnection between a linear system and a nonlinearity verifying a cone bounded sector condition. The stability of such a system has been widely investigated since the seminal work of Lur’e in the continuous-time framework introducing a Lur’e type Lyapunov function [12] and the work of Jury in the discrete-time framework [8], [16], [18]. More recently, a new class of Lyapunov functions has been introduced [4]–[6] for the discrete-time framework. These Lyapunov Lur’e type functions are composed of a quadratic term of the state and a cross term between the state and the nonlinearity. As underlined in [5], [6], they are a tool to obtain sufficient conditions formulated as LMIs for ensuring global or local stability of discrete-time Lur’e systems. In the case of local stability analysis, the unitary level set is an estimate of the basin of attraction. A crucial property of such a Lyapunov function has been pointed out: its level sets may be non convex and disconnected, which is in accordance with the discrete-time nature of the system.

However, questions arise naturally when considering a sampled-data Lur’e system. Actually, the study of the original continuous time system requires to use connected sets. So what become the disconnected level sets in terms of the original continuous time system? and how to manage them?

The main contribution is here to investigate how to use the disconnected level sets issued from the global stability analysis of sampled-data Lur’e system for global stability analysis of the original continuous time system. The chosen discretization scheme is the Euler one.

This paper does not claim to propose a new groundbreaking global stability condition for sampled-data based on the approximate discrete-time models but is focused on the issue of disconnected level sets and their handling in this framework. Results and discussions on digital controllers and sampled-data systems may be found in the literature. See for instance [1], [7], [9]. One of the main related problems is then to guarantee the stability of the continuous-time nonlinear system from the point of view of the discretized one. Notice more particularly the approach considering sufficient conditions taking into account the existence of a Lipschitzian Lyapunov function for stability analysis and control [11], [13]–[15] and for robust control [17].

The paper is organized as follows: in Section II the continuous-time Lur’e system is presented. Definition and properties on the considered Lyapunov functions, inducing non-convex and disconnected level sets, are also given. Discretization issue is reminded. Our choice to use approximate method to define a discrete-time Lur’e system, is explained. The main issue of this article is formulated. In Section III, the design of a sequence of connected sets is proposed and the consequences for the global stability analysis of a sampled-data Lur’e system using an approximate discrete-time model is presented. In Section IV an academic example illustrates the main result. Concluding remarks are presented in Section V.

Notation. For any vector \( x \in \mathbb{R}^n \), \( x \geq 0 \) means that \( \forall i = \{1 \cdots n\}, n \in \mathbb{N} \), its components \( x(i) \) are nonnegative. \( \|x\| \) is related to the Euclidean norm of vector \( x \). For two matrices, \( A \) and \( B \in \mathbb{R}^{n \times n} \), \( A > B \) means that matrix \( A - B \) is positive definite. \( A' \) denotes the transpose of matrix \( A \). \( I_n \) is the \( n \)-order identity matrix \((n \times p\)-order null matrix\), \( + \) means the symmetric blocks in matrices. \( \text{diag}(A;B) \) is a block diagonal matrix of matrices \( A \) and \( B \).

II. PROBLEM STATEMENT

The following class of continuous-time non-linear systems is considered:

\[
\begin{align*}
\dot{x}(t) &= Ax(t) + B\varphi(y(t)) = f(x(t), \varphi(y(t))), \\
y(t) &= Cx(t),
\end{align*}
\]

(1)

where \( x(t) \in \mathbb{R}^n \) is the state vector and \( y(t) \in \mathbb{R}^p \) is the output vector of the system. Matrices \( A \), \( B \) and \( C \) are real ones of appropriate dimensions.

The nonlinearity \( \varphi: \mathbb{R}^p \rightarrow \mathbb{R}^p \) is assumed to be decentralized and verifies a cone bounded sector condition [10]. By
following a conventional abuse of notations [10], the cone bounded sector condition is denoted by $\varphi(0) = 0$ and $\varphi(\cdot)$ is any positive diagonal matrix. Moreover, $\varphi(\cdot)$ is assumed to be Lipschitz to guarantee the existence and the uniqueness of the solution of the system (1) for an initial condition. The stability issue of a non-linear interconnection is usually known as Lur’e problem in continuous-time.

Since the first contributions on the topic [12], the literature is rich and offers several solutions with quadratic and Lur’e type Lyapunov functions for continuous and discrete-time systems [3], [8], [10], [16], [18]. Recently a new class of Lyapunov functions was proposed to study global and local stability of discrete-time Lur’e systems [4]–[6], via the function:

$$ V : \mathbb{R}^n \times \mathbb{R}^p \to \mathbb{R}, \quad (x; \varphi) \mapsto x^T P x + 2 \varphi' \Delta \varphi \Omega C x, \quad (2) $$

where $P \in \mathbb{R}^{n \times n}$ is a symmetric positive definite matrix and $\Delta \in \mathbb{R}^{p \times p}$ is a diagonal positive semidefinite matrix. The Lyapunov function is then chosen as the function composition $x \mapsto V(x; \varphi)(C x)$. The level set of the Lyapunov function associated with $\gamma$ is defined by $L_V(\gamma) = \{ x \in \mathbb{R}^n | V(x; \varphi(C x)) \leq \gamma \}$.

The Lyapunov function depending on the nonlinearity $\varphi(\cdot)$ implies that its level sets may be disconnected and non-convex. This property can be explained by the fact that the trajectory is a discrete-time sequence of states. Level sets contain all next samples of the trajectory. The contraction of these level sets to the singleton $\{ 0 \}$ illustrates the (local or global) stability of the discrete-time system [5].

In continuous-time, level sets should contain the future part of the trajectory. Jumps between different regions are then forbidden.

Nonetheless, when the discrete-time system is a sampled-data model of a continuous-time system, a natural question arises about the become of these disconnected sets. The global stability analysis of a continuous-time Lur’ e system, based on the sampled-data approximate discrete-time model will be used as a support of our questioning.

In order to present a formalized issue, let us introduce some tools and definitions. Consider the exact plant model:

$$ F^e_T(x_k) = x_k + \int_{kT}^{(k+1)T} (A x(\tau) + B \varphi(C x(\tau))) d\tau, \quad (3) $$

where $T$ is the sampling period, $x_k \in \mathbb{R}^n$ is the state vector of the exact discrete-time system at time $kT$, $k \in \mathbb{N}$. Due to the presence of the nonlinearity $\varphi(\cdot)$, determining the analytical expression of $F^e_T(\cdot)$ is a difficult problem, not to say impossible to solve. In order to handle $F^e_T(\cdot)$, an approximation $F^a_T(\cdot)$ is required, under the constraint that the approximation is good enough in a certain sense. The choice which is done here is the Euler explicit method to define the discrete-time non-linear system as follows:

$$ F^a_T(x_k) = x_k + T f(x_k, \varphi(y_k)) = A_d x_k + B_d \varphi_d(C_d x_k), \quad (4) $$

with $A_d = I_n + TA$, $B_d = B$, $C_d = C$ and $\varphi_d(\cdot) = T \varphi(\cdot)$, where $\varphi_d(\cdot)$ satisfies the same assumption that $\varphi(\cdot)$, with $\varphi_d(\cdot) \in [0, \Omega_d]$ and $\Omega_d = T \Omega$.

Remark 1: Note that other choices may be possible, like $B_d = TB$ and $\varphi_d(\cdot) = \varphi(\cdot)$, without loss of generality.

Remark 2: In Equation (3) and (4), $x_k$ is considered as an initial condition to evaluate $F^e_T(x_k)$ and $F^a_T(x_k)$. It will not be required to define different $x_k$ for the exact and the approximate trajectory, to study the global stability of the system (1).

The stability of the discrete-time model is ensured via LMI conditions recalled in Theorem 1 involving the Lyapunov function (2).

Theorem 1 (See [5]): For the class of systems defined by (4), if there exists a matrix $G \in \mathbb{R}^{n \times n}$, a symmetric positive definite matrix $P \in \mathbb{R}^{n \times n}$, a diagonal positive semidefinite matrix $\Delta \in \mathbb{R}^{p \times p}$ and positive diagonal matrices $U, W \in \mathbb{R}^{p \times p}$, such that the LMI

$$ \mathcal{M} = \begin{bmatrix} P - G' - G & G'A_d & G'B_d & 0_{n \times p} \\ * & -P & \Pi_1 & A_d' \Pi_2 \\ * & * & -2U_c & B_d' \Pi_2 \\ * & * & * & -2W \end{bmatrix} < 0, \quad (5) $$

is verified, with $\Pi_1 = C_d' \Omega_d (U_c - \Delta)$ and $\Pi_2 = C_d' \Omega_d (\Delta + W)$, then the origin of the discrete-time system (4) is globally asymptotically stable.

To cope with global stability analysis of non-linear sampled-data systems, the approach proposed in [15] and developed in [11], [14] is reminded in the following proposition.

Proposition 1 (See [15]): Consider a sampled-data system (3) and the sampling period $T > 0$, and $x_k = x(kT) \in \mathbb{R}^n$, $k \in \mathbb{N}$.

- If $\exists \tilde{\beta} \in \mathcal{X}_T$ such that the trajectories of the exact discrete-time system (3) satisfy:

$$ \|x_k\| \leq \tilde{\beta}(\|x_0\|, kT), \quad (6) $$

- And if $\exists \kappa \in \mathcal{X}_T$ such that the solution of the continuous-time system (1) satisfies:

$$ \|x(t) - x_k\| \leq \kappa(\|x_k\|), \quad \forall t \in [kT; (k + 1)T], \quad (7) $$

then there exists $\exists \beta \in \mathcal{X}_T$ such that the trajectories of the continuous-time system satisfy:

$$ \|x(t)\| \leq \beta(\|x_0\|, t), \quad \forall t \geq 0, \quad (8) $$

which guarantees the stability of the system (1).

In this paper, the following problem is addressed:

Problem 1: Consider the continuous-time system (1) and a sampled-data approximate model (4). If the conditions required in Theorem 1 hold, then there exists a decreasing sequence, converging to the origin singleton, of bounded, non convex and disconnected sets which contain all the future sampled states (4). How is thus possible to design a decreasing sequence, converging to the origin singleton, of bounded and connected sets which contain all the future continuous-time trajectory of the system (1)?
III. DESIGN OF A SEQUENCE OF CONNECTED SETS

To provide a solution to Problem 1, we propose to build a sequence of bounded connected sets depending on time, which converges to the origin singleton and containing the future part of the continuous-time trajectory. These connected sets are computed based on the level sets of the discrete-time Lyapunov Lur’e type function and on an upper bound between the discrete-time approximate model (4) and the continuous-time system (1) over a sampling period.

The principle of the design is decomposed into several results which are developed in subsection III-A. The first result, provided by Theorem 2, is to determine an upper bound depending on $|x_k|$ verifying the inequality (7) over a sampling period. The second result, provided by Theorem 3, formalizes the gap between the exact and approximate sampled models in function of the sampling period $T$ and the norm $|x_k|$. The subsection III-B shows how the design of a sequence of connected sets can be involved to prove the global stability of a sampled-data non-linear Lur’e system.

A. Preliminary results

First of all, we prove that trajectory $x(t)$:

$$x(t) = x_k + \int_{kT}^{t} (Ax(\tau) + B\phi(Cx(\tau))) \, d\tau, \forall t \in [kT; (k+1)T]$$

between two samples, verifies Equation (7). In other words, the trajectory $x(t)$, between two samples, is contained in a ball whose radius is $\mathcal{K}_\infty$-class function of $|x_k|$.

Theorem 2: For all $k \in \mathbb{N}$, for all $t \in [kT; (k+1)T]$, there exists a $\mathcal{K}_\infty$ class function depending on $|x_k|$ which is the upper bound of

$$\|x(t) - x_k\| \leq (e^{MT} - 1)\|x_k\| = \kappa(\|x_k\|),$$

where $M$ is a positive constant depending on system parameters $(A, B, C, \Omega)$.

Proof: The proof is organized in two steps, the first one explains how to evaluate the constant $M$ and the second justifies Equation (10).

Part 1:

$$\|Ax(\tau) + B\phi(Cx(\tau))\| \leq \sup_{\Gamma \in \mathcal{D}} \|A + BL_{\phi}(C)\| \|x(\tau)\| = M\|x(\tau)\|,$$

(11)

with $\mathcal{D}$ the set which contains all diagonal matrices in $\mathbb{R}^{p\times p}$ where each element of the diagonal is in the unit interval. $M$ is then given by $M = \sqrt{\mu}$ where $\mu$ is the solution of the optimization problem:

$$\min_{\mu \in \mathbb{R}} \mu, \text{ subject to:}$$

$$\begin{bmatrix} \mu I_n & (A + BL_{\phi}(C))' \\ A + BL_{\phi}(C) \end{bmatrix} < \mu I_n, \forall \Gamma \in \mathcal{D}. \quad (13)$$

Using the Schur complement, (13) is reformulated into:

$$\begin{bmatrix} \mu I_n & (A + BL_{\phi}(C))' \\ A + BL_{\phi}(C) \end{bmatrix} > \mu I_n, \forall \Gamma \in \mathcal{D}. \quad (14)$$

Noticing that

$$A + BL_{\phi}(C) = \sum_{i=1}^{2^p} \frac{\lambda_i}{N_i} N_i + \frac{\lambda_i}{N_i} I_n,$$

(15)

where $\Gamma_i = \text{diag}(\gamma_i)$ with $\gamma_i = 1$ or 0, $j \in \{1; \ldots; p\}$. Indeed (13) could be replaced by

$$\begin{bmatrix} \mu I_n & N_i' \\ N_i & I_n \end{bmatrix} > 0, \forall i \in \{1; \ldots; 2^p\}. \quad (16)$$

Part 2:

First step, using successively, Equation (9), the triangle inequality and Equation (11) as follows:

$$\|x(t) - x_k\| = \left\| \int_{kT}^{t} Ax(\tau) + B\phi(Cx(\tau)) \, d\tau \right\|$$

$$\leq \int_{kT}^{t} \|Ax(\tau) + B\phi(Cx(\tau))\| \, d\tau$$

$$\leq \int_{kT}^{t} M\|x(\tau)\| \, d\tau$$

$$\leq M \left( \int_{kT}^{t} \|x(\tau) - x_k + x_k\| \, d\tau \right)$$

$$\leq M \left( \int_{kT}^{t} \|x(\tau) - x_k\| \, d\tau + (t - kT)\|x_k\| \right),$$

the inequality depends on $\|x(\cdot) - x_k\|$ in both parts. The integral of the exact trajectory is unknown. The second step consists in avoiding this issue, as $x(t)$ is a continuous function in parameter $t$ on $[kT; (k+1)T]$ then $\|x(t) - x_k\|$ and $(t - kT)\|x_k\|$ are positive and continuous in this interval, Gronwall’s lemma [2] is used to yield:

$$\|x(t) - x_k\|$$

$$\leq M(t - kT)\|x_k\| + \int_{kT}^{t} M(t - kT)\|x_k\| e^{M(t - \tau)} \, d\tau$$

$$\leq M(t - kT)\|x_k\| + \int_{kT}^{t} M(t - kT)\|x_k\| e^{M(t - \tau)} \, d\tau$$

$$\leq M(t - kT)\|x_k\| + \left[ e^{M(t - kT)} - 1 \right] \|x_k\|$$

$$\leq e^{M(t - kT) - 1} \|x_k\|$$

with $\kappa(\cdot)$ a $\mathcal{K}_\infty$ class function, allowing Equation (10), $\forall k \in \mathbb{N}, \forall t \in [kT; (k+1)T]$.

Now we give a tool that allows to limit the gap between $F_\tau^C(x_k)$ and $F_\tau^C(\cdot)$.

Theorem 3: For Lur’e system (1), the exact discrete-time model $F_\tau^C(x_k)$ is still close to the approximate discrete-time model $F_\tau^C(x_k)$ such that:

$$\|F_\tau^C(x_k) - F_\tau^C(\cdot)\| \leq \rho(T)\|x_k\||,$$

(17)

with $\rho(T) = \left( \|A\| \left( \frac{e^{MT} - 1}{M} - T \right) + \|B\|\|\Omega C\| \left( \frac{e^{MT} - 1}{M} + T \right) \right)$ a $\mathcal{K}_\infty$ class function.

Proof: Using successively the triangle inequality and Theorem 2, the distance between the exact and the approxi-
mate model becomes:
\[ ||F^e_T(x_k) - F^a_T(x_k)|| \leq \int_{kT}^{(k+1)T} ||A|| ||x(\tau) - x_k|| d\tau + \int_{kT}^{(k+1)T} ||B|| ||C(x(\tau)) - \varphi(Cx_k)|| d\tau \]
\[ \leq \frac{1}{M} \int_{kT}^{(k+1)T} ||A|| (\Delta T + 1) ||x(\tau) - x_k|| d\tau \]
\[ = \rho(T)||x_k||, \]
with \( \rho(T) \) is a \( \mathcal{K}_\infty \) class function. It concludes the proof. ■

The upper bound \( \rho(T)||x_k|| \) in Inequality (17) may be conservative due to the triangle inequality and the fact that it is valid for all the state space \( \mathbb{R}^n \) and for all the non-linearities satisfying the bounded sector condition. An improvement of \( \rho(T) \) may be calculated on a sufficiently limited neighborhood \( \mathcal{N} \) of the origin containing all the possible trajectories : \( \rho(T: \mathcal{A}) \). \( F^e(\cdot) \) can be computed by an oversampling of the Euler approximation.

B. Consequences for global stability analysis of a sampled-data Lur'e system

Theorem 2 checks the second assumption of Proposition 1. The purpose of this part is to prove that to guarantee the stability of the continuous-time system (1), the stability of the exact discrete-time system (3) should be proved.

Stability of the approximate discrete-time system is given by:
\[ V(F^e_T(x_k); \varphi(CF^e_T(x_k))) - V(x_k; \varphi(Cx_k)) \leq 0, \quad \forall x_k \neq 0, \quad (18) \]
where \( V \) is the Lyapunov function (2). The purpose is to prove the stability of the exact discrete-time system:
\[ V(F^e_T(x_k); \varphi(CF^a_T(x_k))) - V(x_k; \varphi(Cx_k)) \leq 0, \quad \forall x_k \neq 0. \]
(19)

By assuming that the approximate and the exact models are close enough, the value \( V(F^e_T(\cdot); \varphi(CF^a_T(\cdot))) \) is also close to \( V(F^e_T(\cdot); \varphi(CF^e_T(\cdot))) \). Nevertheless it does not imply an ordering relation between them. To evaluate their distance, the presence of the function \( f_\alpha(\cdot) \) is used. This function should verify \( f_\alpha(0) = 0 \) and
\[ V(F^e_T(x_k); \varphi(CF^a_T(x_k))) - V(F^a_T(x_k); \varphi(CF^e_T(x_k))) \leq f_\alpha(x_k). \]
(20)

This reasonable assumption implies, for stability of the exact discrete-time system, that condition of Equation (18) is not enough. It is necessary to consider that the Lyapunov function of the approximate discrete-time model decreases fast enough to balance the estimate gap \( f_\alpha(\cdot) \):
\[ V(F^e_T(x_k); \varphi(CF^a_T(x_k))) - V(x_k; \varphi(Cx_k)) \leq -f_\alpha(x_k), \]
(21)
where \( f_\alpha : \mathbb{R}^n \rightarrow \mathbb{R}^+ \) such that \( f_\alpha(x_k) > 0, \quad \forall x_k > 0 \) and \( f_\alpha(0) = 0 \).

Theorem 4: If \( f_\alpha : \mathbb{R}^n \rightarrow \mathbb{R}^+ \) and \( f_\alpha : \mathbb{R}^n \rightarrow \mathbb{R} \) such that \( f_\alpha(0_{n+1}) = 0 \) and \( f_\alpha(0_{n+1}) = 0 \) verify Equations (20) and (21) such that:
\[ f_\alpha(x_k) < f_\alpha(\xi_k), \quad \forall x_k \neq 0, \]
(22)
then the exact discrete-time system (3) is stable.

Proof: 
\[ V(F^e_T(x_k); \varphi(CF^a_T(x_k))) - V(x_k; \varphi(Cx_k)) = V(F^e_T(x_k); \varphi(CF^a_T(x_k))) - V(F^e_T(x_k); \varphi(CF^e_T(x_k))) + V(F^a_T(x_k); \varphi(CF^e_T(x_k))) - V(x_k; \varphi(Cx_k)) \leq f_\alpha(x_k) - f_\alpha(\xi_k) < 0, \quad \forall x_k \neq 0. \]

In order to simplify the problem of determining \( f_\alpha(\cdot) \) and \( f_\alpha(\cdot) \), they are restricted to belong to a particular class of functions, which can be handled in the framework of LMI. We assume that \( f_\alpha(x_k) = w_k^TQ^a\xi_k \) and \( f_\alpha(x_k) = w_k^TQ^e\xi_k \), where \( Q^a \) and \( Q^e \in \mathbb{R}^{(n+2p)\times(n+2p)} \) are symmetric positive definite matrices and \( w_k \) is defined by:
\[ w_k = \begin{pmatrix} \xi_k^T \varphi_\alpha(C\xi_k) \end{pmatrix} \]
(23)
\[ \varphi_\alpha(\cdot) \in \mathbb{R}^{n+2p}. \]

We also introduce the vector \( z_k \) defined by:
\[ z_k = \begin{pmatrix} w_k^T \varphi_\alpha(CF^e_T(x_k)) \end{pmatrix} \]
(24)

Equations (10), (17) and (20) are respectively equivalent to Equations (25), (26) and (27):
\[ z_k^d = \begin{pmatrix} (e^{MT} - 1)\xi_k; 0_{3p}; -I \end{pmatrix} z_k \leq 0, \]
(25)
\[ z_k^e = \begin{pmatrix} -T^2A - Q^e \end{pmatrix} z_k \leq 0, \]
(26)
\[ z_k^a = \begin{pmatrix} \Theta_1 \Theta_2 \Theta_3 \Theta_4 \end{pmatrix} + \varphi_\alpha \]
(27)
with \( \varphi_\alpha = \text{ diag}(Q^a; 0_{p+e}). \)

Sector conditions are introduced by the following equations:
\[ \varphi_\alpha(C\xi_k)^T U \varphi_\alpha(C\xi_k) - \Delta_2 \xi_k \leq 0 \]
\[ \Leftrightarrow z_k^\alpha = \begin{pmatrix} 0_n \end{pmatrix} \]
(28)
\[ \varphi(CF^a_T(x_k))^TW^e \varphi(CF^a_T(x_k)) - \Omega_2 CF^a_T(x_k) \leq 0 \]
\[ \Leftrightarrow z_k^\alpha = \begin{pmatrix} 0_{n+e} \end{pmatrix} \]
(29)
\[ \varphi_d(CF_a^2(x_k))^T W^{\alpha} (\varphi_d(CF_a^2(x_k)) - \Omega_d CF_a^2(x_k)) \leq 0 \iff \\
\begin{bmatrix}
\pi_3 & \pi_4 & \pi_5 & \pi_6 & \pi_7 & \pi_8 \\
0 & 0 & 0 & 0 & 0 & \tau_2 B_d \\
\end{bmatrix} \leq 0 \]
where \( U, W^e \) and \( W^{\alpha} \in R^{p \times p} \) are diagonal positive definite matrices.

**Theorem 5:** For the class of systems defined by (1) and its discretization (4), if there exist symmetric positive definite matrices \( P \in R^{n \times n}, Q^a \) and \( Q^e \in R^{(n+2)p \times (n+2)p} \), a positive diagonal semidefinite matrix \( \Delta \in R^{p \times p} \), positive diagonal matrices \( U_e, U, W_e \) and \( W^{\alpha} \in R^{p \times p} \), a matrix \( G \in R^{n \times n} \) and positive scalars \( \tau_1 \) and \( \tau_2 \) such that:

\[ M + \text{diag}(0_n; Q^e) < 0 \]

are verified, then the origin of system (1) and its Euler discretization is globally asymptotically stable.

**Proof:** Following the proof of [5, theorem 2], Equation (31) implies:

\[ \begin{bmatrix}
A_{d}^T & B_{d}^T \\
0 & 0 \\
\end{bmatrix} P + \begin{bmatrix}
A_{d}^T & B_{d}^T \\
0 & 0 \\
\end{bmatrix}' + \begin{bmatrix}
-P & A_{d}^T P_1 \\
* & -2U_c B_{d}' P_2 \\
* & * & -2W_e \\
\end{bmatrix} < -Q^e, \]

which implies Equation (21).

Equation (32) is obtained by subtracting Equations (28-30) from Equation (27) and using the \( S \)-procedure with Equations (25) and (26) and parameters \( \tau_1 \) and \( \tau_2 \). With the previous equivalences, we get the following inequality:

\[ V(F^2(x_k); \varphi(CF_a^2(x_k)) - V(F^2(x_k); \varphi(CF_a^2(x_k))) \leq w_k Q^{\alpha} W_k + 2\varphi(Cx_k)' U (\varphi(Cx_k) - \Omega C x_k) \]

The end of the proof is deduced by using Theorem 4.

**Definition 1:** The region where the continuous-time trajectories can exist between two samples \( x_k \) and \( x_{k+1} \) is defined by:

\[ \mathcal{B}(x_k) = \{ \tilde{x} \in R^n, \| \tilde{x} - x_k \| \leq \kappa(\| x_k \|) \}. \]

**Definition 2:** The area where all continuous-time trajectories can exist with the initial condition \( x_k \) is defined by:

\[ \mathcal{B}(y) = \bigcup_{x_k \in \mathcal{L}(y)} \mathcal{B}(x_k). \]

Proposition 2 shows that the area \( \mathcal{B}(\cdot) \) verifies by construction some trivial and important properties which guarantee the stability of the continuous-time system.

**Proposition 2:** Consider the sampling period \( T > 0 \) and the level set \( \mathcal{L}(V(x_k; \varphi_d(C_d x_k))) \) found by using Theorem 5. The area \( \mathcal{B}(V(x_k; \varphi_d(C_d x_k))) \) verifies the following properties:

- \( \mathcal{L}(V(x_k; \varphi_d(C_d x_k))) \subset \mathcal{B}(V(x_k; \varphi_d(C_d x_k))) \);
- \( \mathcal{B}(\cdot) \) is bounded, because \( \kappa(\cdot) \) is radially unbounded;
- \( x(t) \in \mathcal{B}(V(x_k; \varphi_d(C_d x_k))), \forall t \geq kT \);
- \( \mathcal{B}(V(x_k; \varphi_d(C_d x_k))) \subset \mathcal{B}(V(x_k; \varphi_d(C_d x_k))) \);
- \( \lim \mathcal{B}(V(x_k; \varphi_d(C_d x_k))) = \{ 0 \} \).

**Remark 3:** \( \mathcal{B}(\cdot) \) is not a level set. It is a new area bounding the continuous-time trajectories whose initial conditions are in the level set \( \mathcal{L}(\cdot) \). Moreover \( \mathcal{B}(\cdot) \) may be non convex.

Fig. 1 summarizes the properties of the area \( \mathcal{B}(\cdot) \) which are listed in Proposition 2.

**IV. ILLUSTRATION**

A numerical example illustrates the handling of the disconnected sets. Consider system (1) where, \( \Omega = 1.9, T = 0.07, \)

\[ A = \begin{bmatrix}
-8 & -2.5 \\
3 & -6 \\
\end{bmatrix}, \quad B = \begin{bmatrix}
0.9 \\
-0.8 \\
\end{bmatrix}, \quad C = \begin{bmatrix}
0.9 & -1 \\
\end{bmatrix}, \quad \varphi(y) = \frac{\Omega y}{2} (1 + \cos(15y + 0.2y^2)) \]

The maximum distance between \( F^2(\cdot) \) and \( F^e(\cdot) \) is obtained by simulation with the grid area \( \mathcal{M} = \{-10; \ldots; 10\}^2 \) with grid spaces equal to 0.5; \( \rho(T, \mathcal{M}) = 0.1944 \).

Applying Theorem 5, the Lyapunov function is defined by the following parameters:

\[ P = \begin{bmatrix}
0.0404 & 0.0158 \\
0.0158 & 0.0514 \\
\end{bmatrix}, \quad \Delta = 0.0828. \]
Fig. 2 illustrates, with simulation, one trajectory with the initial condition $x_0$, the level set $L_V(V(x_0; \phi_d(C_\delta x_0)))$ (thin solid line) and the new area $\mathcal{A}(V(x_0; \phi_d(C_\delta x_0)))$ (thick solid line). The trajectory $x(t)$ is still in the region $\mathcal{A}(V(x_0; \phi_d(C_\delta x_0)))$ and it can go out the level set $L_V(V(x_0; \phi_d(C_\delta x_0)))$ between two samples as shown in Fig. 2. However every sample is in the level set $L_V(V(x_0; \phi_d(C_\delta x_0)))$.

Fig. 3 illustrates the contraction of the different areas $L_V(\cdot)$ and $\mathcal{A}(\cdot)$. $L_V(\cdot)$ is a bounded region which can be connected or not. $\mathcal{A}(\cdot)$ is a bounded and connected area and its convergence to the origin guarantees the stability of the continuous-time system.

V. CONCLUSIONS

In this paper, the question of using disconnected level sets Lyapunov functions in the framework of sampled-data Lur’e systems has been investigated, with an Euler discretization scheme. The key idea consists in building two decreasing sequences of sets converging to the origin and containing respectively the future of the sampled and continuous-time trajectories. The sets of the first sequence are given by level sets of an advanced Lyapunov function and may be disconnected and non convex. They are related to the discrete-time system. The sets of the second sequence are connected and are related to the continuous-time system. An example has been proposed to illustrate this approach.

REFERENCES