Incrementally port-Hamiltonian systems

M.K. Camlibel and A.J. van der Schaft

Abstract—This paper introduces the new class of incrementally port-Hamiltonian systems. This class can be obtained from standard port-Hamiltonian systems by replacing the composition of the Dirac structure and energy-dissipating relation by a maximal monotone relation. After introducing this new class of systems, we study their compositions and show that incrementally port-Hamiltonian systems are closed under composition. Also, we study existence and uniqueness of state trajectories for such systems as well as an energy-based state re-initialization principle.

I. INTRODUCTION

First principles port-based network modeling of physical systems often leads to port-Hamiltonian system descriptions. However, the presence of sources necessarily leads to system models where the Hamiltonian (total energy) cannot be used anymore as a Lyapunov function (since there is a continuous supply of power by the source). One approach to such situations is to consider an incremental version of the property of port-Hamiltonian systems. Similar ideas have been used in incremental passivity theory [1], [2] and in contraction analysis [3].

In this paper we will show how we can define the class of incrementally port-Hamiltonian systems in a way that is very much analogous to the definition of a (standard) port-Hamiltonian system, by replacing the notion of a Dirac structure, or more precisely the composition of a Dirac structure and an energy-dissipating relation, by a maximal monotone relation. Typical examples of incrementally port-Hamiltonian systems are electrical networks containing ideal switching elements (such as ideal diodes and constant sources) and mechanical systems with Coulomb friction.

After setting-up a general framework for incrementally port-Hamiltonian systems, we will show how incrementally port-Hamiltonian systems share the same compositionality properties as ordinary port-Hamiltonian systems. More precisely, we will show that the composition of two incrementally port-Hamiltonian systems is again an incrementally port-Hamiltonian system, based on the fact that the composition of two maximal monotone relations is again maximal monotone (very similar to the property that the composition of two Dirac structures is again a Dirac structure).

Finally we will study existence and uniqueness of solutions of incrementally port-Hamiltonian systems. Based on the classical existence and uniqueness results [4] for differential inclusions with maximal monotone set-valued mappings, we will prove that the incrementally port-Hamiltonian systems admit a unique state trajectory when the initial state belongs to the so-called constraint set. To deal with the initial states that do not belong to the constraint set, we will introduce the set of jump directions. After proving a duality relationship between the constraint set and the set of jump directions, we will introduce a state re-initialization rule based on a certain energy minimization principle in terms of the Hamiltonian of the underlying system. This principle reveals that incrementally port-Hamiltonian systems admit a unique state trajectory for any initial state after a possibly state re-initialization at time $t = 0$. This generalizes the classical charge and flux conservation principle from circuit theory.

The organization of the paper is as follows. In Section II, we quickly review the Dirac structures and ‘ordinary’ port-Hamiltonian systems. This will be followed by the introduction of incrementally port-Hamiltonian systems in Section III. In Section IV, we prove that the composition of two incrementally port-Hamiltonian system is again an incrementally port-Hamiltonian system. Section V studies existence and uniqueness of state trajectories of incrementally port-Hamiltonian systems as well as a certain state re-initialization principle. Finally, the paper closes with the conclusions in Section VI.

II. PORT-HAMILTONIAN SYSTEMS

Underlying the definition of a port-Hamiltonian system is the geometric notion of a Dirac structure, which relates the power variables of the composing elements of the system in a power-conserving manner. The power variables always appear in conjugated pairs (such as voltages and currents, or generalized forces and velocities), and therefore mathematically they are modeled to take their values in dual linear spaces. Next, we introduce the notational conventions that will be in force throughout the paper.

Let $\mathcal{F}$ be a finite-dimensional linear space and $\mathcal{E}$ be its dual space, that is, $\mathcal{E} := \mathcal{F}^*$. We call $\mathcal{F}$ the space of flow variables, and $\mathcal{E} = \mathcal{F}^*$ the space of effort variables. To distinguish flow/effort variables of different components, sometimes we write $\mathcal{F}_x$, $\mathcal{E}_x$, $\mathcal{F}_p$, $\mathcal{E}_p$, etc.

The duality product for the pair $(\mathcal{E}, \mathcal{F})$ is denoted by $\langle \cdot | \cdot \rangle$, that is

$$\langle e | f \rangle = e^T f \in \mathbb{R}$$

for $e \in \mathcal{E}$ and $f \in \mathcal{F}$.

The research leading to these results has received funding from the European Union Seventh Framework Programme [FP7/2007-2013] under grant agreement No.257462 HYCON2 Network of Excellence.

M.K. Camlibel and A.J. van der Schaft are with Johann Bernoulli Institute for Mathematics and Computer Science, University of Groningen, PO Box 407, 9700 AK Groningen, the Netherlands m.k.camlibel@rug.nl, a.j.van.der.schaft@rug.nl
We also define an indefinite bilinear form on \( F \times E \) by
\[
\langle (f_1, e_1), (f_2, e_2) \rangle = \langle e_1 | f_2 \rangle + \langle e_2 | f_1 \rangle
\]
for \((f_i, e_i) \in F \times E\) with \(i \in \{1, 2\}\). For a subspace \(S \subseteq F \times E\), we denote its orthogonal complement with respect to this indefinite bilinear form by \(S^\perp\).

With a slight abuse of notation, the inner product on linear finite dimensional space \(F\) is also denoted by \(\langle \cdot | \cdot \rangle\), that is \(\langle f_1 | f_2 \rangle = f_1^T f_2\) for any \(f_1, f_2 \in F\).

**Definition II.1** A subspace \(D \subseteq F \times E\) is a (constant\(^1\)) Dirac structure if \(D = D^\perp\).

**Remark II.2** An equivalent definition of a Dirac structure is the following [5]–[7]. A Dirac structure is any subspace \(D\) with the property
\[
\langle e | f \rangle = 0, \text{ for all } (f, e) \in D, \tag{1}
\]
which is maximal with respect to this property. That is, there does not exist a subspace \(D'\) with \(D \subsetneq D'\) such that \(\langle e | f \rangle = 0\), for all \((f, e) \in D'\).

For the finite-dimensional case (as will be the case throughout this paper) the maximal dimension of any subspace \(D\) satisfying (1) equals \(\dim F = \dim E\), and thus a Dirac structure is any subspace \(D\) satisfying (1) together with \(\dim D = \dim F\). The property \(\langle e | f \rangle = 0\) for all \((f, e) \in D\) corresponds in applications to power conservation.

For the definition of a conservative port-Hamiltonian system we need the following ingredients (see e.g. [6], [8], [9]). We start with an overall Dirac structure \(D\) on the space of all flow and effort variables involved:
\[
D \subseteq F_x \times E_x \times F_P \times E_P. \tag{2}
\]

The space \(F_x \times E_x\) is the space of flow and effort variables corresponding to the energy-storing elements (to be defined later on) while the space \(F_P \times E_P\) denotes the space of flow and effort variables corresponding to the external ports.

The dynamics of the port-Hamiltonian system is defined by specifying, next to its Dirac structure \(D\), the constitutive relations of the energy-storing elements. Let the Hamiltonian \(H : X \to \mathbb{R}\) denote the total energy at the energy-storage elements with state variables \(x = (x_1, x_2, \ldots, x_n)\); i.e., the total energy is given as \(H(x)\). In the sequel we will throughout take \(X = F_x\), but \(X\) may also denote an \(n\)-dimensional manifold (in which case \(F_x\) is the tangent space to this manifold \(X\) at the state \(x\)). The constitutive relations between the state variables \(x\) and the flow and effort vectors of the energy-storing elements are given by\(^2\)
\[
\dot{x} = -f_x \quad \text{and} \quad e_x = \frac{\partial H}{\partial x}(x). \tag{3}
\]

This immediately implies the energy balance
\[
\frac{d}{dt} H = \frac{\partial^T H}{\partial x}(x) \dot{x} = -e_x^T f_x. \tag{4}
\]

\(^1\)For the definition of Dirac structures on manifolds we refer to [5], [6].

\(^2\)The vector \(\frac{\partial H}{\partial x}(x)\) of partial derivatives throughout denotes a column vector.

The geometric definition of a conservative port-Hamiltonian system is given as follows:

**Definition II.3** Consider a Dirac structure
\[
D \subset F_x \times E_x \times F_P \times E_P
\]
and a Hamiltonian \(H : X \to \mathbb{R}\). Then the dynamics of the corresponding conservative port-Hamiltonian system is given as
\[
( -\dot{x}(t), \frac{\partial H}{\partial x}(x(t)), f_P(t), e_P(t) ) \in D \tag{5}
\]
at almost all time instants \(t\).

It follows from the power-conservation property of Dirac structures and (4) that
\[
\frac{d}{dt} H(x(t)) = e_x^T f_P(t), \tag{6}
\]
This shows that a conservative port-Hamiltonian system is lossless if the Hamiltonian \(H\) is bounded from below.

General port-Hamiltonian systems are defined by including ports associated to energy-dissipation. Thus we consider a Dirac structure
\[
D \subseteq F_x \times E_x \times F_P \times E_P \times F_R \times E_R,
\]
where the flow and effort variables \((f_R, e_R) \in F_R \times E_R\) will be terminated by energy-dissipating (resistive) relations. An energy-dissipating relation is any subset \(R \subset F_R \times E_R\) with the property
\[
\langle e_R | f_R \rangle \leq 0 \text{ for all } (f_R, e_R) \in R
\]
Hence the composition
\[
D \circ R := \{(f_x, e_x, f_P, e_P) \in E_x \times F_P \times E_P | \exists (f_R, e_R) \in R \text{ s.t. } (f_x, e_x, f_P, e_P, f_R, e_R) \in D\} \tag{7}
\]
satisfies the property
\[
e_x^T f_x + e_P^T f_P = -e_R^T f_R \geq 0 \tag{8}
\]
for all \((f_x, e_x, f_P, e_P) \in D \circ R\). In particular it follows that
\[
\frac{d}{dt} H(x(t)) \leq e_R^T f_P(t), \tag{9}
\]
showing passivity of the port-Hamiltonian system if the Hamiltonian \(H\) is bounded from below.

**III. Incrementally Port-Hamiltonian Systems**

Next, we will introduce another class of systems, which is extending, and in some cases modifying, the definition of a port-Hamiltonian system. The basic idea will be to replace the composition of a Dirac structure and an energy-dissipating relation by a maximal monotone relation. To do so, we begin with a quick review of maximal monotone relations and sub gradients of convex functions.

A relation \(M \subseteq F \times E\) is said to be monotone if
\[
\langle v_1 - v_2 | u_1 - u_2 \rangle \geq 0
\]
for all \((u_i, v_i) \in M\) with \(i \in \{1, 2\}\). It is called maximal monotone if it is monotone and the implication

\[ M' \text{ is monotone and } M \subseteq M' \implies M = M' \]

holds.

**Definition III.1** Consider a maximal monotone relation

\[ M \subseteq \mathcal{F}_x \times \mathcal{E}_x \times \mathcal{F}_p \times \mathcal{E}_p \]

and a Hamiltonian \(H : \mathcal{X} \to \mathbb{R}\). Then the dynamics of the corresponding incrementally port-Hamiltonian system is given as

\[ (-\dot{x}_i(t), \frac{\partial H}{\partial x}(x(t)), f_p(t), e_p(t)) \in M \quad (10) \]

for almost all time instants \(t\).

It follows that the dynamics of incrementally port-Hamiltonian systems are characterized by the following ‘dissipation-like inequality’:

\[
\langle \frac{\partial H}{\partial x}(x_1(t)) - \frac{\partial H}{\partial x}(x_2(t)) | \dot{x}_1(t) - \dot{x}_2(t) \rangle 
\leq \langle e_p(t) - e_p^2(t) | f_p(t) - f_p^2(t) \rangle
\]

\[ (11) \]

is satisfied for almost all times \(t\) for any trajectories

\[ (-\dot{x}_i(t), \frac{\partial H}{\partial x}(x_i(t)), f_p^i(t), e_p^i(t)) \in M \]

with \(i \in \{1, 2\}\).

Every conservative port-Hamiltonian system is also incrementally port-Hamiltonian as follows from the following lemma.

**Lemma III.2** Every Dirac structure on \(\mathcal{F} \times \mathcal{E}\) is maximal monotone.

**Proof.** Let \(\mathcal{D} \subseteq \mathcal{F} \times \mathcal{E}\) be a Dirac structure. Let \((f_i, e_i) \in \mathcal{D}\) with \(i = 1, 2\). Since \(\langle e | f \rangle = 0\) for all \((f, e) \in \mathcal{D}\) due to Remark II.2, we get

\[ \langle e_1 - e_2 | f_1 - f_2 \rangle = 0. \]

Therefore, \(\mathcal{D}\) is monotone on \(\mathcal{F} \times \mathcal{E}\). Let \(\mathcal{D}'\) be a monotone relation on \(\mathcal{F} \times \mathcal{E}\) such that \(\mathcal{D} \subseteq \mathcal{D}'\). Let \((f', e') \in \mathcal{D}'\) and \((f, e) \in \mathcal{D}\). Since \(\mathcal{D}'\) is monotone, \(\mathcal{D} \subseteq \mathcal{D}'\), and since \(\mathcal{D}\) is a subspace, we have

\[ 0 \leq \langle e' - \alpha e | f' - \alpha f \rangle = \langle e' | f' \rangle - \alpha(\langle e' | f \rangle + \langle e | f' \rangle) \]

for any \(\alpha \in \mathbb{R}\). This means that

\[ \langle e' | f \rangle + \langle e | f' \rangle = 0 \]

and hence \((f', e') \in \mathcal{D}^\perp = \mathcal{D}\). Therefore, we get \(\mathcal{D}' \subseteq \mathcal{D}\). Since the reverse inclusion already holds, we further get \(\mathcal{D}' = \mathcal{D}\). Consequently, \(\mathcal{D}\) is maximal monotone.

\[ \]

**A. Examples**

In the non-conservative case the relation between port-Hamiltonian systems and incrementally port-Hamiltonian systems is more subtle, as can be seen from the following examples.

**Example III.3 (Mechanical systems with friction)** Consider a mechanical system with standard kinetic energy and arbitrary potential energy, subject to friction. The friction characteristic corresponds to a constitutive relation between certain port variables \(f_R, e_R\). Assume for simplicity that \(f_R, e_R\) are scalar variables, that is, consider a single friction component with velocity \(f_R\) and friction force \(-e_R\).

In the case of linear friction \(-e_R = df_R\) with \(d > 0\), the resulting system is both port-Hamiltonian and incrementally port-Hamiltonian. In the case of a friction characteristic

\[ -e_R = R(f_R) \]

the system will be port-Hamiltonian if the graph of the function \(R\) is in the first and third quadrant. On the other hand, it will be incrementally port-Hamiltonian if the relation is the function \(R\) is a monotonically non-decreasing function.

For example, the Striebeck friction characteristic defines a port-Hamiltonian system, but *not* an incrementally port-Hamiltonian system.

**Example III.4 (Circuit with tunnel diode)** Consider an electrical LC-circuit (possibly with nonlinear capacitors and inductors) together with a resistor corresponding to an electrical port \(f_R = -I, e_R = V\) (current and voltage). For a linear resistor (conductor) \(I = GV, G > 0\), the system is both port-Hamiltonian and incrementally port-Hamiltonian. For a nonlinear conductor \(I = G(V)\) the system is port-Hamiltonian if the graph of the function \(G\) is in the first and third quadrant while incrementally port-Hamiltonian if \(G\) is monotonically non-decreasing.

For example, a tunnel diode characteristic

\[ I = \Phi(V - V_0) + I_0, \]

for certain positive constants \(V_0, I_0\), and a function \(\Phi(z) = \gamma z^3 - \alpha z, \alpha, \gamma > 0\), defines a system which is port-Hamiltonian but *not* incrementally port-Hamiltonian.

**Example III.5 (Sources)** Physical examples of systems which are incrementally port-Hamiltonian but *not* port-Hamiltonian are provided by systems with constant sources. Consider for example any nonlinear LC-circuit with passive resistors (conductors) and constant voltage and/or current sources. The same holds for an arbitrary mechanical system with damping and constant actuation: all are incrementally port-Hamiltonian but not port-Hamiltonian.

**B. Connection with incrementally and differentially passive systems**

When the Hamiltonian is quadratic, that is \(H(x) = \frac{1}{2}x^TQx\) for some symmetric positive definite matrix \(Q\), the
inequality (11) is equivalent to
\[
H \left( x_1(t_1) - x_2(t_1) \right) \leq H \left( x_1(t_0) - x_2(t_0) \right) + \int_{t_0}^{t_1} \langle e_1 P(t) - e_2 P(t), x_1(t) - x_2(t) \rangle \, dt \quad (12)
\]
for almost all \( t_0 \) and \( t_1 \) with \( t_0 < t_1 \). Hence an incrementally port-Hamiltonian system with a quadratic Hamiltonian
\[
H(x) = \frac{1}{2} x^T Q x
\]
with \( Q \geq 0 \) is necessarily incrementally passive in the sense of [10]; see also [1], [2].

Furthermore, consider the infinitesimal version of (11). In fact, let \( (f_1^p, e_1^p, x_1) \) and \( (f_2^p, e_2^p, x_2) \) be two triples of system trajectories arbitrarily near each other. Taking the limit we deduce from (11)
\[
(\partial x)^T \frac{\partial^2 H}{\partial x^2} (x) \partial \dot{e} \leq (\partial e)^T \partial f
\]
where \( \partial x \) denotes the variational state, and \( \partial f, \partial e \) the variational external variables (e.g., variational inputs and outputs); see [11], [12]. Hence if the Hamiltonian \( H \) is a quadratic function \( H(x) = \frac{1}{2} x^T Q x \) then this amounts to the differential dissipativity inequality
\[
\frac{d}{dt} \frac{1}{2} (\partial x)^T Q \partial x \leq (\partial e)^T \partial f
\]
(13)
implying that the incrementally port-Hamiltonian system is differentially passive in the sense of [11], [12], with differential storage function \( \frac{1}{2}(\partial x)^T Q \partial x \).

IV. COMPOSITION OF MAXIMAL MONOTONE RELATIONS

A cornerstone of port-Hamiltonian systems theory is the fact that the power-conserving interconnection of port-Hamiltonian systems defines again a port-Hamiltonian system. This is based on the fact that the composition of Dirac structures is again a Dirac structure. In this section we will show that the same property holds for incrementally port-Hamiltonian systems, but now based on the fact that the composition of maximal monotone relations is again maximal monotone.

Consider two maximal monotonous relations \( M_a \subset F_a \times F_a^* \times V_a \times V_a^* \) with typical element denoted by \( (f_a, e_a, v_a, w_a) \) and \( M_b \subset F_b \times F_b^* \times V_b \times V_b^* \) with typical element denoted by \( (f_b, e_b, v_b, w_b) \), where \( V_a = V_b = V \), and thus \( V_a^* = V_b^* = V^* \) (shared variables). Define the composition of \( M_a \) and \( M_b \), denoted as \( M_a \circ M_b \), by
\[
M_a \circ M_b := \{ (f_a, e_a, f_b, e_b) \in F_a \times F_a^* \times F_b \times F_b^* \mid \exists v \in V, w \in V^* \text{ s.t. } (f_a, e_a, v, w) \in F_a \times F_a^* \times V \times V^*, \ (f_b, e_b, -v, w) \in F_b \times F_b^* \times V \times V^* \}
\]
(14)
Thus the composition of \( M_a \) and \( M_b \) is obtained by imposing on the vectors \( (f_a, e_a, v_a, w_a) \in M_a \) and \( (f_b, e_b, v_b, w_b) \in M_b \) the interconnection constraints
\[
v_a = -v_b, \quad w_a = w_b,
\]
and looking at the resulting vectors \( (f_a, e_a, f_b, e_b) \in F_a \times F_a^* \times F_b \times F_b^* \).

The main result of this section is that, whenever \( M_a \circ M_b \) has a non-empty relative interior, then the composition \( M_a \circ M_b \) is again a maximal monotone relation. The main ingredient in the proof of this statement will be the following theorem from [13, Ex. 12.46], which we recall here for completeness (in a slightly specialized form).

**Theorem IV.1** Let \( M \subset F_a \times F_a^* \times F_b \times F_b^* \) be maximal monotone. Assume that the reduced relation
\[
M_r := \{ (f_a, e_a) \mid \exists f_b \text{ s.t. } (f_a, e_a, f_b, e_b) = 0) \in M \}
\]
has a non-empty relative interior. Then, \( M_r \) is maximal monotone.

This theorem can be applied to the situation at hand after applying the following transformation.
\[
y_v := \frac{v_a + v_b}{\sqrt{2}}, \quad z_v := \frac{v_a - v_b}{\sqrt{2}}, \quad y_v, z_v \in V
\]
(16a)
\[
y_w := \frac{w_a + w_b}{\sqrt{2}}, \quad z_w := \frac{w_a - w_b}{\sqrt{2}}, \quad y_w, z_w \in V^*.
\]
(16b)
The motivation for this transformation lies in the fact that the interconnection constraints (15) take in the new variables the simple form \( y_v = 0, z_w = 0 \). Furthermore, by direct computation one obtains
\[
\langle y_w v_1 - y_w v_2, y_v \rangle = \langle y_w v_1 - y_w v_2, y_v \rangle = \langle v_a - v_a, v_a - v_a \rangle = \langle v_a - v_a, v_a - v_a \rangle = \langle w_a - w_a, v_a - v_a \rangle = \langle v_a - v_a, v_a - v_a \rangle = \langle v_a - v_a, v_a - v_a \rangle.
\]
(17)

**Theorem IV.2** Let \( M_a \subset F_a \times F_a^* \times V_a \times V_a^* \) and \( M_b \subset F_b \times F_b^* \times V_b \times V_b^* \) be maximal monotone relations such that \( M_a \circ M_b \) is non-empty. Then \( M_a \circ M_b \) is maximal monotone.

**Proof** It is evident that the direct sum \( M_a \oplus M_b \subset F_a \times F_a^* \times V_a \times V_a^* \times F_b \times F_b^* \times V_b \times V_b^* \) of \( M_a \) and \( M_b \) defined by
\[
M_a \oplus M_b := \{ (f_a, e_a, f_b, e_b, v_a, w_a, v_b, w_b) \mid (f_a, e_a, v_a, w_a) \in M_a, (f_b, e_b, v_b, w_b) \in M_b \}
\]
(18)
is a maximal monotone relation. By (17) we deduce that \( M_a \oplus M_b \) is also maximal monotone with respect to the coordinates \( (f_a, e_a, f_b, e_b, v_a, w_a, y_w, z_w) \). It now follows from Theorem IV.1 (with \( e_b = (y_w, z_w) \)) that \( M_a \circ M_b \) is maximal monotone.

This leads immediately to the following corollary.

**Corollary IV.3** Consider two incrementally port-Hamiltonian systems \( \Sigma_a \) and \( \Sigma_b \) with external port variables respectively \( (f_a^0, e_a^0) \in V \times V^* \) and \( (f_b^0, e_b^0) \in V \times V^* \), interconnected by the interconnection constraints
\[
f_a^0 = -f_b^0, \quad e_a^0 = e_b^0.
\]
The resulting interconnected system is incrementally port-Hamiltonian.
In this section we study existence and uniqueness of solutions for the incrementally port-Hamiltonian systems of the form

$$\left( -\dot{x}(t), \frac{\partial H}{\partial x}(x(t)) \right) \in \mathcal{M}$$

where $\mathcal{M} \subseteq \mathcal{F} \times \mathcal{E}$ is maximal monotone. As before, $\mathcal{F}$ is a finite-dimensional linear space and $\mathcal{E}$ is its dual, and the Hamiltonian $H(x) = \frac{1}{2} x^T Q x$ for some symmetric positive definite matrix $Q$.

We say that an absolutely continuous function $x : \mathbb{R}_+ \rightarrow \mathcal{E}_x$ is a solution of the incrementally port-Hamiltonian system (19) for the initial state $x_0$ if $x(0) = x_0$ and the relation (19) is satisfied for almost all $t \geq 0$.

We define the constraint set as

$$\mathcal{C}_M := \{ e_x \mid \exists f_x \text{ such that } (f_x, e_x) \in \mathcal{M} \}. \quad (20)$$

The following theorem is a consequence of the classical existence and uniqueness result [4, Thm. 3.4 and Prop. 3.8] for differential inclusions with maximal monotone set-valued mappings.

**Theorem V.1** Suppose that $\mathcal{C}_M$ has non-empty interior. Then, for each initial state $x_0 \in Q^{-1} \text{cl}(\mathcal{C}_M)$ there exists a unique solution for the system (19).

**Proof.** Let

$$z(t) = Q^{\frac{1}{2}} x(t)$$

where $Q^{\frac{1}{2}}$ is the unique symmetric and positive definite matrix satisfying $Q^{\frac{1}{2}} Q^{\frac{1}{2}} = Q$. Then, the relation (19) is equivalent to

$$\left( -\dot{z}(t), z(t) \right) \in Q^{\frac{1}{2}} \mathcal{M} Q^{\frac{1}{2}} \quad (21)$$

since $\frac{\partial H}{\partial x}(\bar{e}) = Q \bar{e}$. It follows from [13, Thm. 12.43] that $Q^{\frac{1}{2}} \mathcal{M} Q^{\frac{1}{2}}$ is maximal monotone. Then, the claim follows from [4, Thm. 3.4 and Prop. 3.8].

To extend the solution concept we adopted to initial states that do not belong to the constraint set, we need to introduce some notation. Let $\mathcal{F}$ be a finite-dimensional linear space and let $\mathcal{E}$ be its dual. For a non-empty set $\mathcal{K} \subseteq \mathcal{F}$, we define its quasi-recession cone

$$\mathcal{K}_\infty := \{ f' \in \mathcal{F} \mid \exists f \in \mathcal{K} \text{ s.t. } f + \lambda f' \in \mathcal{K} \forall \lambda \geq 0 \}. \quad (22)$$

When $\mathcal{K}$ is closed and convex, $\mathcal{K}_\infty$ coincides with the recession cone of $\mathcal{K}$ (see e.g. [14, Thm. 8.3]). Also, we define the barrier cone of $\mathcal{K}$ as

$$\mathcal{K}^b := \{ e \in \mathcal{E} \mid \sup_{f \in \mathcal{K}} \langle e \mid f \rangle < +\infty \}. \quad (23)$$

Now, define the set of jump directions corresponding to a relation $\mathcal{M} \subseteq \mathcal{F} \times \mathcal{E}$ as

$$\mathcal{J}_M := \{ f' \mid (f', 0) \in -\mathcal{M}_\infty \}. \quad (24)$$

With these preparations, we can prove the following duality relation between the constraint set $\mathcal{C}_M$ and the set of jump directions $\mathcal{J}_M$ when $\mathcal{M}$ is a maximal monotone relation.

**Theorem V.2** Suppose that $\mathcal{C}_M$ is closed. Then, the following duality relationship holds:

$$-\mathcal{J}_M = (\mathcal{C}_M)^b.$$

**Proof.** First, we claim that $-\mathcal{J}_M \subseteq (\mathcal{C}_M)^b$. To prove this claim, let $f' \in -\mathcal{J}_M$ and $e \in \mathcal{C}_M$. Then, there exists $(f, e) \in \mathcal{M}$ such that $(f + \lambda f', e) \in \mathcal{M}$ for all $\lambda \geq 0$ and there exists $\tilde{f}$ such that $(\tilde{f}, e) \in \mathcal{M}$. Since $\mathcal{M}$ is maximal monotone, we have

$$\langle e - \tilde{e} \mid f + \lambda f' - \tilde{f} \rangle \geq 0$$

for all $\lambda \geq 0$. Hence, we get

$$\langle e - \tilde{e} \mid f' \rangle \geq 0.$$

This means that $\langle \tilde{e} \mid f' \rangle \leq \langle e \mid f' \rangle$. Since $\tilde{e} \in \mathcal{C}_M$ is arbitrary, this proves that $-\mathcal{J}_M \subseteq (\mathcal{C}_M)^b$. To show the reverse inclusion, let $f' \in (\mathcal{C}_M)^b$. Suppose, for the moment, that there exists $\tilde{e} \in \mathcal{C}_M$ such that

$$\sup_{e \in \mathcal{C}_M} \langle e \mid f' \rangle = \langle \tilde{e} \mid f' \rangle. \quad (25)$$

This yields that $\langle e \mid f' \rangle \leq \langle \tilde{e} \mid f' \rangle$ for all $e \in \mathcal{C}_M$. Since $\mathcal{M}$ is maximal monotone and $\mathcal{C}_M$ is closed, it follows from [13, Thm. 12.37] that there must exist $\tilde{f}$ such that $(\tilde{f}, e) \in \mathcal{M}$ and $(\tilde{f} + \lambda f', e) \in \mathcal{M}$ for all $\lambda \geq 0$. Therefore, $(\tilde{f}', 0) \in \mathcal{M}_\infty$ and hence $f' \in -\mathcal{J}_M$. Then, what remains to be proven is the existence of $\tilde{e} \in \mathcal{C}_M$ satisfying (25). Let $g : \mathcal{E} \rightarrow \mathbb{R} \cup \{-\infty, +\infty\}$ be the extended-real valued function defined by $g(x) = -(x \mid f') + \delta(x \mid \mathcal{C}_M)$ where $\delta(x \mid \mathcal{C}_M)$ denotes the indicator function of the set $\mathcal{C}_M$. It can be verified that $g$ is a closed proper convex function. It follows from [14, Thm. 27.3] that the directions of recession of the function $g$ are the common directions of recession of the function $x \mapsto -(x \mid f')$ and the set $\mathcal{C}_M$. Let $y$ be such a direction of recession. This would mean that $\langle y \mid f' \rangle \geq 0$ since $y$ is a direction of recession for the function $x \mapsto -(x \mid f')$. Since it is also a direction of recession for the set $\mathcal{C}_M$, we have $x + \lambda y \in \mathcal{C}_M$ for all $x \in \mathcal{C}_M$ and $\lambda \geq 0$. As $f' \in (\mathcal{C}_M)^b$, this implies that $\langle y \mid f' \rangle \leq 0$. Therefore, we get $\langle y \mid f' \rangle = 0$. Consequently, $y$ is a direction in which the function $g$ is constant. Now, it follows from [14, Thm. 27.1] that the infimum of $g$ is attained. In other words, there exists $\tilde{e} \in \mathcal{C}_M$ satisfying (25).

Next, we extend the solution concept defined earlier in order to deal with the initial states that do not belong to the constraint set.

**Theorem V.3** Suppose that $\mathcal{C}_M$ is closed and has non-empty interior. Then, for each initial state $x_0$ there exist a unique re-initialized state $x_0^+$ satisfying

$$x_0^+ - x_0 \in \mathcal{J}_M \quad \text{and} \quad Q x_0^+ \in \mathcal{C}_M \quad (26)$$
and a unique absolutely continuous function \( x : \mathbb{R}_+ \to \mathcal{E}_x \) with \( x(0) = x_0^+ \) satisfying (19) for almost all \( t \geq 0 \). Moreover, the reinitialized state \( x_0^- \) satisfies the inequality \( H(x_0^-) \leq H(x_0) \) whenever \( 0 \in C_M \).

**Proof.** For a given \( x_0 \), consider the minimization problem

\[
\min_{Q \in \mathcal{C}_M} \frac{1}{2} (\xi - x_0)^T Q (\xi - x_0).
\]

Since \( Q \) is positive definite, the function \( z \mapsto z^T Q z \) is convex and has no direction of recession. Then, it follows from [14, Thm. 27.3] that there exists \( x_0^+ \) with \( Q x_0^+ \in C_M \) such that

\[
(x_0^+ - x_0)^T Q (x_0^+ - x_0) \leq (\xi - x_0)^T Q (\xi - x_0)
\]

for any \( \xi \) with \( Q \xi \in C_M \). In view of [14, Thm. 27.4], this holds if and only if \(-Q(x_0^- - x_0)\) is normal to \( C_M \) at \( x_0^+ \). This means that

\[
(x_0^+ - x_0)^T Q (x_0^+ - x_0) \geq 0
\]

(27)

for all \( x \in C_M \), or equivalently, \((x_0^+ - x_0)^T (\bar{y} - Q x_0^+) \geq 0\) for all \( \bar{y} \in C_M \). Then, we obtain \(-Q(x_0^- - x_0) \in (C_M)^b\) and hence \( x_0^- - x_0 \in J_M \). Uniqueness of \( x_0^+ \) readily follows from positive definiteness of \( Q \). To prove the rest, note that \( H \) is a convex differentiable function. Then, it follows from [13, Thm. 2.13] that

\[
H(x_0) \geq H(x_0^+) + (x_0 - x_0^+)^T \frac{\partial H}{\partial x}(x_0^+) = H(x_0^+) + (x_0 - x_0^+)^T Q x_0^+.
\]

By using (27), we get \( H(x_0) \geq H(x_0^+) + (x_0 - x_0^+)^T Q \bar{x} \) for all \( \bar{x} \in C_M \). In particular, this yields \( H(x_0) \geq H(x_0^+) \) whenever \( 0 \in C_M \).

**Remark V.4** In this remark we will indicate how the above jump rule generalizes the jump rule for port-Hamiltonian systems with linear resistive relations as given before in [15].

As alluded to before, in our previous paper [15] the maximal monotone relation \( M \) is given as \( D \circ R \), where \( D \) is a Dirac structure and \( R \) is a linear resistive (energy-dissipating) relation \( e_R = -R f_R \) with \( R > 0 \). The definition of the constraint subset \( C_M \) can be seen to be the same as in [15], (Note furthermore that in case \( M = D \circ R \) the subset \( C_M \) is actually a linear subspace.)

Also the definition of the jump space \( J_M \) directly extends the definition given in [15]. First we note that in case of a linear subspace \( K \) the set \( K_{\infty} \) equals \( K \). It follows that for \( M = D \circ R \) the jump space \( J_M \) is the linear subspace given as

\[
J_M = \{ f_x \mid \exists f_R, e_R \text{ s.t. } e_R = -R f_R, (f_x, 0, f_R, e_R) \in D \}
\]

On the other hand, \((f_x, 0, f_R, e_R) \in D\) implies that \( 0^T f_x + e_R^T f_R = 0 \), and thus \( e_R^T R e_R = 0 \). Hence, since \( R \) is positive definite, this implies \( e_R = f_R = 0 \). Hence \( J_M \) is also given as

\[
J_M = \{ f_x \mid (f_x, 0, 0, 0) \in D \},
\]

which is exactly the definition given in [15]. Note that in this case \( J_M = (C_M)^b \), while for a general maximal monotone relation we only obtain the weaker (but more general) property \(-J_M = (C_M)^b\).

VI. CONCLUSIONS

In this paper we introduced incrementally port-Hamiltonian systems in a way that is very much analogous to the definition of a (standard) port-Hamiltonian system, by replacing the composition of a Dirac structure and an energy-dissipating relation by a maximal monotone relation. Very similar to the property that the port-Hamiltonian structure is preserved under compositions, we proved that the composition of two incrementally port-Hamiltonian systems is again an incrementally port-Hamiltonian system. Also, we studied existence and uniqueness of state trajectories for this class of systems as well as an energy-based state re-initialization principle for those initial state which do not yield a ‘smooth’ solution.

Further research directions include extensions of incrementally port-Hamiltonian systems to infinite dimensional spaces as well as investigation of the relationships between the proposed state re-initialization principle and the charge/flux conservation principles that are employed in the context of electrical circuits.

REFERENCES


