Delay-Independent Decentralized Fixed Modes for Multi-Channel LTI Systems Subject to Input and Output Delays

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Abstract—This paper concerns stabilizability of a multi-channel linear time-invariant (LTI) system with delay in inputs and outputs by means of a finite-dimensional decentralized output feedback controller. This problem can be tackled by obtaining the set of unstable decentralized fixed modes (DFMs) of the system. Although the set of DFMs for LTI time-delay systems is generally delay-dependent, there exists a certain type of DFMs, which are delay-independent. In this work, the notion of delay-independent DFM (DIDFM) is defined, and algebraic conditions are provided under which a mode of a multi-channel LTI system subject to input and output delays is a DIDFM. The result of this paper is more general than that of [1], due to the fact that all the modes of the system in [1] are assumed distinct.

I. INTRODUCTION

Decentralized control design has been an important challenge in control theory for the past few decades (for example, see [2], [3], [4], [5], [6] and references therein). In the literature of decentralized control systems, two classes of models have been considered: multi-channel, and interconnected systems [7]. These models are distinguished based on the internal structure of the system (for a mathematical description of these models, see [7]).

Decentralized stabilization of multi-channel finite-dimensional linear time-invariant (LTI) systems (the no-delay case) has been investigated for several years (e.g., see [2], [8], [9]). The work [2] introduces the notion of decentralized fixed mode (DFM) which provides a necessary and sufficient condition for the decentralized stabilizability of multi-channel LTI systems; in particular, it is shown in [2] that a DFM remains fixed in the complex plane, using any type of LTI decentralized dynamic controller. An algebraic characterization of DFMs was first presented in [8] for strictly proper system, and was then extended to the case of general proper systems in [9].

Decentralized stabilization of interconnected LTI time-delay systems has been investigated in a number of works [10], [11], [12], [13], [14], [15], [16]. Stabilization of interconnected LTI time-delay systems using local state feedback control is studied in [10] where matching conditions are assumed, and the time-delays appear only in the interconnections of the system. The work [11] attempts to relax these matching conditions for a more restrictive class of systems [12]. Using an LMI technique, delay-dependent sufficient stabilizability conditions are proposed for interconnected LTI time-delay systems in [13], and the case of time-varying delays is discussed in [14], [15], [16].

The paper [7] studies stabilizability of interconnected and multi-channel LTI systems, where sufficient conditions for stabilization of a multi-channel LTI system subject to time-varying delays using a decentralized state-feedback control law is presented. The notion of DFM for multi-channel LTI systems subject to commensurate delays was proposed in [17] for the first time in the literature, where necessary and sufficient conditions for the stabilizability of the system by means of decentralized LTI output feedback controllers is obtained. It was later shown in [18] that similar results hold for a more general class of multi-channel LTI systems with multiple distinct delays in dynamics. As discussed in [1], the set of DFMs for a LTI time-delay system, generally speaking, depends on the length of delay. Nevertheless, a DFM of a LTI time-delay system may remain fixed for any value of delay. Algebraic conditions are provided in [1] to determine when a distinct mode of a multi-channel LTI system subject to input and output (I/O) delays is a delay-independent DFM (DIDFM).

This work also studied the characterization of DIDFMs for the class of multi-channel LTI systems subject to I/O delays. In contrast with [1], it is not assumed that the modes of the system have to be distinct in the paper, i.e. repeated modes for the system are allowed. Regarding the main contribution of the paper, rank conditions are presented under which a mode of a multi-channel LTI system with I/O delays is DIDFM. To arrive at the main result of the work, two linear-algebraic lemmas are initially obtained. The equivalence of the conditions given in this paper with the results obtained for the special case studied in [1] is discussed. A numerical example is provided to demonstrate the importance of the main result of the paper.

The organization of the remainder of the paper is as follows. In Section II, the problem statement is first presented, and the main result of the paper regarding DIDFMs is then shown. A numerical example is given in Section III, and some concluding remarks are made in Section IV.

II. DIDFMS FOR MULTI-CHANNEL LTI SYSTEMS WITH I/O DELAYS

A. Problem statement

Notation: The set of real and complex numbers are denoted by \( \mathbb{R} \) and \( \mathbb{C} \), respectively. \( 0_{m \times n} \) is the zero matrix of the size \( m \times n \), and \( I_n \) is the identity matrix of the size \( n \). When no size is specified, the dimension can be
understood from the context. Given an integer number \( n \geq 1 \),
the following definition will be used:

\[
\text{diag}[a_1, a_2, \ldots, a_n] := \begin{pmatrix}
a_1 & 0 & \cdots & 0 \\
0 & a_2 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & a_n
\end{pmatrix}
\]

where \( a_i \in \mathbb{C} (\in \mathbb{R}) \), \( i = 1, 2, \ldots, n \).

Consider the following multi-channel LTI time-delay system with \( \nu \) subsystems with delay \( h \) in the input and output

\[
\dot{x}(t) = Ax(t) + \sum_{i=1}^{\nu} [B_{i,0}u_i(t) + B_{i,1}u_i(t-h)]
\]

\[
y_i(t) = C_{i,0}x(t) + C_{i,1}x(t-h), \quad i \in \tilde{\nu} := \{1, 2, \ldots, \nu\}
\]

where \( x(t) \in \mathbb{R}^n \) is the state, \( u_i(t) \in \mathbb{R}^{m_i} \) and \( y_i(t) \in \mathbb{R}^{p_i} \) are the input and output of the \( i \)-th local subsystem, respectively. The matrices \( A \in \mathbb{R}^{n \times n} \), \( B_{i,l} \in \mathbb{R}^{n \times m_i} \) and \( C_{i,l} \in \mathbb{R}^{p_i \times n} \), \( i \in \tilde{\nu} \) and \( l = 0, 1 \), are assumed to be real and constant. Clearly, \( h \geq 0 \).

The mode \( \sigma \in \text{sp}(A) \) (spectrum of \( A \)) is a DFM if for any \( K_i \in \mathbb{R}^{m_i \times p_i} \) [18]

\[
\det \left( \sigma I - A - \sum_{i=1}^{\nu} B_{i}(e^{-\sigma h})C_{i}(e^{-\sigma h}) \right) = 0
\]

where

\[
B_{i}(e^{-\sigma h}) = B_{i,0} + B_{i,1}e^{-\sigma h} \\
C_{i}(e^{-\sigma h}) = C_{i,0} + C_{i,1}e^{-\sigma h}
\]

According to [18], if system (1) has no unstable DFM, there exists a decentralized controller with the \( i \)-th local control station of the form

\[
\dot{z}_i(t) = \Gamma_i z_i(t) + R_i y_i(t) \\
u_i(t) = Q_i z_i(t) + K_i y_i(t), \quad i \in \tilde{\nu}
\]

which stabilizes system (1), and vice versa. In (3), \( z_i(t) \in \mathbb{R}^{\overline{\nu}} \) is the state of the local controller, and \( \Gamma_i, \ R_i, \ Q_i, \ \text{and} \ K_i \) are real constant matrices of appropriate size.

The set of DFMs for system (1) is most often delay-dependent; i.e., \( \sigma \in \text{sp}(A) \) could be a DFM for \( h = h_1 \), but not for \( h = h_2 \), where \( h_1 \neq h_2 \). Nevertheless, system (1) may possess a mode \( \sigma \in \text{sp}(A) \) which is a DFM for any value of \( h \). Such a mode is called a DIDFM (delay-dependent DFM). If system (1) has an unstable DIDFM, no decentralized controller with the local control station of the form (3) can be found to stabilize the system, for any value of \( h \). Thus, it is of interest to identify if a mode is DIDFM. In this paper, rank conditions are presented under which a mode of the system (1) is a DIDFM.

B. Main result

The main result of this paper which characterizes DIDFMs is given in this subsection. To obtain this result, two lemmas (Lemmas 1 and 2) are first derived. The following theorem, which is directly borrowed from [19, pp. 122], is required for the proof of Lemma 1.

**Theorem 1:** Consider a function \( f : \mathbb{C}^\kappa \rightarrow \mathbb{C} \) (\( f : \mathbb{R}^\kappa \rightarrow \mathbb{R} \)) for a given integer number \( \kappa \geq 1 \), and let \( f \) be an analytic function in a connected open set \( D \subseteq \mathbb{C}^\kappa \) (\( D \subseteq \mathbb{R}^\kappa \)), and assume \( x_0 \in D \). Then, the following two statements are equivalent:

i) \( f \) is identically zero in a neighborhood of \( x_0 \).

ii) \( f \) is identically zero in \( D \).

**Lemma 1:** Given positive integers \( n \) and \( \pi_i \)'s, \( i \in \tilde{\nu} \), consider matrices \( P \in \mathbb{C}^{n \times n} \), \( N_i \in \mathbb{C}^{n \times \pi_i} \), \( M_i \in \mathbb{C}^{\pi_i \times \pi_i} \), and \( N_i \in \mathbb{C}^{\pi_i \times n} \). Then the \( n \times n \) matrix

\[
P - \sum_{i=1}^{\nu} M_i L_i N_i
\]

has rank \( < n \) for any \( L_i \in \mathbb{C}^{\pi_i \times \pi_i} \), if and only if the square matrix

\[
\begin{bmatrix}
P & M_1 & M_2 & \cdots & M_\nu \\
N_1 & L_1 & 0 & \cdots & 0 \\
N_2 & 0 & L_2 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
N_\nu & 0 & 0 & \cdots & L_\nu
\end{bmatrix}
\]

is not full-rank for any \( L_i \in \mathbb{C}^{\pi_i \times \pi_i} \), \( i \in \tilde{\nu} \).

**Proof of necessity:** First, assume that the \( L_i \)'s, \( i \in \tilde{\nu} \), matrices are invertible. In this case, one can conclude that

\[
\begin{bmatrix}
P & M_1 & M_2 & \cdots & M_\nu \\
N_1 & L_1 & 0 & \cdots & 0 \\
N_2 & 0 & L_2 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
N_\nu & 0 & 0 & \cdots & L_\nu
\end{bmatrix}
\times
\begin{bmatrix}
R_1 \cdots R_\nu
\end{bmatrix}
\]

which implies that

\[
\begin{bmatrix}
P & M_1 & M_2 & \cdots & M_\nu \\
N_1 & L_1 & 0 & \cdots & 0 \\
N_2 & 0 & L_2 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
N_\nu & 0 & 0 & \cdots & L_\nu
\end{bmatrix}
\]

\[
= \frac{\det \left( P - \sum_{i=1}^{\nu} M_i L_i^{-1} N_i \right)}{\prod_{i=1}^{\nu} \det L_i^{-1}}
\]

(6)
Thus, if (4) is not full-rank, the determinant of (5) is zero for all full-rank $L_i$’s. Next, it will be shown that if the determinant of (5) is zero for full-rank $L_i$, it is zero for any $L_i \in \mathbb{C}^{\bar{\nu} \times \bar{\nu}}$, whether $L_i$ is invertible or not. Now, let $L_i^0$, $i \in \bar{\nu}$, be invertible; i.e. det $L_i^0 \neq 0$. Denote the elements of matrix $L_i^0$ with $l_i^0(\alpha, \beta)$, $\alpha, \beta = 1, 2, \ldots, \pi_i$. From the continuity property of a determinant as a function of the matrix elements, one can conclude that for any $i \in \bar{\nu}$, there are positive real numbers $\delta_i(\alpha, \beta)$’s, $\alpha, \beta = 1, 2, \ldots, \pi_i$, such that for any $L_i$ with the property that its $(\alpha, \beta)$ element satisfies $|l_i(\alpha, \beta) - l_i^0(\alpha, \beta)| < \delta_i(\alpha, \beta)$, det $L_i \neq 0$. This implies that for any $i \in \bar{\nu}$, there are positive real numbers $\delta_i(\alpha, \beta)$’s such that the determinant of (5) is zero for any $L_i$ with the property that its $(\alpha, \beta)$ element satisfies $|l_i(\alpha, \beta) - l_i^0(\alpha, \beta)| < \delta_i(\alpha, \beta)$. Since the determinant of (5) is an analytic function in terms of the elements of $L_i$’s, Theorem 1 concludes that determinant of (5) is zero for any $L_i \in \mathbb{C}^{\pi_i \times \pi_i}$, $(\in \mathbb{R}^{\pi_i \times \pi_i})$.

Proof of sufficiency: First, assume that $L_i$’s, $i \in \bar{\nu}$, matrices are invertible. From (6), it follows that

$$\det \left( P - \sum_{i=1}^{\nu} M_i L_i N_i \right) = \det (P - \sum_{i=1}^{\nu} M_i L_i N_i)$$

Thus, if (5) is not full-rank, determinant of (4) is zero for full-rank $L_i$’s. Similar to the proof of necessity, it can be shown that determinant of (4) is zero for any $L_i \in \mathbb{C}^{\pi_i \times \pi_i}$, $(\in \mathbb{R}^{\pi_i \times \pi_i})$, whether it is invertible or not.

The following theorem is borrowed from [8, Theorem 2.1], which is required for the proof of Lemma 2.

Theorem 2: Given a positive integer $r$, define $\bar{\rho} := \{1, 2, \ldots, r\}$. Let $M_i$, and $N_i$, $i \in \bar{\rho}$, be real (complex) matrices of size $\rho \times \gamma_i$ and $\rho \times \delta_i$, respectively. Assume also that $\rho = \sum_{i=1}^{\bar{\rho}} \gamma_i$. Then a necessary and sufficient condition for the following square matrix

$$[ M_1 + N_1 L_1 \ M_2 + N_2 L_2 \ \cdots \ M_r + N_r L_r ]$$

to not being full-rank for all $\delta_i \times \gamma_i$ real (complex) matrices $L_i$, $i \in \bar{\rho}$, is that there exists a nonempty subset $\Phi := \{i_1, i_2, \ldots, i_k\}$ of $\bar{\rho}$ so that

$$\text{rank} \left[ M_{i_1} \ N_{i_1} \ \cdots \ M_{i_k} \ N_{i_k} \right] < \sum_{i \in \Phi} \gamma_i$$

Lemma 2: Given positive integers $n$, $m_i$’s, and $p_i$’s, $i \in \bar{\nu}$, consider matrices $P \in \mathbb{C}^{n \times n}$, $(\in \mathbb{R}^{n \times n})$, $M_i \in \mathbb{C}^{n \times m_i}$, $(\in \mathbb{R}^{n \times m_i})$, and $N_i \in \mathbb{C}^{p_i \times n}$, $(\in \mathbb{R}^{p_i \times n})$. Then the $n \times n$ matrix

$$P - \sum_{i=1}^{\nu} M_i L_i N_i$$

is not full-rank, where $\pi_i := \max\{m_i, p_i\}$. Then, one can conclude from Lemma 1 that

$$\begin{bmatrix} P & M_1 & \cdots & M_k \\ N_{i+1} & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ N_{i\nu} & 0 & \cdots & 0 \end{bmatrix} < n$$

Proof: First, the proof of necessity is given. Construct the matrices $\hat{M}_i$ and $\hat{N}_i$, $i \in \bar{\nu}$, as follows

$$\hat{M}_i = \begin{cases} M_i & p_i > m_i \\ M_i, & m_i \geq p_i \end{cases}$$

$$\hat{N}_i = \begin{cases} N_i & m_i \geq p_i \\ 0_{(m_i - p_i) \times n}, & p_i > m_i \end{cases}$$

Assume that (7) is not full rank for any $L_i \in \mathbb{C}^{m_i \times p_i}$, $(\in \mathbb{R}^{m_i \times p_i})$, then it is easy to show that for any $L_i \in \mathbb{C}^{\pi_i \times \pi_i}$, $(\in \mathbb{R}^{\pi_i \times \pi_i})$, $i \in \bar{\nu}$,

$$P - \sum_{i=1}^{\nu} \hat{M}_i \hat{L}_i \hat{N}_i$$

is not full-rank, where $\pi_i := \max\{m_i, p_i\}$. Then, one can conclude from Lemma 1 that

$$\begin{bmatrix} P & \hat{M}_1 & \cdots & \hat{M}_\nu \\ \hat{N}_1 & \hat{L}_1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \hat{N}_\nu & 0 & \cdots & \hat{L}_\nu \end{bmatrix}$$

is not full-rank for any $L_i \in \mathbb{C}^{\pi_i \times \pi_i}$, $(\in \mathbb{R}^{\pi_i \times \pi_i})$, $i \in \bar{\nu}$. On the other hand, (8) can be written as

$$\begin{bmatrix} M_1 + N_1 \hat{L}_0 & M_2 + N_2 \hat{L}_1 & M_3 + N_3 \hat{L}_2 & \cdots \\ \vdots & \ddots & \vdots & \vdots \\ M_{\nu+1} + N_{\nu+1} \hat{L}_\nu \end{bmatrix}$$

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where

\[
\begin{align*}
\mathcal{M}_1 &= \begin{bmatrix} P & \hat{N}_1 & \cdots & \hat{N}_\nu \end{bmatrix}, \\
\mathcal{M}_2 &= \begin{bmatrix} \hat{M}_1 & \cdots & 0 \pi_1 \times \pi_1 & \cdots & 0 \pi_\nu \times \pi_1 \end{bmatrix}, \\
& \vdots \\
\mathcal{M}_{\nu+1} &= \begin{bmatrix} \hat{N}_{\nu+1} & 0 \pi_1 \times \pi_1 & \cdots & 0 \pi_\nu \times \pi_1 \end{bmatrix}, \\
& \vdots \\
\mathcal{M}_l &= \begin{bmatrix} 0 \pi_1 \times \pi_1 & \cdots & 0 \pi_\nu \times \pi_1 \end{bmatrix}, \\
\mathcal{N}_1 &= \begin{bmatrix} \hat{N}_1 & \cdots & \hat{N}_\nu \end{bmatrix}, \\
\mathcal{N}_2 &= \begin{bmatrix} \hat{M}_1 & \cdots & 0 \pi_1 \times \pi_1 & \cdots & 0 \pi_\nu \times \pi_1 \end{bmatrix}, \\
& \vdots \\
\mathcal{N}_{\nu+1} &= \begin{bmatrix} \hat{N}_{\nu+1} & 0 \pi_1 \times \pi_1 & \cdots & 0 \pi_\nu \times \pi_1 \end{bmatrix}, \\
& \vdots \\
\mathcal{N}_l &= \begin{bmatrix} 0 \pi_1 \times \pi_1 & \cdots & 0 \pi_\nu \times \pi_1 \end{bmatrix},
\end{align*}
\] (10a)

and \( \hat{L}_0 \) is an arbitrary \( n \times n \) complex (real) matrix. Now, let Theorem 2 be applied to (9) where in this case,

\[
r = \nu + 1, \quad \gamma_1 = \delta_1 = n \quad \rho = n + \sum_{i=1}^{\nu} \pi_i
\]

\[
\gamma_i = \delta_i = \pi_{i-1}, \quad i = 2, \ldots, \nu + 1
\]

Since (9) is not full-rank for any \( \hat{L}_i, i \in \bar{\nu} \cup \{0\} \), it is obtained from Theorem 2 that at least one of the following cases occurs:

Case I: \( \Phi = \{1\} \). In this case, Theorem 2 implies that

\[
\text{rank} \begin{bmatrix} P & 0 & \cdots & \hat{N}_1 & 0 \end{bmatrix} < \gamma_1,
\]

which is equivalent to

\[
\text{rank} \begin{bmatrix} P & \hat{N}_1 \end{bmatrix} < \gamma_1
\]

and one can easily show that this implies

\[
\text{rank} \begin{bmatrix} P & \hat{N}_1 \end{bmatrix} < \gamma_1
\]

which is equivalent with

\[
\begin{align*}
\text{rank} \begin{bmatrix} P & \hat{M}_1 & \cdots & \hat{M}_l \\
\hat{N}_1 & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
\hat{N}_\nu & 0 & \cdots & 0
\end{bmatrix} + \begin{bmatrix} 0 & \cdots & 0 \pi_1 \times \pi_1 \\
\vdots & \ddots & \vdots & \vdots \\
0 & \cdots & 0 \pi_\nu \times \pi_1 \\
\hat{N}_{\nu+1} & \cdots & 0 \pi_1 \times \pi_1 & \cdots & 0 \pi_\nu \times \pi_1
\end{bmatrix}
\end{align*}
\]

which is equivalent to

\[
\begin{align*}
\text{rank} \begin{bmatrix} P & \hat{M}_1 \end{bmatrix} + \begin{bmatrix} 0 & \cdots & 0 \pi_1 \times \pi_1 \\
\hat{N}_1 & 0 & \cdots & 0 \pi_\nu \times \pi_1
\end{bmatrix}
\end{align*}
\]

and then

\[
\text{rank} \begin{bmatrix} P & \hat{M}_1 \end{bmatrix} < \gamma_1
\]

Case II: \( \Phi = \{1, 2, \ldots, \nu + 1\} \). In this case, Theorem 2 results that

\[
\begin{align*}
\text{rank} \begin{bmatrix} P & \hat{M}_1 & \cdots & \hat{M}_l \\
\hat{N}_1 & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
\hat{N}_\nu & 0 & \cdots & 0 \pi_1 \times \pi_1 & \cdots & 0 \pi_\nu \times \pi_1
\end{bmatrix} < n + \sum_{i=1}^{\nu+1} \pi_i,
\end{align*}
\]

One can easily show that this concludes

\[
\text{rank} \begin{bmatrix} P & \hat{M}_1 & \cdots & \hat{M}_l \\
\hat{N}_1 & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
\hat{N}_\nu & 0 & \cdots & 0 \pi_1 \times \pi_1 & \cdots & 0 \pi_\nu \times \pi_1
\end{bmatrix} < n
\]

Case III: \( \Phi = \{1, 2, \ldots, l + 1\} \), where \( l \) is an integer satisfying 0 < \( l < \nu \). In this case, Theorem 2 implies that

\[
\begin{align*}
\text{rank} \begin{bmatrix} P & \hat{M}_1 & \cdots & \hat{M}_l \\
\hat{N}_1 & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
\hat{N}_l & 0 & \cdots & 0 \pi_1 \times \pi_1 & \cdots & 0 \pi_\nu \times \pi_1
\end{bmatrix} + \begin{bmatrix} 0 & \cdots & 0 \pi_1 \times \pi_1 \\
\vdots & \ddots & \vdots & \vdots \\
0 & \cdots & 0 \pi_\nu \times \pi_1 \\
\hat{N}_{l+1} & \cdots & 0 \pi_1 \times \pi_1 & \cdots & 0 \pi_\nu \times \pi_1
\end{bmatrix}
\end{align*}
\]

which is equivalent with

\[
\begin{align*}
\text{rank} \begin{bmatrix} P & \hat{M}_1 & \cdots & \hat{M}_l \\
\hat{N}_1 & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
\hat{N}_l & 0 & \cdots & 0 \pi_1 \times \pi_1 & \cdots & 0 \pi_\nu \times \pi_1
\end{bmatrix}
\end{align*}
\]

and then

\[
\text{rank} \begin{bmatrix} P & \hat{M}_1 & \cdots & \hat{M}_l \\
\hat{N}_1 & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
\hat{N}_l & 0 & \cdots & 0 \pi_1 \times \pi_1 & \cdots & 0 \pi_\nu \times \pi_1
\end{bmatrix}
\]

\[< n\]
Moreover, it is straightforward to show that
\[
\text{rank} \begin{bmatrix}
P & M_1 & \cdots & M_l \\
N_{l+1} & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
N_{\nu} & 0 & \cdots & 0 \\
\end{bmatrix} < n
\]
In the sequel, it is shown that the selection of \( \Phi \) such that \( 1 \notin \Phi \) is an impossible case. Let \( \Phi \) be represented as \( \{i_1, \ldots, i_l\} \) where \( i_s \neq 1, s = 1, \ldots, l \). From Theorem 2,
\[
\text{rank} \begin{bmatrix}
M_{i_1} & \cdots & M_{i_l} & N_{i_1} & \cdots & N_{i_l} \\
\end{bmatrix} < \sum_{i \in \Phi} \pi_{i-1}
\]
where matrices \( M_i \) and \( N_i \) are defined according to (10a) and (10b), respectively. This implies that
\[
\text{rank} \begin{bmatrix}
N_{i_1} & \cdots & N_{i_l} \\
\end{bmatrix} < \sum_{i \in \Phi} \pi_{i-1}
\]
On the other hand, it is simple to verify from definition of matrices \( N_i \) in (10b) that
\[
\text{rank} \begin{bmatrix}
N_{i_1} & \cdots & N_{i_l} \\
\end{bmatrix} = \sum_{i \in \Phi} \pi_{i-1}
\]
This leads to a contradiction. Furthermore, it is noted that any choice of \( \Phi \) other than the cases investigated above, can be translated to Case III by a simple reordering of indices. This completes the proof of necessity of Lemma 2. The proof of sufficiency directly follows by reversing the above argument. \( \blacksquare \)

Now, the main result of this paper is given.

Theorem 3: Consider system (1). If there exists at least one partition of the set \( \nu \) into disjoint (not necessarily, nonempty) subsets \( \{i_1, \ldots, i_k\} \) and \( \{i_{k+1}, \ldots, i_{\nu}\} \) such that the rank of
\[
\begin{bmatrix}
\sigma I - A - \sum_{i=1}^{\nu} [B_{i_0} B_{i_1} \cdots B_{i_{k0}} B_{i_{k1}}] & C_{i_0} \\
C_{i_{k+1,0}} & 0 & \cdots & 0 \\
C_{i_{k+1,1}} & \vdots & \ddots & \vdots \\
C_{i_{\nu,0}} & 0 & \cdots & 0 \\
\end{bmatrix} (11)
\]
is less than \( n \), then \( \sigma \) is a DIDFM.

Proof: If (11) holds, one can deduce from Lemma 2
\[
\sigma I - A - \sum_{i=1}^{\nu} [B_{i_0} B_{i_1}] L_i \begin{bmatrix} C_{i_0} \\ C_{i_1} \end{bmatrix} 
= \begin{bmatrix} K_i & K_i e^{-\sigma h} \\ K_i e^{-\sigma h} & K_i e^{-2\sigma h} \end{bmatrix} L_i \begin{bmatrix} C_{i_0} \\ C_{i_1} \end{bmatrix}
\]
is a full-rank for any \( L_i \in \mathbb{C}^{2m_i \times 2p_i} \). Now let
\[
L_i = \begin{bmatrix} K_i & K_i e^{-\sigma h} \\ K_i e^{-\sigma h} & K_i e^{-2\sigma h} \end{bmatrix}
\]
where \( K_i \) is an arbitrary \( m_i \times p_i \) real matrix, and \( h \) is an arbitrary delay value. In this case, it is straightforward to show that
\[
\begin{bmatrix} B_{i_0} B_{i_1} \end{bmatrix} L_i \begin{bmatrix} C_{i_0} \\ C_{i_1} \end{bmatrix} = (B_{i_0} + B_{i_1} e^{-\sigma h}) K_i (C_{i_0} + C_{i_1} e^{-\sigma h}) = B_i (e^{-\sigma h}) K_i C_i (e^{-\sigma h})
\]
Therefore, it follows from (12) that
\[
\det \left( \sigma I - A - \sum_{i=1}^{\nu} B_i (e^{-\sigma h}) K_i C_i (e^{-\sigma h}) \right) = 0
\]
for any \( K_i \in \mathbb{R}^{m_i \times 2p_i} \) and any arbitrary delay \( h \), which implies that \( \sigma \) is a DIDFM. \( \blacksquare \)

Remark 1: As an important special case, assume that
\[
A = \text{diag} \{\sigma_1, \ldots, \sigma_j, \ldots, \sigma_n\},
\]
where \( \sigma_k \neq \sigma_i \), for \( k \neq l \), and \( k, l = 1, 2, \ldots, n \). Suppose also that (11) holds for \( \sigma = \sigma_j \). Let the \((\eta_1, \eta_2)\) entry of the matrices \( B_{i,l} \) and \( C_{i,l} \), \( i \in \nu, l = 0, 1 \), be denoted by \( b_{\eta_1, \eta_2}^{i,l} \) and \( c_{\eta_1, \eta_2}^{i,l} \), respectively. Now, one can show, similar to the proof of Theorem 3 in [1], that condition (11), in this case, is equivalent to the following two conditions being both satisfied:

i. \( b_{\eta_1, \eta_2}^{i,l} = c_{\eta_1, \eta_2}^{i,l} = 0 \)

ii. \( \sum_{j=1, j \neq i}^{n} b_{\eta_1, \eta_2}^{i,j} (\sigma_j - \sigma) = 0 \)

for any \( \eta \in \{i_1, \ldots, i_k\} \), \( \gamma \in \{i_{k+1}, \ldots, i_{\nu}\} \), \( \alpha \in \{1, \ldots, m_i\} \), \( \beta \in \{1, \ldots, p_i\} \), and \( l, l' \in \{0, 1\} \). It is interesting to note that the above conditions are indeed the ones presented in Theorem 4 of [1].

III. AN ILLUSTRATIVE EXAMPLE

Consider the following two-input two-output system
\[
\begin{align*}
\dot{x}(t) &= Ax(t) + \sum_{i=1}^{2} [B_{i,0} u_i(t) + B_{i,1} u_i(t - h)] \\
y_i(t) &= C_{i,0} x(t) + C_{i,1} x(t - h), \quad i = 1, 2
\end{align*}
(13)
\]
where
\[
A = \begin{bmatrix} 1 & -2 & 1 \\
-3 & -4 & 3 \\
2 & 4 & 0 \end{bmatrix},
\]
\[
B_{1,1} = \begin{bmatrix} -1 \\
-3 \\
2 \end{bmatrix}, \quad B_{2,0} = \begin{bmatrix} 0 \\
0 \\
1 \end{bmatrix},
\]
\[
C_{1,0} = \begin{bmatrix} 1 & 0 & -1 \end{bmatrix}, \quad C_{2,0} = \begin{bmatrix} 1 & 2 & 0 \end{bmatrix}, \quad C_{2,1} = \begin{bmatrix} -1 & -2 & 0 \end{bmatrix},
\]
\[
B_{1,0} = B_{2,1} = 0_{3 \times 1}, \quad \text{and } C_{1,1} = 0_{1 \times 3}.
\]
According to (2), for this system,
\[
B_1 (e^{-\sigma h}) = \begin{bmatrix} -1 e^{-\sigma h} \\
-3 e^{-\sigma h} \\
2 e^{-\sigma h} \end{bmatrix}, \quad B_2 (e^{-\sigma h}) = \begin{bmatrix} 0 \\
0 \\
1 \end{bmatrix}, \quad C_1 (e^{-\sigma h}) = \begin{bmatrix} 1 & 0 & -1 \end{bmatrix}, \quad C_2 (e^{-\sigma h}) = \begin{bmatrix} 1 & -e^{-\sigma h} & 2 - 2 e^{-\sigma h} & 0 \end{bmatrix}.
\]
It is easy to verify that $\sigma = 2$ is a repeated mode of the system. Since

$$
\text{rank} \begin{bmatrix}
\sigma I - A & B_{1,0} & B_{1,1} \\
C_{2,0} & 0 & 0 \\
C_{2,1} & 0 & 0
\end{bmatrix} = \text{rank} \begin{bmatrix}
\sigma I - A & B_{1,1} \\
C_{2,0} & 0 \\
C_{2,1} & 0
\end{bmatrix} = \begin{bmatrix}
1 & 2 & -1 & -1 \\
3 & 6 & -3 & -3 \\
-2 & -4 & 2 & 2 \\
1 & 2 & 0 & 0 \\
-1 & -2 & 0 & 0
\end{bmatrix}
$$

$= 2 < 3,$

it can be concluded from Theorem 3 that $\sigma = 2$ is a DIDFM. This can also be checked from definition of DFM. For this purpose, consider the following matrix where $k_1$ and $k_2$ are arbitrary real scalars:

$$
\sigma I - A - \sum_{i=1}^{2} B_i(e^{-\sigma h})k_i C_i(e^{-\sigma h}) = 
\begin{bmatrix}
1 + k_1 z & 2 \\
3 + 3k_1 z & 6 \\
-2 - 2k_1 z - k_2(1 - z) & -4 - 2k_2(1 - z) \\
-1 - k_1 z & -3 - 3k_1 z \\
2 + 2k_1 z
\end{bmatrix} = 0
$$

for any $k_1$, $k_2$, and $h$. This means that no decentralized output feedback controller can be found to stabilize (13), independent of the delay. It is worth mentioning that the conditions introduced in [1] cannot be used to determine if $\sigma$ is a DIDFM.

IV. CONCLUSIONS

The problem of stabilizability of a multi-channel LTI time-delay system with respect to the class of decentralized LTI output feedback controllers can be addressed using the notion of DFM. Generally speaking, the set of DFMs of a multi-channel LTI time-delay system depends on the length of the delay. However, such a system may have a DFM which remains a fixed mode for any value of delay. In this work, a multi-channel system subject to input and output delays is considered. The notion of a delay-independent DFM (DIDFM) is defined for this class of systems, and rank conditions are given under which a mode of the system is a DIDFM. It is also shown that for the special case investigated in [1], the conditions are equivalent to the ones obtained in [1]. A numerical example is provided to illustrate application of the main result (Theorem 3).

REFERENCES
