TCP Reno and queue management: 
local stability and Hopf bifurcation analysis

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Abstract—We study local stability and the local Hopf bifurcation properties of a fluid model for TCP Reno coupled with queue management schemes in routers. We analyse models for the widely deployed Drop-Tail queue policy, over a small and an intermediate buffer regime. We also study a model for a threshold based queue policy. In small Drop-Tail buffers, stability depends crucially on the buffer size. With intermediate buffer Drop-Tail, the system can be locally unstable for large values of the feedback delay and link capacities. The threshold based policy highlights the importance of setting queue thresholds guided by stability analysis. We show that variations in system parameters can produce Hopf induced limit cycles. It is practically important to characterise the existence, uniqueness and stability of the bifurcating periodic solutions. Using the theory of normal forms and the center manifold theorem, we establish that the Hopf bifurcation is indeed supercritical. Packet-level simulations for the Drop-Tail queue policy corroborate our theoretical analysis.

I. INTRODUCTION

The Transmission Control Protocol (TCP) plays a key role in providing end-to-end performance of applications that run over the Internet. There is growing consensus that latency, in the network, is on the rise which could be rather detrimental to end-to-end Quality of Service (QoS); for example, see [2], [9], [11]. A key contributor to latency is large, and unmanaged, router buffers. Thus the choice and design of queue management policies form an integral aspect of network design. Our focus, in this paper, is on the stability and bifurcation properties of some queue management policies with TCP Reno.

When queues in routers get overloaded, packets may get dropped. Packets that are dropped serve to provide indications of congestion, in the network, to end-systems. Thus the choice of the variant of TCP, the router buffer dimensioning rules, and the choice of queue policies all have an impact on network performance. For models of TCP Reno see [10], [12], [14], for some early work on sizing router buffers see [15], [17], and for motivation of some queuing models see [6], [15]. There is continued interest in the design of new variants of TCP; for some very recent work, see [16]. Today Cubic TCP is implemented in the Linux operating system, and Compound TCP in the Windows operating system. Both these variants build on TCP Reno, which is a standardised transport layer protocol. Thus, for now, our work focuses on this variant of TCP.

In this paper, we study TCP Reno in combination with three different queue models. The models we study are a Drop-Tail queue policy, in a small and an intermediate buffer regime, and a threshold based queue management policy. Using a combination of queuing, control and bifurcation theory, supplemented with numerical computations and packet-level simulations, we analyse the models for their local stability and Hopf bifurcation properties. We first outline sufficient conditions for local stability, and then also characterise the necessary and sufficient conditions. Transition from stability to instability is shown to occur via a Hopf bifurcation as system parameters may vary. A practical manifestation of the Hopf bifurcation, in communication networks, is the formation of limit cycles in the queue size. Employing the theory of normal forms and the center manifold theorem [3], [5], we explicitly show that the Hopf bifurcation is supercritical. Thus we are able to theoretically characterise the existence, uniqueness and orbital stability of the bifurcating periodic solutions. The existence of the predicted stable limit cycles are confirmed with packet-level simulations using the Network Simulator (NS2) [18].

The rest of this paper is organised as follows. In Section II, we describe a non-linear fluid model for TCP Reno and the queue policies which we study. Section III contains the local stability analysis. In Section IV, we show that the local Hopf bifurcation is supercritical. In Section V, with packet-level simulations, we show the existence of stable limit cycles in the queue size. In Section VI, we provide a summary of our contributions. To support the results in Section IV, the analysis associated with the local Hopf bifurcation is presented in the Appendix.

II. TCP RENO AND QUEUE MANAGEMENT

We first outline a non-linear, time-delayed, many flows fluid model for TCP Reno. Assume that the average window size of all the long-lived TCP flows be \( w(t) \), and so the average rate is \( x(t) = \frac{w(t)}{\tau} \). A fluid model for the congestion avoidance phase of TCP Reno is [14]

\[
\frac{dw(t)}{dt} = \frac{(1 - p(t - \tau))}{\tau} - \frac{w(t)}{2} x(t - \tau)p(t - \tau),
\]

where \( \tau \) is the round-trip time between end-systems and \( p(t) \) is the packet loss probability at the queues. The first term represents Additive Increase (AI) of the window size for
each of the flows. The second term represents Multiplicative Decrease (MD) of the window size when packet losses are detected. This captures the well known AIMD dynamics of TCP's congestion avoidance phase. The TCP Reno model has to be combined with models for queue management in order to study the dynamics of the system. We study the Drop-Tail queue policy, in a small and an intermediate buffer regime, and a threshold based queue policy. We now describe models for these queue management policies.

Model I) Small buffer Drop-Tail: In a Drop-Tail policy, all packets that arrive to find the router buffer being full get dropped. It has been proposed that with a large number of flows, the blocking probability of a M/M/1 queue is a reasonable approximation for small buffer Drop-Tail queues [14], [15]. Thus, a fluid model for the queue is

\[ p(x(t)) = \left( \frac{x(t)}{C} \right)^B, \]

where \( C \) is the link capacity, and \( B \) is the router buffer size.

Model II) Intermediate buffer Drop-Tail: Consider a M/M/1/B queue, where \( B \) is the router buffer size. For such a queueing model, it is well known that

\[ p(x(t)) = \frac{(1 - \rho)x^B}{1 - \rho x^{B+1}}, \]

where \( \rho = x(t)/C \) [7]. If the arrival rate \( x(t) \), the link capacity \( C \), and the buffer size \( B \) are all scaled by a factor \( \beta \), as in a many-sources large-deviation scaling, then by letting \( \beta \to \infty \) one obtains [8]

\[ \lim_{\beta \to \infty} \frac{(1 - \rho)x^B}{1 - \rho x^{B+1}} = \left( \frac{x(t) - C}{x(t)} \right)^+, \]

where \((z)^+\) is defined as \( \max(z, 0) \). This simply denotes the fraction of fluid lost when the arrival rate exceeds the capacity. This simple model represents an approximation of a Drop-Tail policy implemented in a router with a large buffer. For the relationship between this model and an intermediate buffer Drop-Tail policy, see [15].

Model III) Threshold based policy: Consider a queue policy where if the workload exceeds a threshold level \( B \), then the incoming packet is either dropped or marked with a congestion indication signal. Assume that the workload arriving over a time period \( \delta \) is Gaussian, with mean \( x(t) \delta \) and variance \( x(t) \delta \sigma^2 \). From the stationary distribution of a reflected Brownian motion [4], it has been deduced that [6]

\[ p(x(t)) = \exp \left( -\frac{2B(C - x(t))}{x(t)\sigma^2} \right). \]

These fluid models for queue policies combined with the non-linear model for TCP Reno yields a non-linear time-delayed dynamical system.

III. LOCAL STABILITY

In this section, we determine sufficient, and necessary and sufficient, conditions for local stability for TCP Reno along with the three queue policies. Equation (1) can be written in terms of the rate \( x(t) \), as

\[ \frac{dx(t)}{dt} = \frac{1 - p(x(t - \tau))}{\tau^2} - \frac{x(t)x(t - \tau)p(x(t - \tau))}{2}. \]  

The equilibrium of (5) is at

\[ x_s = \frac{1}{\tau} \sqrt{\frac{2(1 - p(x_s))}{p(x_s)}}. \]

Let \( x(t) = u(t) + x_s \). Linearising (5), about the equilibrium (6), we get

\[ \frac{du(t)}{dt} = \frac{x_s}{2} p(x_s) u(t) - \left( \frac{x_s}{2} \left( x_s p'(x_s) + p(x_s) \right) + \frac{p'(x_s)}{\tau^2} \right) u(t - \tau). \]

This linear delay equation can be represented as

\[ \frac{du(t)}{dt} = -au(t) - bu(t - \tau), \]

where

\[ a = \frac{x_s p(x_s)}{2}, \]
\[ b = x_s \left( x_s p'(x_s) + p(x_s) \right) + \frac{p'(x_s)}{\tau^2}. \]

For \( a \geq 0, b > 0, b > a \) and \( \tau > 0 \), it has been shown that

\[ b\tau < \frac{\pi}{2}, \]

is a sufficient condition for the stability of system (8) and that the necessary and sufficient condition for stability is [13]

\[ \tau \sqrt{b^2 - a^2} < \cos^{-1} \left( \frac{a}{b} \right). \]

Using conditions (9) and (10), we have tabulated the stability conditions in Tables I and II for all the three queue policies. For Model II, for ease of exposition, we defined \( x_s = \gamma C \), for some \( \gamma > 1 \).

We now summarize our key deductions:

- Model I represents a Drop-Tail policy, in routers with small buffer sizes. The stability results for Model I highlight that local stability can be ensured for small buffer sizes, and for large window sizes. Thus variations in the choice of buffer size can drive the system into a locally unstable state. Interestingly, this condition for local stability does not depend on the feedback delay, or the link capacity.
- Model II also represents a Drop-Tail policy, but over a large buffer. Note that, in this case, system stability can be easily lost as the feedback delay, or the link capacity, get large. Thus both the queue policy and the router buffer dimensioning rule do play a decisive role in maintaining stability.
TABLE I
SUFFICIENT CONDITIONS FOR LOCAL STABILITY: MODELS I, II, AND III

<table>
<thead>
<tr>
<th>I</th>
<th>$B + (1 - p(x_s)) &lt; \frac{\pi}{w_s}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>II</td>
<td>$\gamma C \tau (1 + p(x_s)) &lt; \pi$</td>
</tr>
<tr>
<td>III</td>
<td>$\frac{2H}{\sigma^2} + (1 - p(x_s)) &lt; \frac{\pi}{w_s}$</td>
</tr>
</tbody>
</table>

TABLE II
NECESSARY AND SUFFICIENT CONDITIONS FOR LOCAL STABILITY: MODELS I, II, AND III

<table>
<thead>
<tr>
<th>I</th>
<th>$\frac{1}{w_s} \sqrt{B(B + 2(1 - p(x_s)))} &lt; \cos^{-1} \left( \frac{(1 - p(x_s))}{B + (1 - p(x_s))} \right)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>II</td>
<td>$\frac{\gamma C \tau}{\sigma^2} \sqrt{1 + 2p(x_s)} &lt; \cos^{-1} \left( \frac{p(x_s)}{1 + p(x_s)} \right)$</td>
</tr>
<tr>
<td>III</td>
<td>$\frac{1}{w_s} \sqrt{\frac{2H}{\sigma^2} + 2(1 - p(x_s))} &lt; \cos^{-1} \left( \frac{(1 - p(x_s))}{\frac{2H}{\sigma^2} + (1 - p(x_s))} \right)$</td>
</tr>
</tbody>
</table>

- Model III represents a simple threshold based queue policy. The stability results clearly highlight the importance of choosing this parameter very carefully, if stability is desired. Again smaller values of the parameter $B$ help stability, and variations in this parameter can induce local instability.

Clearly, system parameters readily influence stability and should be chosen prudently. It is quite natural to ask what may happen if these conditions for stability are just violated as system parameters vary. To that end, in the next section, we also perform a detailed local bifurcation analysis.

IV. LOCAL HOPF BIFURCATION

For the local bifurcation analysis, instead of using any of the system parameters as the bifurcation parameter, we introduce an exogenous non-dimensional parameter $\kappa > 0$. It will be assumed that the system is at the edge of stability, and then it is this non-dimensional parameter which drives the system just beyond the locally stable region. The non-linear model for TCP Reno is now essentially of the following form:

$$\frac{d}{dt} \tau(t) = \kappa f \left( x(t), x(t - \tau) \right),$$

where $\kappa$ is the new exogenous parameter.

Let $\kappa_c$ denote the critical point where a local Hopf bifurcation occurs and so at the Hopf condition, $\kappa_c = 1$. The characteristic equation, for the associated linearised system, is of the form

$$\lambda + \kappa a + \kappa b e^{-\lambda \tau} = 0,$$

where $a$ and $b$ are defined in (8). We can show that the transversality condition

$$\text{Re} \left( \frac{d\lambda}{d\kappa} \right)_{\kappa=\kappa_c} \neq 0,$$

is indeed satisfied. We get

$$\left. \frac{d\lambda}{d\kappa} \right|_{\kappa=\kappa_c} = -\left( a + b e^{-\lambda \tau} \right) \frac{1}{1 - \kappa b \tau e^{-\lambda \tau}}_{\kappa=\kappa_c},$$

from which we obtain,

$$\text{Re} \left( \frac{d\lambda}{d\kappa} \right)_{\kappa=\kappa_c} = \frac{\kappa_c \tau (b^2 - a^2)}{1 + 2\kappa_c a \tau + \kappa_c^2 b^2 \tau^2} > 0,$$

which shows that the system undergoes a Hopf bifurcation.

We now perform the requisite computations to determine the type of the Hopf bifurcation for Model I. Similar computations, using the analysis in the Appendix, can be performed for the other models as well. The Hopf condition for TCP Reno and small buffer Drop-tail queues is outlined in [12]. Here we provide an explicit analytical characterisation for the orbital stability of the bifurcating limit cycles, when the Hopf condition is just violated.

Following the analysis in [5], [13], we take a Taylor series expansion of (11), and keep the linear, quadratic and cubic terms, about equilibrium. This yields

$$\frac{d}{dt} u(t) = \kappa \xi_{x} u(t) + \kappa \xi_{y} u(t - \tau) + \kappa \xi_{x+y} u(t) u(t - \tau) + \kappa \xi_{x+y+y} u(t) u(t - \tau)^2 + \kappa \xi_{x+y+y+y} u(t) u(t - \tau)^3 + O(u^4),$$

where $u(t) = x(t) - x_s$ and

$$\xi_{x+y} = \frac{1}{(i + j)!} \frac{\partial^{i+j}}{\partial x^i \partial y^j} \biggr|_{x=x_s,y=y_s}.$$

In Table III we have listed the linear, quadratic, and cubic terms for Models I and II, to highlight the differences in their non-linear structure. However for the computations, in this paper, we focus on Model I. The $\xi_{x+y+y}'$'s are used to analytically characterise the Hopf bifurcation. The local Hopf bifurcation analysis is outlined in the Appendix. The quantities $\mu_2$ and $\beta_2$, defined in the Appendix (see (43, 44)), are required to characterise the type of the Hopf bifurcation and the stability of the bifurcating periodic solutions.

Figure 1 shows the values of $\mu_2$ and $\beta_2$ for Model I, as the non-dimensional parameter $\kappa$ is increased beyond the Hopf condition. We note that as $\mu_2 > 0$ and $\beta_2 < 0$, the bifurcation would be supercritical and that the bifurcating limit cycles would be orbitally stable. In Table IV we also outline the period of the emergent periodic orbits at the Hopf bifurcation for the various models. The theoretical predictions about the emergence of stable limit cycles will be confirmed, via packet-level simulations, in the next section.
TABLE III
LINEAR, QUADRATIC AND CUBIC TERMS OF MODELS I AND II

<table>
<thead>
<tr>
<th>Model I</th>
<th>Model II</th>
</tr>
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<tbody>
<tr>
<td>$\xi_x$</td>
<td>$-\frac{1}{w_x^r}(1 - p(x_s))$</td>
</tr>
<tr>
<td>$\xi_y$</td>
<td>$-\frac{1}{w_x^r}(B + (1 - p(x_s)))$</td>
</tr>
<tr>
<td>$\xi_{xy}$</td>
<td>$\frac{(1 - p(x_s))\tau}{\sqrt{w_x}}(B + 1)$</td>
</tr>
<tr>
<td>$\xi_{yy}$</td>
<td>$-\frac{B}{2w_x^r}(B + (1 - 2p(x_s)))$</td>
</tr>
<tr>
<td>$\xi_{xyy}$</td>
<td>$\frac{(1 - p(x_s))\tau}{2w_x^r}B(B + 1)$</td>
</tr>
<tr>
<td>$\xi_{yy}$</td>
<td>$-\frac{B(B - 1)\tau}{6w_x^r}(B + (1 - 3p(x_s)))$</td>
</tr>
</tbody>
</table>

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V. PACKET-LEVEL SIMULATIONS

We now conduct packet-level simulations, using the Network Simulator 2 (NS2) [18], with TCP Reno under the Drop-Tail queue management policy. The simulations are conducted on a single bottleneck topology. The following parameters are used for our simulations: buffer size = 10, 100, and 200 packets, with round-trip time $\tau = 200$ ms, bottleneck link capacity $C = 100$ Mbps, number of flows $N = 60$, and packet size $= 1500$ bytes. In the simulations, we focus on the dynamics of the queue size as the buffer size is varied.

In Figure 2, as the buffer size varies from 10 to larger values, we can clearly observe the emergence of stable limit cycles in the queue size. As per the stability analysis, smaller values of the parameter $B$ should help stability. To that end, with a buffer size of 10 packets we do not observe deterministic oscillations, in the form of limit cycles, in the queue size. Figure 3 shows the phase plot, for the queue size dynamics, with 100 and 200 packet buffers.

VI. CONTRIBUTIONS

We conducted a detailed local stability and local Hopf bifurcation analysis of a fluid model for TCP Reno coupled with some queue management policies. We considered the widely deployed Drop-Tail queue policy, over a small and an intermediate buffer regime. We also studied a threshold based queue management policy. A key contribution was to exhibit, using a combination of analysis and packet-level simulations, the existence and stability of limit cycles in the queue size as system parameters vary. The transition from stability to instability was shown to occur via a Hopf bifurcation. The asymptotic orbital stability of the bifurcating periodic solutions was established using the theory of normal
forms and the center manifold theorem.

Natural extensions of our work would be to analyse models with multiple bottlenecks and heterogeneous feedback delays. Additionally, the study of other queue management policies like the well known RED policy [1] also merits investigation.

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APPENDIX

Following the framework outlined in [5], we perform the requisite analysis to determine the type of the Hopf bifurcation and the orbital stability of the bifurcating periodic solutions.

Consider the following general functional differential equation

$$
\frac{d}{dt} u(t) = L_\mu u(t) + F(u_t, \mu) \quad \mu \in \mathbb{R},\quad (16)
$$

where for $\tau > 0$

$$
\begin{align*}
\mu(t) &= u(t + \theta) \quad u : [-\tau, 0] \to \mathbb{R}, \quad \theta \in [-\tau, 0].
\end{align*}
$$

$L_\mu$ is a one-parameter family of continuous linear operators $L_\mu : C[-\tau, 0] \to \mathbb{R}$ . The operator $F(u_t, \mu) : C[-\tau, 0] \to \mathbb{R}$ contains the non-linear terms. Let $F$ and $L_\mu$ depend analytically on the bifurcation parameter $\mu$. The objective is to rewrite (16) as

$$
\frac{d}{dt} u(t) = A(\mu) u(t) + R u(t) \quad (17)
$$

which has $u_t$ instead of both $u$ and $u_t$. Our intent is now to transform the linear problem $(d/dt)u(t) = L_\mu u(t)$. By the Riesz representation theorem, there exists an $n \times n$ matrix-valued function $\eta(\cdot, \mu) : [-\tau, 0] \to \mathbb{R}$, such that each component of $\eta$ has bounded variation and for all $\phi \in C[-\tau, 0]$

$$
L_\mu \phi = \int_{-\tau}^{0} \eta(\theta, \mu) \phi(\theta) d\theta.
$$

In fact

$$
L_\mu u_t = \int_{-\tau}^{0} \eta(\theta, \mu) u(t + \theta) d\theta. \quad (18)
$$

Note that

$$
d\eta(\theta, \mu) = \kappa(\xi_0 \delta(\theta) + \xi_\theta \delta(\theta + \tau)) d\theta,
$$

where $\delta(\theta)$ is the Dirac delta function, would satisfy (18).

For $\phi \in C^1[-\tau, 0]$, we define

$$
A(\mu) \phi(\theta) = \begin{cases} \frac{d\phi(\theta)}{d\theta}, & \theta \in [-\tau, 0) \\ \int_{-\tau}^{0} d\eta(s, \mu) \phi(s) \equiv L_\mu \phi, & \theta = 0 \end{cases}
$$

and

$$
R \phi(\theta) = \begin{cases} 0, & \theta \in [-\tau, 0) \\ F(\phi, \mu), & \theta = 0. \end{cases}
$$

As $du_t/d\theta \equiv du_t/dt$, system (16) becomes (17). Let $q(\theta)$ be the eigenfunction for $A(0)$ corresponding to $\lambda(0)$.

$$
A(0) q(\theta) = i\omega_0 q(\theta).
$$

We also define the adjoint operator $A^*(0)$ as

$$
A^*(0) \alpha(s) = \begin{cases} -\frac{d\alpha(s)}{ds}, & s \in (0, \tau] \\ \int_{-\tau}^{0} d\eta^T(t, 0) \alpha(-t), & s = 0, \end{cases}
$$

where $\eta^T$ denotes the transpose of $\eta$.

Note that the domains of $A$ and $A^*$ are $C^1[-\tau, 0]$ and $C^1[0, \tau]$. As

$$
A q(\theta) = \lambda(0) q(\theta)
$$

$\lambda(0)$ is an eigenvalue for $A^*$, and

$$
A^* q^* = -i\omega_0 q^*
$$

for some nonzero vector $q^*$. For $\phi \in C[-\tau, 0]$ and $\psi \in C[0, \tau]$ define an inner product

$$
\langle \psi, \phi \rangle = \psi(0) \cdot \phi(0) - \int_{\theta = -\tau}^{0} \int_{\zeta = 0}^{\theta} \psi^T(\zeta - \theta) d\eta(\theta) \phi(\zeta) d\zeta, \quad (20)
$$

where $a \cdot b$ means $\sum_{i=1}^{n} a_i b_i$. Then $\langle \psi, A \phi \rangle = \langle A^* \psi, \phi \rangle$ for $\phi \in \text{Dom}(A)$, $\psi \in \text{Dom}(A^*)$. Let $q(\theta) = e^{i\omega_0 \theta}$ and $q^*(s) = D e^{i\omega_0 s}$ be the eigenvectors for $A$ and $A^*$ corresponding to the eigenvalues $+i\omega_0$ and $-i\omega_0$. With

$$
D = \frac{1}{1 + \tau \kappa \xi_0 e^{i\omega_0 \tau}},
$$

we get $\langle q^*, q \rangle = 1$ and $\langle q^*, \bar{q} \rangle = 0$. Using (20) we can verify that $\langle q^*, \bar{q} \rangle = 1$. Similarly, we can readily show that $\langle q^*, \bar{q} \rangle = 0$. For $u_t$, a solution of (17) at $\mu = 0$, define

$$
z(t) = \langle q^*, u_t \rangle,
$$

and

$$
w(t, \theta) = u_t(\theta) - 2Re\{z(t) q(\theta)\}. \quad (21)
$$

Then, on the manifold, $C_0$, $w(t, \theta) = w(z, \bar{z}, t, \theta)$ where

$$
w(z, \bar{z}, \theta) = w_{20}(\theta)\frac{z^2}{2} + w_{11}(\theta) z \bar{z} + w_{02}(\theta)\frac{\bar{z}^2}{2} + \cdots.
$$

Observe that, $z$ and $\bar{z}$ are local coordinates for $C_0$ in the directions of $q^*$ and $\bar{q}^*$. The existence of the center manifold $C_0$ enables the reduction of (17) to an ODE for a single complex variable on $C_0$. At $\mu = 0$, this yields

$$
z'(t) = \langle q^*, A u_t + R u_t \rangle = i\omega_0 z(t) + \bar{q}^*(0) \cdot F_0(z, \bar{z})
$$

which can be written in abbreviated form as

$$
z'(t) = i\omega_0 z(t) + g(z, \bar{z}).
$$

The key objective is to expand $g$ in powers of $z$ and $\bar{z}$. We need to determine the coefficients $w_{ij}(\theta)$ in (21). Once the
coefficients $w_{ij}$ are determined, the differential equation (22) for $z$ would be explicit. Expanding the function $g(z, \bar{z})$ in powers of $z$ and $\bar{z}$, we get

$$
g(z, \bar{z}) = \bar{g}^* (0) \cdot F_0(z, \bar{z}) = g_{20} \frac{z^2}{2} + g_{11} z \bar{z} + g_{02} \frac{\bar{z}^2}{2} + g_{21} \frac{z^2 \bar{z}}{2} + \cdots.
$$

Following [5], we have

$$w' = u'_i - z' q - \bar{z}' \bar{q}$$

and using (17) and (23) we get

$$w' = \left\{ \begin{array}{ll}
Aw - 2 \text{Re}\{q^*(0) \cdot F_0 q(\theta)\}, & \theta \in (-\tau, 0) \\
Aw - 2 \text{Re}\{q^*(0) \cdot F_0 q(0)\} + F_0, & \theta = 0,
\end{array} \right.$$

which is rewritten as

$$w' = Aw + H(z, \bar{z}, \theta), \quad (24)$$

where

$$H(z, \bar{z}, \theta) = H_{20}(\theta) \frac{z^2}{2} + H_{11}(\theta) z \bar{z} + H_{02}(\theta) \frac{\bar{z}^2}{2} + \cdots. \quad (25)$$

Note that on $C_0$, near the origin,

$$w' = w_z z' + w_{\bar{z}} \bar{z}'.
$$

Use (21) and (23) to replace $w_z$, $z'$ and equating this with (24), we get

$$(2i\omega_0 - A)w_{20}(\theta) = H_{20}(\theta) - Aw_{11}(\theta) = H_{11}(\theta) - (2i\omega_0 + A)w_{02}(\theta) = H_{02}(\theta).$$

Note that

$$u_i(\theta) = w(z, \bar{z}, \theta) + zq(\theta) + \bar{z}\bar{q}(\theta) = w_{20}(\theta) \frac{z^2}{2} + w_{11}(\theta) z \bar{z} + w_{02}(\theta) \frac{\bar{z}^2}{2} + ze^{i\omega_0 \theta} + \bar{z}e^{-i\omega_0 \theta} + \cdots,$$

from which we get $u_i(0)$ and $u_i(-\tau)$. We will only be requiring the coefficients of $z^2$, $z \bar{z}$, $\bar{z}^2$, and $z^2 \bar{z}$. Thus we only keep these desired terms in the following expansions:

$$u_i(0)u_i(-\tau) = (w(z, \bar{z}, 0) + z + \bar{z}) \times (w(z, \bar{z}, -\tau) + z e^{-i\omega_0 \tau} + \bar{z} e^{i\omega_0 \tau})$$

$$= z^2 \frac{ze^{i\omega_0 \theta} + \bar{z}e^{-i\omega_0 \theta}}{2} + \frac{z^2 \bar{z}}{2}(2w_{11}(0)e^{-i\omega_0 \tau} + w_{20}(0)e^{i\omega_0 \tau})$$

$$+ 2w_{11}(-\tau) + w_{20}(-\tau) + \cdots.$$
We have already noted that
\[(2i\omega_0 - A)w_{20}(\theta) = H_{20}(\theta) \quad (31)\]
\[-A\bar{w}_{11}(\theta) = H_{11}(\theta) \quad (32)\]
\[-(2i\omega_0 + A)w_{02}(\theta) = H_{02}(\theta). \quad (33)\]

From (19), (31) and (32) we get
\[w_{20}(\theta) = 2i\omega_0 w_{20}(\theta) + g_{20}q(\theta) + \bar{g}_{02}\bar{q}(\theta) \quad (34)\]
\[w_{11}(\theta) = g_{11}q(\theta) + \bar{g}_{11}\bar{q}(\theta). \quad (35)\]

Solving the equations (34) and (35) yields
\[w_{20}(\theta) = \frac{g_{20}}{i\omega_0} q(0)e^{i\omega_0\theta} - \frac{\bar{g}_{02}}{3i\omega_0} \bar{q}(0)e^{-i\omega_0\theta} + E_1e^{2i\omega_0\theta} \quad (36)\]
\[w_{11}(\theta) = \frac{g_{11}}{i\omega_0} q(0)e^{i\omega_0\theta} - \frac{\bar{g}_{11}}{i\omega_0} \bar{q}(0)e^{-i\omega_0\theta} + E_2 \quad (37)\]

for some $E_1$, $E_2$. For
\[H(z, \bar{z}, 0) = -2\Re(q(0)\cdot F_0q(0)) + F_0, \quad (38)\]
\[H_{20}(0) = -g_{20}q(0) - \bar{g}_{02}\bar{q}(0) + \kappa(2\xi_y e^{-i\omega_0\tau} + 2\xi_{yy} e^{-2i\omega_0\tau}) \quad (39)\]
\[H_{11}(0) = -g_{11}q(0) - \bar{g}_{11}\bar{q}(0) + \kappa(\xi_y e^{i\omega_0\theta} + e^{-i\omega_0\theta} + 2\xi_{yy}) \quad (40)\]

From (36) and (37), we will have the solution for $w_{20}(\theta)$ and $w_{11}(\theta)$ which will allow us to obtain expressions for $E_1$ and $E_2$. Thus the complete solutions for $w_{20}(\theta)$ and $w_{11}(\theta)$ can be obtained. All the quantities required for the computations associated for the stability analysis of the Hopf bifurcation are finally completed. The analysis can be performed using [5]
\[c_1(0) = \frac{i}{2\omega_0} \left( g_{20}g_{11} - 2|g_{11}|^2 - \frac{1}{3}|g_{02}|^2 \right) + \frac{g_{21}}{2} \quad (41)\]
\[\mu_2 = \frac{-\Re c_1(0)}{\alpha'(0)} \quad (42)\]
\[\beta = \epsilon^2 \beta_2 + O(\epsilon^4), \quad \beta_2 = 2\Re c_1(0), \quad \epsilon = \sqrt{\mu_2} \quad (43)\]

where $c_1(0)$ is the Lyapunov coefficient and $g_{20}$, $g_{11}$, $g_{02}$, $g_{21}$ are defined by (26)-(29). If $\mu_2 > 0$ then the Hopf bifurcation is supercritical; it is subcritical if $\mu_2 < 0$. The periodic solutions are asymptotically orbitally stable when $\beta_2 < 0$ and unstable if $\beta_2 > 0$.

**REFERENCES**


