\(H_\infty\) Synthesis with Unstable Weighting Filters: An LMI Solution

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Abstract—Synthesis of a controller is considered for achieving a desired level of guaranteed worst-case energy gain performance in the case of unstable (input and output) weighting filters. The solution of this problem is shown to require a proper replication of the unstable filter dynamics accompanied by a controller of free structure. By merging the replicated filter dynamics with the plant, the problem is reformulated as the design of the free part of the controller in a way to stabilize an extended plant and ensure the considered performance objective. A solution is then provided based on LMI conditions accompanied by affine equation constraints. A procedure is also outlined for the synthesis of a controller whose order is equal to the order of the plant plus the orders of the filters.

I. INTRODUCTION

Generalized plant framework emerged as a convenient setting for controller synthesis in the robust control era (see e.g. [14]). In this setting, the performance objectives are conveniently formulated as the minimization of a preferred norm \(H_2, H_\infty\) of a transfer function (or matrix), weighted with input and/or output filters. Although weighting is usually done with stable filters, unstable weighting filters are also of interest as the generators of constant, sinusoidal or unbounded reference/disturbance inputs. They can also be used at the output to enforce the placement of unstable zeros in the transfer function to be shaped. The problem becomes nonstandard for unstable filters due to the violation of stabilizability and detectability assumptions.

A number of approaches are available in the literature to handle unstable weighting filters. The polynomial approach to \(H_\infty\) synthesis has some advantages in this respect, also allowing to handle improper weights as well [6], [7]. Nevertheless, it is not applicable to the filters with imaginary axis poles and might not be preferable for multi-variable systems. One can consider absorbing the filters into the loop and thereby modifying the problem formulation into a convenient form. A simplification of this approach that avoids unnecessary increase in the controller order is provided in [10] though for particular \(H_\infty\) synthesis problems. The problem is treated in a generalized plant setting by [9], [11] as an extended synthesis and solutions are provided in terms of the quasi-stabilizing solutions of Riccati equations. Recently, extended synthesis problems are considered for descriptor systems in [1], [2] and solutions are provided based on generalized algebraic Riccati equations and generalized Sylvester equations. Descriptor system setting makes it possible to consider improper weights easily. Nevertheless, solutions of standard as well as extended synthesis problems based on the Riccati equation approach all require the original plant not to have imaginary axis zeros in certain channels.

Synthesis with unstable input filters is in fact closely related with asymptotic output regulation problem with additional performance objectives (see [12]). Indeed, it is straightforward to adapt the approach developed in [5] to obtain a linear matrix inequality (LMI) solution to the \((H_\infty, H_2)\) synthesis problems with unstable input filters. This requires a suitable replication of the unstable input filter dynamics in the controller. By merging the replicated dynamics with the plant dynamics, one is able to reformulate the problem as the synthesis of a controller for an extended plant. LMI conditions are derived for the solvability of this problem with a controller whose order is equal to the order of the original plant plus the order of the filter. This solution can also be adapted for the case of unstable output filters based on the observation that a transfer function has the same norm with its transpose. To the best of our knowledge, there is no LMI solution in the literature for the synthesis problem in the joint presence of unstable input and output filters.

This paper considers the \(H_\infty\) synthesis problem formulated in the following section for unstable input/output filters. In Section III, we derive the required controller structure together with the necessary and sufficient conditions for solvability. This leads us to a reformulation of the problem as a synthesis for a properly extended plant. The solution of this new problem based on LMI optimization is derived in Section IV by favor of a novel approach, which facilitates the synthesis of a controller whose order is not unnecessarily large. The paper is concluded by some final remarks.

II. PROBLEM STATEMENT

Consider a linear time-invariant (LTI) plant as

\[
\Sigma_p: \begin{bmatrix} \dot{x} \\ e \\ y \end{bmatrix} = \begin{bmatrix} A & B_p & B_c
\
C_p & D_p & D_{pc}
\
C & D_{cp} & 0 \end{bmatrix} \begin{bmatrix} x \\ v \\ u \end{bmatrix}, x(0) = 0, \quad (1)
\]

where \(x(t) \in \mathbb{R}^k\) denotes the state, \(u(t) \in \mathbb{R}^n\) represents the control input and \(y(t) \in \mathbb{R}^m\) is the measured output. In this paper, we will consider a disturbance input \(v(t) \in \mathbb{R}^l\) that is generated by an anti-stable input filter of the form

\[
\Sigma_i: \begin{bmatrix} \dot{x}_i \\ v \end{bmatrix} = \begin{bmatrix} A_i & B_i \\ C_i & D_i \end{bmatrix} \begin{bmatrix} x_i \\ w \end{bmatrix}, x_i(0) = 0, \quad (2)
\]
where \( x_i(t) \in \mathbb{R}^{k_i} \) is the input filter state and \( w \in \mathbb{R}^p \) represents a finite-energy disturbance input. We indicate the transfer function of this filter as
\[
T_{uv}(s) \triangleq C_i(sI - A_i)^{-1}B_i + D_i.
\]
The performance of the system will be evaluated based on an output signal \( z(t) \in \mathbb{R}^q \). This signal is obtained by filtering \( e(t) \in \mathbb{R}^r \) by an anti-stable output filter represented as
\[
\Sigma_o : \begin{bmatrix} \dot{x}_o \\ z \end{bmatrix} = \begin{bmatrix} A_o & B_o \\ C_o & D_o \end{bmatrix} \begin{bmatrix} x_o \\ e \end{bmatrix}, x_o(0) = 0,
\]
where \( x_o(t) \in \mathbb{R}^{k_o} \) is the output filter state. The transfer function of the output filter is found as
\[
T_{ze}(s) \triangleq C_o(sI - A_o)^{-1}B_o + D_o.
\]

For a concise expression of the standing assumptions of the paper and the solvability conditions of the considered problem, we need to introduce extended realization matrices by merging the dynamics of the plant and the weighting filters. The first extended plant is obtained by merging the dynamics of the plant with the dynamics of the output filter:
\[
\tilde{\Sigma}_p : \begin{bmatrix} \dot{x} \\ z \end{bmatrix} = \begin{bmatrix} A & B_o C_p & x_o \\ \tilde{A} & \tilde{B} & \tilde{D}_p \end{bmatrix} \begin{bmatrix} x \\ \tilde{x} \end{bmatrix} + \begin{bmatrix} B_p & D_p u \\ \tilde{B}_p & \tilde{D}_p \end{bmatrix} v + \begin{bmatrix} B_o | \tilde{B} & \tilde{D}_pc \end{bmatrix} u,
\]
\[
y = \begin{bmatrix} C_o & \tilde{C}_o \end{bmatrix} \tilde{x} + D_i w + D_{pc} u,
\]
where \( \tilde{\Sigma}_p \) is the output filter state. The transfer function of this filter as
\[
\Sigma_e : \begin{bmatrix} \dot{\xi} \\ \xi \end{bmatrix} = \begin{bmatrix} A_e & B_e \\ C_e & D_e \end{bmatrix} \begin{bmatrix} \xi \\ \eta \end{bmatrix}, \xi(0) = 0;
\]
which stabilizes the plant \( \Sigma_p \) and which ensures an \( \mathcal{H}_\infty \) performance objective for the transfer function
\[
T_{ze}(s) = T_{ze}(s) \begin{bmatrix} C (sI - A)^{-1}B + D \end{bmatrix} T_{uv}(s).
\]

The realization matrices of \( T_{ze} \) are identified from
\[
\chi = \begin{bmatrix} A + BD_c C & B C_c \\ B_c C & A_c \end{bmatrix} \begin{bmatrix} x \\ \xi \end{bmatrix} + \begin{bmatrix} B_p \tilde{B}_cp \end{bmatrix} v,
\]
\[
e = \begin{bmatrix} C_p + D_{pc} \tilde{C}_p & D_{pc} C_c \end{bmatrix} \chi + \begin{bmatrix} \tilde{B} & \tilde{B}_cp \end{bmatrix} v,
\]
\[
\text{We state the problem precisely as follows:}
\]

**Problem 1:** Given a plant \( \Sigma_p \) as in (1) and weighting filters \( \Sigma_i, \Sigma_o \) as in (2),(4) that fulfill the assumptions A1-A5, design a controller \( \Sigma_c \) as in (9) such that:

C.1. The feedback system formed by \( \Sigma_p \) and \( \Sigma_c \) is internally stable, i.e. \( A \) is Hurwitz.

C.2. The transfer function \( T_{zw} \) is stable, i.e. \( T_{zw} \in \mathcal{RH}_\infty^{q \times p} \).

C.3. \( \|T_{zw}\|_\infty \triangleq \sup_{s \in \mathbb{R}} \|T_{zw}(j\omega)\| < \gamma \).

C.2 requires \( T_{zw} \) to cancel the unstable dynamics of \( T_{uv} \) and \( T_{ze} \). The combination of C.1 and C.2 is referred to as **comprehensive stability** in [11].

**III. NECESSARY CONDITIONS FOR SOLVABILITY**

We investigate the conditions for comprehensive stability and the controller structure it requires in two subsections.

**A. Elimination of Input Filter Dynamics**

In this subsection, we derive the necessary conditions under which the dynamics of the input filter can be eliminated by a proper replication in the controller. We start by forming the feedback interconnection of \( \Sigma_p \) and \( \Sigma_c \) to express the dynamics of the system proceeded by the input filter as
\[
\dot{\chi} = \begin{bmatrix} A + \tilde{B} D_c \tilde{C} & \tilde{B} C_c \\ B_c C & A_c \end{bmatrix} \begin{bmatrix} \dot{\xi} \\ \xi \end{bmatrix} + \begin{bmatrix} \tilde{B}_p \tilde{B}_cp \end{bmatrix} v,
\]
\[
z = \begin{bmatrix} \tilde{C}_p + D_{pc} \tilde{C} & D_{pc} C_c \end{bmatrix} \dot{\chi} + D_o Dv.
\]

The standing assumptions of the paper are stated as follows:

A.1. \( A_i \) is anti-Hurwitz, i.e. it has no eigenvalues in the left half-plane, and \( (A_i, B_i) \) is controllable.

A.2. \( A_o \) is anti-Hurwitz and \( (A_o, C_o) \) is observable.

A.3. \( A_i \) and \( A_o \) do not have any common eigenvalues.

A.4. \( (\hat{A}, \hat{B}) \) is stabilizable.

A.5. \( (\hat{A}, \hat{C}) \) is detectable.
The overall dynamics of the system are then described by
\[
\begin{bmatrix}
\dot{\hat{x}}_i \\
\dot{x}_i \\
z
\end{bmatrix} =
\begin{bmatrix}
\hat{A} & \hat{B}C_i \\
\hat{B}D_i & 0 & 0 \\
\hat{C} & D_o \hat{D}C_i & D_o \hat{D}D_i
\end{bmatrix}
\begin{bmatrix}
\hat{x}_i \\
x_i \\
w
\end{bmatrix}.
\] (13)

With a state transformation of the form
\[
\begin{bmatrix}
\phi_o \\
\phi \\
z
\end{bmatrix} = \begin{bmatrix}
\phi \\
\hat{x} \\
\hat{x}
\end{bmatrix} \Delta + \begin{bmatrix}
\hat{\Pi}_i \\
\Pi_i \\
\Pi_i
\end{bmatrix} x_i = \begin{bmatrix}
\phi_o \\
\hat{x}_i \\
\hat{x}_i
\end{bmatrix},
\] (14)

the overall dynamics are expressed equivalently as
\[
\begin{bmatrix}
\dot{\hat{x}} \\
\dot{x}_1 \\
z
\end{bmatrix} =
\begin{bmatrix}
\hat{A} & \hat{B}C_i + \hat{A} \hat{\Psi}_i - \hat{A} \hat{\Psi}_i \\
0 & \hat{A}_i & \hat{B}_i \\
\hat{C} & D_o \hat{D}C_i - \hat{C} \hat{\Psi}_i & D_o \hat{D}D_i
\end{bmatrix}
\begin{bmatrix}
\hat{x} \\
x_1 \\
w
\end{bmatrix}.
\] (15)

We now choose \(\Psi_i\) in a way to satisfy the Sylvester equation
\[
\hat{B}C_i + \hat{A} \hat{\Psi}_i - \hat{A} \hat{\Psi}_i = 0.
\] (16)

A closer look into the structure of \(\hat{A}\) reveals that its eigenvalues are formed by the eigenvalues of \(A\) and \(A_o\). When \(A\) is Hurwitz and \(A_o\) is satisfied, \(A\) and \(A_o\) have no common eigenvalues. In this case, equation (16) will have a unique solution for \(\Psi_i\). It is then easy to observe that the transfer function from \(w\) to \(z\) can be expressed in terms of the new realization matrices as
\[
T_{zw}(s) = \hat{C} \begin{bmatrix} sI - \hat{A} \end{bmatrix}^{-1} \begin{bmatrix} \hat{B}D_i + \hat{A} \hat{\Psi}_i \\
0 & \hat{A}_i & \hat{B}_i \\
\hat{C} & D_o \hat{D}C_i - \hat{C} \hat{\Psi}_i & D_o \hat{D}D_i
\end{bmatrix} \begin{bmatrix} x_1 \\
w
\end{bmatrix}.
\] (17)

Since the spectra of \(\hat{A}\) and \(\hat{A}_i\) are disjoint, any component of the term in the second line of the equation (which is anti-stable) cannot be canceled by the terms in the first line. As a result, \(T_{zw}\) can be stable only when the term in the second line of the equation totally vanishes. Since \((\hat{A}_i, \hat{B}_i)\) is controllable, this will be the case if and only if
\[
\hat{C} \hat{\Psi}_i - D_o \hat{D}C_i = 0.
\] (18)

We thus conclude that, when \(T_{zw}\) and \(T_{zy}\) are both stable, there must exist a matrix \(\Psi_i\) that satisfies (16) and (18).

Using the closed-loop matrices in (12) and the partition of \(\Psi\) from (14), we reorganize (16) and (18) as
\[
\begin{bmatrix}
\hat{A} & \hat{B} & \hat{B}_c C_i \\
\hat{B}_c D_o & D_o & D_o \hat{D}C_i \\
\hat{C} & D_o \hat{D}C_i & D_o \hat{D}D_i
\end{bmatrix}
\begin{bmatrix}
\hat{\Pi}_i \\
\Pi_i \\
\Pi_i \\
\end{bmatrix} =
\begin{bmatrix}
\hat{B}_c C_i \\
\hat{B}_c D_o & D_o \hat{D}C_i \\
\hat{C} & D_o \hat{D}C_i & D_o \hat{D}D_i
\end{bmatrix}
\begin{bmatrix}
\hat{\Pi}_i \\
\Pi_i \\
\Pi_i \\
\end{bmatrix} x_i.
\] (19)

Equation (20) in fact expresses the structural constraints on the controller. Indeed, if \(\Phi_i\) is a matrix with full column rank, one can transform the states of the controller in such a way that the associated \(\Phi_i\) matrix is given by
\[
\Phi_i = \begin{bmatrix}
0 \\
I_{k_i \times k_i}
\end{bmatrix}.
\] (22)

When \(\Phi_i\) does not have full column rank, it can be modified suitably into a version with full column-rank, by introducing a non-minimal realization of the controller that still preserves the stability of the closed-loop (see [12] for the details of this argument). Considering a compatible partition, we can then identify the structure of the controller as
\[
\begin{bmatrix}
\dot{\hat{\xi}}_a \\
\dot{\xi}_a \\
u
\end{bmatrix} =
\begin{bmatrix}
\hat{A}_a & -\hat{B}_a Z_i & \hat{B}_a \\
\hat{C}_a^2 & \hat{D}_a Z_i & \hat{D}_a \\
\hat{C}_a & \hat{D}_a & \hat{D}_a
\end{bmatrix}
\begin{bmatrix}
\hat{\xi}_a \\
\xi_a \\
y
\end{bmatrix}.
\] (23)

We now isolate the replicated dynamics of the input filter as
\[
\hat{\xi}_a = A_{i} x_1 + \hat{C}_a \hat{\xi}_a + D_{a} y_2,
\] (24)

In this equation, we have introduced the measurement to be fed to and a control input generated by the accompanying controller, respectively, as \(y_2\) and \(u_1\). Introducing the other control input as
\[
u = A_{i} x_1 + \hat{C}_a \hat{\xi}_a + D_{a} y_2,
\] (25)

we can identify the accompanying controller as
\[
\hat{\Sigma}_a : 
\begin{bmatrix}
\dot{\hat{\xi}}_a \\
\dot{\xi}_a \\
u
\end{bmatrix} =
\begin{bmatrix}
\hat{A}_a & \hat{B}_a \\
\hat{C}_a & \hat{D}_a \\
\hat{C}_a & \hat{D}_a
\end{bmatrix}
\begin{bmatrix}
\hat{\xi}_a \\
\xi_a \\
y
\end{bmatrix}.
\] (26)

As will be important in the sequel, we note that \(D_{a}^2 = D_{c}\).

The new extended plant can simply be formed by appending the plant state \(x\) with \(\xi_a\), which evolves as in (24). Note that this extended plant will generate a new measurement vector \(y_2\) in response to two control inputs \(u_1\) and \(u_2\) (as well as the disturbance input \(w\), which were all introduced above. For the convenience of our derivations, we will express the dynamics of the extended plant and the output filter with
\[
\begin{bmatrix}
\psi_o \\
\psi \\
\xi_i
\end{bmatrix} =
\begin{bmatrix}
\psi_o \\
x_o \\
\hat{x}_i
\end{bmatrix} + \begin{bmatrix}
\hat{\Phi}_i \\
\Pi_i \\
I
\end{bmatrix} \begin{bmatrix}
\hat{\xi}_a \\
\xi_a \\
y
\end{bmatrix}.
\] (27)

By inserting (25) in (6)-(7) and then using the definitions of the new states, we can exploit (19) to arrive at
\[
\begin{align*}
\dot{\psi}_o &= A_o \psi_o + B_o \hat{C}_p \psi + B_o D_o \hat{D}_c \psi + C_o \psi + D_o \hat{D}_c \psi \left[ \begin{array}{c} 0 \\
I
\end{array} \right] \psi, \\
\dot{\psi} &= A^\ast \psi + \hat{B}_p \psi \left[ \begin{array}{c} 0 \\
I
\end{array} \right] \psi, \\
z &= C_o \psi_o + D_o \hat{C}_p \psi + D_o \hat{D}_c \psi \left[ \begin{array}{c} 0 \\
E_i
\end{array} \right] \psi, \\
y_2 &= \tilde{C}_p \psi + D_c \hat{D}_c \psi.
\end{align*}
\] (28)
We have thus identified the necessary conditions for the elimination of the input filter modes together with the proper controller structure. As there are no constraints on the realization matrices of $\Sigma_a$, we can now consider its proper choice for the elimination of the output filter dynamics.

### B. Elimination of Output Filter Dynamics

In order to derive the conditions necessary for the elimination of the output filter dynamics, we start by adopting the structured controller derived in the previous section. Let us hence consider the design of $\Sigma_a$ in (26) for the extended plant identified by (28). We can express the dynamics of the feedback interconnection as

$$
\dot{\psi}_o = A_o \psi_o + B_o \dot{C} \dot{z} + B_o D D_i w,
$$

$$
\dot{z} = \begin{bmatrix} \dot{A} + B^e D^2 a B^e C_a A_a \\ \dot{B}_p + B^e D^2 a D c_p D \end{bmatrix} w,
$$

$$
z = C_o \psi_o + D_o \begin{bmatrix} \dot{C}_p + D_{pc} C \dot{D}_p a C_a & D_{pc} C^2_a \end{bmatrix} \dot{z} + D_o D D_i w,
$$

where $D^2_a \triangleq \begin{bmatrix} (D^1_a)^T & (D^2_a)^T \end{bmatrix}^T$ is the feedthrough matrix of $\Sigma_a$. The overall system is described by

$$
\begin{bmatrix} \dot{\psi}_o \\ \dot{z} \end{bmatrix} = \begin{bmatrix} A_o & B_o \dot{C} - \Lambda \\ 0 & \dot{A} \end{bmatrix} \begin{bmatrix} \psi_o \\ \dot{z} \end{bmatrix} + \begin{bmatrix} B_o D D_i - \Omega \\ \dot{B} \end{bmatrix} w.
$$

The ensuing derivations are based on

$$
\varphi_o \triangleq \psi_o - \begin{bmatrix} \Pi_o & \Theta_o & \Phi_o \end{bmatrix} \begin{bmatrix} \psi_o \\ \xi_o \\ \xi_o \end{bmatrix} = \psi_o - \begin{bmatrix} \Pi_o & \Phi_o \end{bmatrix} \begin{bmatrix} \psi_o \\ \xi_o \end{bmatrix},
$$

where $\Psi_o$ is the unique solution of

$$
B_o \dot{C} - \Lambda + A_o \Psi_o - \Psi_o \dot{A} = 0.
$$

For the dynamics of the output filter not to be observable, the solution of this equation has to also satisfy

$$
\Psi_o \dot{B} - B_o \dot{D} D_i + \Omega = 0.
$$

Recalling the partition of $\Psi_o$ as in (31) together with (12), we again reorganize the equations to be satisfied as

$$
[\Pi_o, \Gamma_o] \begin{bmatrix} \dot{A} \\ \dot{B}_p \\ \dot{C} \\ D_{cp} D_i \end{bmatrix} - A_o [\Pi_o 0] = [B_o \dot{C}_p B_o D_p D_i].
$$

Equation (34) clearly has a dual form if compared to (19).

The additional structural constraints on the controller are expressed by (35). We now assume (again) without loss of generality that $\Phi_o$ is a matrix of the form

$$
\Phi_o = \begin{bmatrix} I_{k_a \times k_o} & 0 \end{bmatrix}.
$$

In this case, we identify the structure of $\Sigma_a$ as

$$
[\begin{bmatrix} \xi_o \\ \xi_a \end{bmatrix} \begin{bmatrix} y_2 \\ y_2 \end{bmatrix}] = \begin{bmatrix} A_o - \dot{Z}_o D^2_a - \dot{Z}_o C_a \Gamma_o - \dot{Z}_o D_2^a \\ B^2_1 A_o \\ D^2_1 C_a \\ D^2_2 C^2_a \end{bmatrix} \begin{bmatrix} \xi_o \\ \xi_a \end{bmatrix} \begin{bmatrix} y_2 \\ y_2 \end{bmatrix}.
$$

The suppressed partitions in this expression are given by

$$
[ D^1_a C_a | D^2_a ] = [ D^1_a C^1_a | D^2_a C^2_a ] = [ \hat{C}^1_a | \hat{D}^2_a ].
$$

Let us now isolate the replicated dynamics of the output filter and thereby completely reveal the rationale behind our notation. To this end, we first express the evolution of $\xi_o$

$$
\begin{bmatrix} \dot{\xi}_o \\ \dot{\xi}_a \end{bmatrix} = \begin{bmatrix} A_o \xi_o + \Gamma_o \begin{bmatrix} \hat{C} \psi + D_{cp} D_i w \\ - \dot{Z}_o \begin{bmatrix} C_o \xi_a + D^1_a \xi_a + D^2_a y_2 \end{bmatrix} \end{bmatrix} \end{bmatrix}.
$$

By appending the state $\xi_o$ to the measurement vector as

$$
y_o \triangleq \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} \xi_o \\ y_2 \end{bmatrix},
$$

we identify the free part of the controller in the form

$$
\Sigma_a : \begin{bmatrix} \dot{\xi}_o \\ \dot{\xi}_a \\ \dot{y}_e \end{bmatrix} = \begin{bmatrix} A_o \xi_a + B^1_1 \xi_a + B^2_2 \xi_a \\ C^2_a \xi_a + D^1_a \xi_a + D^2_a y_2 \end{bmatrix} \begin{bmatrix} \xi_o \\ \xi_o \\ y_2 \end{bmatrix}.
$$

In order the introduce the second extension in the plant and thereby obtain the plant to be controlled by $\Sigma_a$, we first introduce a state transformation as

$$
\xi_o \triangleq \xi_o + \Pi_o \dot{\psi} \leftrightarrow \xi_o = \begin{bmatrix} I & -\Pi_o \end{bmatrix} \begin{bmatrix} \xi_o \\ \dot{\psi} \end{bmatrix}.
$$

One can verify by using (34) and (28) that $\xi_o$ obeys the same differential equation as $\psi_o$ (see (28)). This implies that, with $\xi_o(0) = \psi_o(0) = 0$, we will have $\xi_o(t) = \psi_o(t) = 0$ for all $t \geq 0$. Hence, by replacing $\psi_o$ with $\xi_o$ in (28) and adding
the extra measurement, we form the extended plant to be stabilized and optimally controlled by $\Sigma_a$ as
\[
\Sigma_p^e : \begin{cases}
\hat{\psi} = \bar{A}\hat{\psi} + \bar{B}_p w + \left[ \begin{array}{c}
\bar{P}
\end{array} \right] u_e,
\end{cases}
\] (44)
\[
z = \bar{C}_p \hat{\psi} + D_p \bar{D}_p w + D_{pc} E_i u_e,
\]
\[
y_e = \left[ \begin{array}{c}
0
\end{array} \right] \bar{C} \hat{\psi} + \left[ \begin{array}{c}
0
\end{array} \right] D_{pc} D_i w.
\]
Note that $\Sigma_p^e$ differs from $\Sigma_p$ only in the control and measurement channels.

Our results so far can now be summarized as follows:

**Lemma 1:** There exists a solution to Problem 1 with C.3 removed if and only if there exist matrices
\[
\tilde{\Pi}_i = \left[ \begin{array}{c}
\Theta_i \\
\Pi_i
\end{array} \right] \in \mathbb{R}^{(k_i + k) \times k_i}, \quad \Gamma_i \in \mathbb{R}^{n \times k_i},
\]
\[
\tilde{\Pi}_o = \left[ \begin{array}{c}
\Pi_o \\
\Theta_o
\end{array} \right] \in \mathbb{R}^{k_o \times (k + k_o)}, \quad \Gamma_o \in \mathbb{R}^{k_o \times m},
\] (45)
that satisfy equations (19) and (34). With $\Sigma_a$ in (42) representing a controller that stabilizes the extended plant in (44), all controllers that ensure C.1 and C.2 can be realized as
\[
\begin{bmatrix}
A_o - \bar{Z}_o D^1 a - \bar{Z}_o C_a - (\Gamma_o - \bar{Z}_o D^2 a) Z_o \Gamma_o - \bar{Z}_o D^2 a \\
D^1 a & D^2 a
\end{bmatrix}
\begin{bmatrix}
B^1 a & B^2 a \\
D^1 a & D^2 a
\end{bmatrix}
\begin{bmatrix}
A_i - D^1 a \bar{Z}_o & B^1 a \\
D_a & D^2_a
\end{bmatrix}
\begin{bmatrix}
\Gamma_a - \bar{Z}_o D^3 a \\
D^3 a
\end{bmatrix},
\] (46)
where $C_a, D^1 a, D^2 a$ are formed as in (39), while $Z_i$ and $\bar{Z}_o$ are obtained according to (21) and (36) respectively.

**Proof:** Sufficiency proof is omitted.

**IV. $\mathcal{H}_\infty$ Synthesis with Unstable Weighting Filters**

Problem 1 is reduced by Lemma 1 to the design of $\Sigma_a$ for the stabilization of the extended plant $\Sigma_p^e$ and achievement of C.3. There are two important observations that make this problem nonstandard: (i) the plant has dependence on the matrix variables $\Pi_i$ and $\Pi_o$ in the control and measurement channels; (ii) the order of the plant is increased due to the replication of the dynamics of the input and output filters. In this section, we develop a novel approach that leads to convex synthesis conditions and a synthesis procedure that avoids an unnecessary increase in the controller order.

Let us recall a condition for $\|T_{zw}\|_\infty < \gamma$ from [13] as
\[
\begin{bmatrix}
\text{He}(\gamma^T \mathcal{X} \mathcal{A}_1 \mathcal{Y}) \\
\gamma^T \mathcal{X} \mathcal{A}_1 \\
\mathcal{A}_1 \mathcal{Y} \mathcal{B} \\
\mathcal{C}_y \\
\mathcal{D}
\end{bmatrix}
\begin{bmatrix}
\gamma^T \mathcal{X} \mathcal{A}_1 \\
\mathcal{A}_1 \mathcal{Y} \mathcal{B} \\
\mathcal{C}_y \\
\mathcal{D}
\end{bmatrix}
\begin{bmatrix}
\gamma^T \mathcal{X} \mathcal{A}_1 \\
\mathcal{A}_1 \mathcal{Y} \mathcal{B} \\
\mathcal{C}_y \\
\mathcal{D}
\end{bmatrix} < 0,
\] (47)
where $\text{He}(L) = L + L^T$ and *’s represent entries that are identifiable from symmetry. In this condition, $\mathcal{X} > 0$ is a positive-definite matrix, $\gamma$ is a square as well as invertible matrix and $(\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D})$ represents a set of realization matrices for the closed-loop system formed by $\Sigma_p^e$ and $\Sigma_a$. In the sequel, we describe the proper choice of $\gamma$ in terms of the sub-blocks of $\mathcal{X}$ that forms the basis of our novel approach.

We first express the considered realization matrices as
\[
\begin{bmatrix}
\bar{A} & \bar{B} \\
\bar{C} & \bar{D}
\end{bmatrix} = \begin{bmatrix}
A & B_p \\
D_p & D_a D_p D_i
\end{bmatrix}
\]
\[
\begin{bmatrix}
0 & \bar{B}^c(\bar{\Pi}_o) \\
\bar{C}_e & \bar{D}_p D_p D_i
\end{bmatrix} \begin{bmatrix}
A_n B_n \\
C_n D_n
\end{bmatrix} \begin{bmatrix}
\bar{C}^c(\bar{\Pi}_o) \\
E_o D_c D_p D_i
\end{bmatrix},
\] (48)
Let us now partition $\mathcal{X}$ and its inverse compatibly with $\bar{A}$ as
\[
\mathcal{X} = \begin{bmatrix}
\bar{S}^{-1} \bar{X} & \bar{S}^{-1} \\
\bar{U}^T \bar{S}^{-1} & W^{-1} + \bar{U}^T \bar{X}^{-1} \bar{U}
\end{bmatrix},
\]
\[
\mathcal{X}^{-1} = \begin{bmatrix}
\bar{T}^{-1} \bar{Y} \bar{T}^{-T} & \bar{T}^{-1} \bar{V} \\
\bar{V}^T \bar{T}^{-T} & W
\end{bmatrix},
\] (50)
where $\bar{T}$ and $\bar{S}$ are defined as
\[
\bar{T} = \begin{bmatrix}
\bar{P}_o & \bar{\Pi}_i \\
0 & I
\end{bmatrix},
\]
\[
\bar{S} = \begin{bmatrix}
\bar{\Pi}_p p^T & I \\
0 & I
\end{bmatrix}.
\] (52)
We now introduce a partition of $\bar{X}$ that is compatible with $\bar{S}$ and a partition of $\bar{Y}$ that is compatible with $\bar{T}$ as
\[
\bar{X} = \begin{bmatrix}
X_o & Q_o \\
Q_o^T & X_i
\end{bmatrix}, \quad \bar{Y} = \begin{bmatrix}
Y & Q_i \\
Q_i^T & Y_i
\end{bmatrix}.
\] (53)
In terms of the variables introduced so far, we choose
\[
\gamma = \begin{bmatrix}
\bar{T}^{-1} \bar{Y} \bar{P}_o & S \bar{P}_i \\
\bar{V}^T \bar{P}_o & 0
\end{bmatrix}.
\] (54)
It follows from $\mathcal{X} \mathcal{X}^{-1} = I$ that
\[
\gamma^T \mathcal{X} \mathcal{X} = \begin{bmatrix}
\bar{T} & \bar{T}^T \bar{P}_o & S \bar{P}_i \\
\bar{T}^T \bar{Y} \bar{P}_o & 0 & \bar{T}^T \bar{X} \bar{P}_i
\end{bmatrix},
\]
\[
\gamma^T \mathcal{X} \mathcal{X} = \begin{bmatrix}
\bar{T} & \bar{T}^T \bar{P}_o & S \bar{P}_i \\
\bar{T}^T \bar{Y} \bar{P}_o & 0 & \bar{T}^T \bar{X} \bar{P}_i
\end{bmatrix}.
\] (56)
By using these together with the realization considered in (48) and then exploiting (19) and (34), we derive
\[
\begin{bmatrix}
\gamma^T \mathcal{X} \mathcal{A}_1 & \gamma^T \mathcal{X} \mathcal{A}_2 \\
\bar{C}_y & \bar{D}
\end{bmatrix}
\begin{bmatrix}
\bar{A} Y & \bar{P}_o \bar{T} \bar{A} \bar{S} \bar{P}_i \\
0 & \bar{X} \bar{A}
\end{bmatrix}
\begin{bmatrix}
\bar{P}_o & \bar{T} \bar{A} \bar{S} \bar{P}_i \\
C_o Y & C_o P_i + D_o C_p
\end{bmatrix}
\begin{bmatrix}
D_o D_p D_i \\
0
\end{bmatrix}
\begin{bmatrix}
\bar{B} \Gamma_i \bar{Q}_i^T \\
\bar{B} \Gamma_i \bar{Q}_i^T
\end{bmatrix}
\begin{bmatrix}
\bar{P}_o & \bar{T} \bar{A} \bar{S} \bar{P}_i \\
C_o Y & C_o P_i + D_o C_p
\end{bmatrix}
\begin{bmatrix}
\bar{P}_o & \bar{T} \bar{A} \bar{S} \bar{P}_i \\
C_o Y & C_o P_i + D_o C_p
\end{bmatrix}
\begin{bmatrix}
D_o D_p D_i \\
0
\end{bmatrix}
+ \begin{bmatrix}
0 & \bar{B} \\
I & 0
\end{bmatrix} \begin{bmatrix}
K_a & M_a \\
N_a & D
\end{bmatrix} \begin{bmatrix}
I & 0 & 0 \\
0 & \bar{C} & D_c D_p D_i
\end{bmatrix}.
\] (57)
The transformed controller parameters can be defined as
\[
\begin{bmatrix}
K M \\
N D
\end{bmatrix} = \begin{bmatrix}
K_0 + \tilde{P}^T X S^{-1} \tilde{A} \tilde{T}^{-1} \tilde{Y} P_o M_o + Q_1 Q_1 & 0 \\
0 & N_o
\end{bmatrix}.
\]

(59)

In this fashion, we arrive at the following result:

**Theorem 1:** There is a controller of order \(k + k_1 + k_o\) that solves Problem 1 iff there exist matrices as in (45) and
\[
Y = Y^T \in \mathbb{R}^{(k_1+k_1) \times (k_1+k_1)}, X = X^T \in \mathbb{R}^{(k_1+k) \times (k_1+k)},
\]
\[
M \in \mathbb{R}^{(k_1+k) \times m}, N \in \mathbb{R}^n \times (k_1+k), D \in \mathbb{R}^{n \times m},
\]

that satisfy (19) and (34) as well as
\[
\begin{bmatrix}
Y \\
\Xi^T X \\
\Xi
\end{bmatrix} = \begin{bmatrix}
Y_{11} & Y_{12} & \Pi_o & \Theta_o & \Xi \\
Y_{12} & Y_{22} & \Pi_i & \Xi
\end{bmatrix} \succ 0, (61)
\]

\[
\begin{bmatrix}
\text{He}(\dot{A}Y + \dot{BN}) & * \\
(\Pi_i B_i + (B_p + BDD_{cp}) D_i)^T & \gamma \Theta(D, \gamma)
\end{bmatrix} \prec 0, (62)
\]

\[
\begin{bmatrix}
\text{He}(X \dot{A} + M \dot{C}) & * \\
(X B_p + M D_{dp} D_i)^T & \gamma \Theta(D, \gamma)
\end{bmatrix} \prec 0, (63)
\]

where \(\Theta(D, \gamma) = \dot{Y}^T (D, \gamma)\) is used to represent
\[
\Theta(D, \gamma) = \begin{bmatrix}
-\gamma I & * \\
D_o (D_p + D_{dp} D_{cp}) D_i & -\gamma I
\end{bmatrix}.
\]

A controller can then be constructed as follows:

1. Introduce \(T\) and \(S\) with associated partitions as
\[
T \triangleq \begin{bmatrix}
\tilde{P} & \Pi_i \\
\Pi_i & I
\end{bmatrix},
\]

(65)

\[
S \triangleq \begin{bmatrix}
P_o & \Pi_o \\
\Pi_o & I
\end{bmatrix},
\]

(66)

2. Choose \(Q_o\) and \(Q_1\) as
\[
Q_o = 0 \text{ and } Q_1 = \Xi X^{-1} P_i,
\]

(67)

3. Obtain \(K\) as
\[
K = \begin{bmatrix}
\tilde{P} & I
\end{bmatrix} \tilde{T} \begin{bmatrix}
I \\
\Pi_i
\end{bmatrix}^T
+ \begin{bmatrix}
X \tilde{B}_p + M D_{cp} D_i \\
\Pi_o \Pi_o + D_o \tilde{C}_p + D_{dp} D_{cp} \tilde{C}_p
\end{bmatrix} D_i^T \dot{Y} (D, \gamma)^{-1} \begin{bmatrix}
\Pi_i B_i + (B_p + BDD_{cp} D_i)^T \\
C_p Y + D_o D_{dp} N
\end{bmatrix}.
\]

(68)

4. Obtain \(K_o\), \(M_o\), and \(N_o\) as
\[
K_o = K - X \dot{A} \begin{bmatrix}
Y_2 - \Pi_i Q_1 Q_1^T \\
Q_1^T
\end{bmatrix},
\]

(69)

\[
M_o = M - Q_1 Q_1^T \Gamma_o,
\]

(70)

\[
N_o = N - \Gamma_i Q_1^T,
\]

(71)

where \(Y_2 \triangleq \begin{bmatrix}
Y_{21}^T \\
Y_{22}
\end{bmatrix}\).

5. Obtain \(H\) and \(R\) as
\[
H \triangleq Y - \Xi X^{-1} \Xi^T,
\]

(72)

\[
R \triangleq H^{-1} \begin{bmatrix}
-\Pi_o \\
P_o
\end{bmatrix}.
\]

(73)

6. Obtain a realization of \(\Sigma_a\) as
\[
\begin{bmatrix}
A_o & B^1_o \\
C^1_o & D^1_{11} & D^1_{12}
\end{bmatrix}
= \begin{bmatrix}
TX - \hat{B}^T \Pi_o M_o \Pi_i \\
0 & I
\end{bmatrix} \begin{bmatrix}
R \cr 0 & I
\end{bmatrix}.
\]

(74)

7. Construct a realization of \(\Sigma_c\) as in (46).

**Proof:** Omitted for reasons of space.

**Remark 1:** By applying the projection lemma [3] to eliminate the controller parameters, we reduce (62) and (63) to
\[
\begin{bmatrix}
\hat{N}_s^T \\
N_s
\end{bmatrix} \succ 0, (75)
\]

\[
\begin{bmatrix}
\hat{N}_d^T \\
N_d
\end{bmatrix} \succ 0, (76)
\]

where \(\hat{N}_s\) and \(\hat{N}_d\) represent the bases for the null spaces of \([\hat{B}^T D^1_{11} D^1_{12}]\) and \([\hat{C} D_{cp} D_i]\) respectively. Standard conditions [3, 4] are recovered with \(W_1 = I, W_o = I\).

**V. CONCLUDING REMARKS**

We have provided an LMI solution to the problem of \(\mathcal{H}_\infty\) synthesis with anti-stable weighting filters. It is straightforward to adapt the approach for \(\mathcal{H}_2\) synthesis. Extending the solution to handle improper weighting filters requires a study of the problem with weighting filters in descriptor form.

**REFERENCES**


