Stability analysis with integral quadratic constraints: A dissipativity based proof

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Abstract—In this paper we formulate a dissipativity based proof of the well-known integral quadratic constraint (IQC) theorem under mild assumptions. For general dynamic (frequency dependent) IQC-multipliers it is shown that, once the conditions of the IQC-theorem are satisfied, it is possible to construct a nonnegative Lyapunov function that satisfies a dissipation inequality. This not only shows that IQCs can be interpreted as dynamic supply functions, but also opens the way to merge frequency-domain techniques with time-domain conditions known from Lyapunov-theory.

I. INTRODUCTION

A powerful framework for the systematic analysis of uncertain systems is the integral quadratic constraint (IQC) approach [1]. IQCs are very useful in capturing the properties of a rich class of uncertainties, such as e.g. time-invariant or (rate-bounded) time-varying parametric uncertainties, linear time-invariant (LTI) dynamic uncertainties, delay uncertainties or static sector-bounded nonlinearities (see e.g. [1], [2], [3], [4], [5]). Furthermore, the IQC-framework can be employed for synthesis purposes too. Recent years have witnessed a variety of synthesis results allowing not only for static (frequency independent) (see e.g. [6], [7], [8], [9], [10]), but also for general dynamic (frequency dependent) IQC-multipliers (see e.g. [11], [12], [13], [14], [5], [15]).

Another central notion in systems theory is dissipativity [16], [17]. Roughly speaking, a dissipative dynamical system is characterized by the property that, at any time, the amount of energy that is stored in the system never exceeds the amount of energy that has been supplied to the system. This can be formalized for a general class of systems by means of the so-called dissipation inequality which involves a storage and a supply function. An appealing aspect for the analysis of linear dissipative systems is that linear matrix inequalities (LMIs) very naturally emerge. This makes the framework attractive from a computational point-of-view. In fact, one can often relatively easily generalize the notions from linear to time-varying, uncertain and/or nonlinear dissipative systems.

It is relevant to note that there is a common interest in finding a connection between the IQC-framework and the dissipativity approach [18], [19], [20], [21], [22], [23], [24], [25], [26], [27] (see also [28] for another link between the IQC and the multiplier approach [2], [29]). A connection between the two approaches is of particular importance, because it would open the way to merge frequency-domain techniques with time-domain conditions known from Lyapunov-theory. This could lead to generalizations that would be hard to obtain directly in the state-space. So far, a link has only been established for the special case of static and a rather restrictive class of dynamic IQC-multipliers that are satisfied on all finite time horizons. In case of general dynamic IQCs, which only need to hold on infinite time horizons, it remains unclear how to proceed. Here we would like to stress that [26] contains a technical glitch and that [27] has been submitted at a later date.

As a first contribution, we provide (to the best of our knowledge) a novel reformulation of the IQC-theorem which is of independent interest. We will exploit the result for the main contribution of this paper, which is a dissipativity based proof of the IQC-theorem. Indeed, for a rather general class of dynamic IQC-multipliers it will be shown that, once the conditions of the IQC-theorem are satisfied, it is possible to construct a nonnegative Lyapunov function that satisfies a dissipation inequality. The proof relies on (i) the reformulation of the IQC-theorem; (ii) a symmetric Wiener-Hopf factorization [30], which is guaranteed to exist and can be easily constructed through the solution of an algebraic Riccati equation (ARE) [31] (see also [32]); (iii) the gluing lemma, which describes how certain operations on frequency domain conditions can be performed in the state-space [33]; and (iv) on dissipativity theory.

The paper is organized as follows: After a short recap of stability analysis with IQCs in Section II-A–II-C, we formally state the problem in Section II-D. Subsequently, in Section III we present a reformulation of the IQC-theorem which serves as a preparation of the main results in Section IV: A dissipativity based proof of the IQC-theorem. We conclude the paper by giving a brief sketch of possible generalizations and applications in Section V.

Notation: We denote by $\mathcal{L}_2$ the extended space of vector-valued locally square integrable functions on $[0, \infty)$, while $\mathcal{L}_2 \subset \mathcal{L}_2$ is the subspace of functions with finite energy, equipped with the standard inner product $(\cdot, \cdot)$ and the corresponding norm $\| \cdot \|$. Further we denote by $\mathcal{M}_{\mathcal{L}_2}$ the space of real-rational and proper matrix functions without poles on the extended imaginary axis $\mathbb{C}^0 := i\mathbb{R} \cup \{\infty\}$ (in the closed right-half complex-plane). The induced-$\mathcal{L}_2$-norm of a bounded and causal operator is denoted by $\| \cdot \|_{\mathcal{L}_2}$ and realizations of LTI operators are denoted by $G = [\Phi(\omega)]$ or $G = (A, B, C, D)$. Finally, if necessary, we abbreviate $G^* + G$ and $G^* PG$ as $\text{He}(G)$ and $(\star)^* PG$ respectively.
II. A SHORT RECAP ON INPUT-OUTPUT STABILITY

A. The basic setup

Consider the standard feedback interconnection
\[
\begin{align*}
q &= Gp + \mu \\
p &= \Delta(q) + \eta 
\end{align*}
\]  
(1)
where (typically) \( G \in \mathbb{R}^{H} \) represents a nominal and fixed LTI model, \( \Delta : \mathcal{L}_2 \rightarrow \mathcal{L}_2 \) is the trouble-making component that captures the uncertainties which are allowed to vary in a certain class \( \Delta \), and \( \mu, \eta \in \mathcal{L}_2 \) are the exogenous disturbance inputs. Given \( G \) and \( \Delta \), the central task in robust stability analysis is to characterize whether the feedback interconnection (1) remains stable for all \( \Delta \in \Delta \).

For this purpose, let us rewrite (1) as
\[
\begin{pmatrix}
I \\
\Delta
\end{pmatrix}
\begin{pmatrix}
q \\
p
\end{pmatrix} := \begin{pmatrix}
q \\
G
\end{pmatrix} + \begin{pmatrix}
\Delta(q) \\
\mu
\end{pmatrix} + \begin{pmatrix}
\eta
\end{pmatrix}
\]  
(2)
and introduce the following definitions.

Definition 1 (Well-posedness): The feedback interconnection (1) is well-posed if for each \( \operatorname{col}(\mu, \eta) \in \mathcal{L}_2 \) there exists a unique response \( \operatorname{col}(q, p) \in \mathcal{L}_2 \) satisfying (2) such that the map \( \operatorname{col}(\mu, \eta) \rightarrow \operatorname{col}(q, p) \) is causal.

Definition 2 (Stability): The feedback interconnection (1) is stable if it is well-posed and if the \( \mathcal{L}_2 \)-gain of the map \( \operatorname{col}(\mu, \eta) \rightarrow \operatorname{col}(q, p) \) is bounded.

Definition 3 (Robust stability): The feedback interconnection is robustly stable if it is stable for all \( \Delta \in \Delta \).

B. The IQC-framework

As mentioned in the introduction, the IQC-framework offers a systematic way to study the stability properties of complex system interconnections. The basic concept can be summarized as follows [1]: Two signals \( v, w \in \mathcal{L}_2 \) are said to satisfy the IQC defined by the IQC-multiplier \( \Pi = \Pi^* \in \mathbb{R}^{2 \times 2} \) if
\[
\Sigma(\Pi, v, w) := \begin{pmatrix}
v \\
w
\end{pmatrix} \Pi \begin{pmatrix}
v \\
w
\end{pmatrix} \succeq 0. 
\]  
(3)
IQC's can be employed for capturing the properties of uncertainties by imposing suitable constraints on the IQC-multiplier \( \Pi \) such that \( \Sigma(\Pi, q, \Delta(q)) \succeq 0 \) holds for all \( q \in \mathcal{L}_2 \). If we assume \( \Pi \) to belong to the set
\[
\Pi = \{ \Pi = \Pi^* \in \mathbb{R}^{2 \times 2} : \Pi_{11} \succeq 0, \Pi_{22} \preceq 0 \}
\]  
then we can state the following variant of the main result of [1].

Theorem 1: Assume that
1) the feedback interconnection of \( G \) and \( \Delta \) is well-posed;
2) the IQC defined by \( \Pi \in \Pi \) is satisfied for \( \Delta \);
3) \( \begin{pmatrix}
G^* \\
I
\end{pmatrix} \Pi \begin{pmatrix}
G \\
I
\end{pmatrix} \prec 0 \) on \( \mathbb{C}^0 \). 
(4)
Then the feedback interconnection (1) is stable.

Recall that the results in [1] have been proven by means of a homotopy method, without any inertia properties on \( \Pi_{11} \) and \( \Pi_{22} \). For this reason, the first two conditions in Theorem 1 need to be satisfied with \( \tau \Delta \) for all \( \tau \in [0, 1] \), which implies that \( \Pi_{11} \succeq 0 \) on \( \mathbb{C}^0 \). In addition, if we assume that \( \Pi_{22} \preceq 0 \) on \( \mathbb{C}^0 \) then \( \Sigma(\Pi, q, \Delta(q)) \succeq 0 \) for all \( q \in \mathcal{L}_2 \) automatically implies that \( \Sigma(\Pi, q, \tau \Delta(q)) \succeq 0 \) holds for all \( \tau \in [0, 1] \) and \( q \in \mathcal{L}_2 \). Although this might seem restrictive, we still cover most practical IQC-multipliers that are found in the literature. Indeed, apart from certain full-block multiplier relaxation schemes, all IQCs presented in [1], [2], [3], [4], [5] satisfy the above-mentioned inertia constraints. Without too much loss of generality we can hence assume that \( \Pi \in \Pi \). Further note that we only require the feedback interconnection of \( G \) and \( \tau \Delta \) to be well-posed for \( \tau = 1 \). Consequently, it is possible to consider more general uncertainty sets, and even allow for singletons.

C. Robust stability

A key feature of the IQC-framework involves the computational search for a whole family of IQC-multipliers that are parameterized as \( \Pi_p := \Psi^* \Pi \psi \subseteq \Pi \) with a suitable (LMIs) set of symmetric matrices \( P \) and with a typically tall \( \Psi \in \mathbb{R}^{\mathcal{H}} \) such that the IQC defined by \( \Pi = \Psi^* \Pi \psi \) with \( P \in P \) holds for all \( \Delta \in \Delta \). Robust stability can then be guaranteed as follows.

Corollary 1: Suppose that the first two conditions in Theorem 1 hold for all \( \Delta \in \Delta \), with \( \Pi \) replaced by \( \Pi_p \). Then (1) is robustly stable if there exists some \( P \in P \) for which the following frequency domain inequality (FDI) holds:
\[
\begin{pmatrix}
G^* \\
I
\end{pmatrix} \Psi^* P \Psi \begin{pmatrix}
G \\
I
\end{pmatrix} \prec 0 \text{ on } \mathbb{C}^0. 
\]  
(5)
If \( P \) has a nice description, condition (5) can be checked numerically by exploiting the KYP-lemma [34]. Indeed, if we assume \( \Psi \operatorname{col}(G, I) \) to admit the realization \( (A, B, C, D) \) with \( \operatorname{eig}(A) \subset \mathbb{C}^- \), then (5) is equivalent to the existence of some matrix \( V = V^T \) for which the following LMI holds:
\[
\begin{pmatrix}
V A + A^T V V B & V B^T \\
B^T V & (C D)^T P (C D)
\end{pmatrix} \prec 0. 
\]  
(6)
D. Key difficulties to resolve

Let us briefly discuss the main difficulties which occur when one tries to prove Theorem 1 with dissipation arguments. For this purpose, consider the open sets \( X \in \mathbb{R}^n, U \subset \mathbb{R}^n, Y \subset \mathbb{R}^y \) and define the sufficiently smooth mappings \( f : X \times U \rightarrow \mathbb{R}^n, g : X \times U \rightarrow Y \) and \( s : U \times Y \rightarrow \mathbb{R} \). Then we can formally introduce the following definition of dissipativity [16].

Definition 4 (Dissipativity): The system \( \dot{x} = f(x, u), y = g(x, u) \) is dissipative with respect to the supply rate \( s(\cdot, \cdot) \) if there exists a storage function \( V : X \rightarrow \mathbb{R} \) such that
\[
V(x(t_2)) + \int_{t_1}^{t_2} s(u(t), y(t)) dt \leq V(x(t_1)) 
\]  
(7)
holds for all trajectories with \( x(t) \in X, u(t) \in U, y(t) \in Y \) for \( t \in [t_1, t_2] \) and for all \( t_1 < t_2 \).

Let us first consider the special case of static IQC-multipliers (i.e. \( \Psi = I \)). For any trajectory of (1) with initial condition \( x(0) \) we can right and left-multiply (6) with \( \operatorname{col}(x(t), p(t)) \) and its transpose. Then, after integration on \( [0, T] \), we obtain
\[ x(T)^TVx(T) + \int_0^T y(t)^TPy(t)dt \leq x(0)^TVx(0) \quad \forall T \geq 0. \]

Here \( y = \text{col}(Gp, p) = \text{col}(q, \Delta(q)) \). Now observe that the left-upper block of \( P \) in the partition \( \text{col}(G, I) \) is nonnegative just because \( \Psi^*PP = P \in \Pi_P \). Therefore, (6) implies that \( V > 0 \) if and only if \( \text{eig}(A) \subset \mathbb{C}^- \). Further note that \( \int_0^T y(t)^TPy(t)dt \geq 0 \) holds with \( y = \text{col}(q, \Delta(q)) \) for all \( T > 0 \), again, just because \( \Psi^*PP = P \in \Pi_P \). These two properties allow us to prove Theorem 1 for the special case of static IQC-multipliers by classical arguments. On the other hand, for the general case of dynamic IQC-multipliers (i.e. \( \Psi \) is an arbitrary tall and stable transfer matrix), one cannot directly relate the trajectories of \( \Psi \text{col}(G, I) \) to the trajectories of the feedback interconnection (1), since \( \Psi \) is not stably invertible. Moreover, \( P \) is in general an indefinite matrix. Hence, \( \text{eig}(A) \subset \mathbb{C}^- \) neither implies, nor is implied by positive definiteness of \( V \). Further note that \( \int_0^T y(t)^TPy(t)dt \) is in general only nonnegative for \( T = \infty \). All this prevents us to prove stability as for static multipliers. In the sequel we will show how to resolve these difficulties.

III. A Reformulation of Theorem 1

As a preparation for the proof of Theorem 1, we will first reformulate the feedback interconnection (1) as well as the conditions in Theorem 1. For this purpose, let us introduce the signals \( q = q_1 = p_1 \) and \( p = p_2 = q_2 \) with \( p_1, q_1, p_2, q_2 \in \mathbb{L}_2 \). Then (1) can be expressed as

\[
\begin{align*}
\begin{pmatrix} q_1 \\ q_2 \end{pmatrix} &= \begin{pmatrix} I & 0 \\ 2\Delta & -I \end{pmatrix} \begin{pmatrix} p_1 \\ p_2 \end{pmatrix} + \begin{pmatrix} 2\mu_1 \\ 2\mu_2 \end{pmatrix}, \\
\begin{pmatrix} p_1 \\ p_2 \end{pmatrix} &= \begin{pmatrix} I & 0 \\ 2\Delta & -I \end{pmatrix} \begin{pmatrix} q_1 \\ q_2 \end{pmatrix} + \begin{pmatrix} 0 \\ 2\mu_2 \end{pmatrix}.
\end{align*}
\]

Moreover, by replacing the structured external disturbances \( \mu \) and \( \eta \) by the unstructured external disturbances \( \mu_e := \text{col}(\mu_1, \mu_2) \) and \( \eta_e := \text{col}(\eta_1, \eta_2) \) respectively, we obtain the following extended feedback interconnection:

\[
\begin{align*}
\begin{pmatrix} q_e \\ p_e \end{pmatrix} &= \begin{pmatrix} G_e & \mu_e \\ \sigma_e & \eta_e \end{pmatrix} \begin{pmatrix} p_1 \\ p_2 \end{pmatrix} + \begin{pmatrix} 2\mu_1 \\ 2\mu_2 \end{pmatrix},
\end{align*}
\]

(9)

If we also introduce the extended IQC-multipliers

\[
\begin{align*}
\Pi_e := \begin{pmatrix} 0 \\ \Pi_e \end{pmatrix}, \Pi_e = \begin{pmatrix} \Pi_{11e} & \Pi_{12e} \\ \Pi_{21e} & \Pi_{22e} \end{pmatrix} = \begin{pmatrix} \Pi_{11} & \Pi_{12} \\ \Pi_{21} & \Pi_{22} \end{pmatrix} \begin{pmatrix} 0 \\ I \end{pmatrix} + \begin{pmatrix} \varepsilon I \\ 0 \end{pmatrix}
\end{align*}
\]

for \( \varepsilon \geq 0 \), we can state the following result.

Theorem 2.

1) The feedback interconnection (1) is stable iff the feedback interconnection (9) is stable.
2) For all \( \varepsilon \geq 0 \) the IQC defined by \( \Pi_e \) is satisfied for \( \Delta \) iff the IQC defined by \( \Pi_\varepsilon \) is satisfied for \( \Delta_e \).
3) The FDI (4) is satisfied iff there exists a small \( \varepsilon \geq 0 \) such that

\[
\begin{pmatrix} G_e^* \\ I \end{pmatrix} \Pi_e \begin{pmatrix} G_e \\ I \end{pmatrix} \leq 0 \quad \text{on} \quad \mathbb{C}^0.
\]

Proof: Let us start proving the first statement. For this purpose, observe that \( \mathcal{I}(G_e, \Delta_e) \) can be written as

\[
\mathcal{I}(G_e, \Delta_e) = T_2T_3T_4 = T_2T_3T_5T_4, \quad \text{with} \quad T_1 = \text{diag}(-2I, 2I), T_3 = \text{diag}(\mathcal{I}(G, \Delta), -I), T_5 = \text{diag}(I, -I), T_2 = \begin{pmatrix} G_1I & -G_e \\ 0 & I \end{pmatrix} \quad \text{and} \quad T_4 = \begin{pmatrix} I \\ -\Delta_e \end{pmatrix}.
\]

Hence, if \( \mathcal{I}(G, \Delta) \) has a causal inverse on \( \mathbb{L}_2 \), we have that \( \mathcal{I}(G_e, \Delta_e) = T_2T_3T_5T_4 \). Also note that there exist constants \( \kappa_2, \kappa_2, \kappa_4, \kappa_4 > 0 \) with \( ||T_2|| \leq \kappa_2, ||T_4|| \leq \kappa_4, ||T_4|| \leq \kappa_4 \), respectively, we obtain the

\[
\begin{align*}
\mathcal{I}(G_e, \Delta_e) &\leq (T_2T_3)^{-1}T_4T_3^{-1}T_2^{-1} \leq \kappa_2, \kappa_4, \kappa_4 \leq \kappa_2, \kappa_4, \kappa_4.
\end{align*}
\]

Therefore, if we assume that (1) is well-posed and that there exists some \( \kappa_1 > 0 \) such that \( ||T_2T_3|| \leq \kappa_1 \), then

\[
\begin{align*}
\mathcal{I}(T_5T_3)|| \leq \kappa_1 + \kappa_3
\end{align*}
\]

(10)
\[ \tilde{p}_e = \Delta_e(\tilde{q}_e) = \text{col}(q, \Delta(q)) = \tilde{q}_e. \] Hence, for \( q_e = \tilde{q}_e \) the lifted IQC (12) simplifies into
\[ \Sigma(\Pi_e, \tilde{q}_e, \tilde{q}_e) \geq 0, \] \( \tilde{q}_e = \ldots \rightarrow \infty. \)

It remains show the third statement. For this purpose, observe that the FDI (4) can be written as
\[ G^*\Pi_{11}G + G^*\Pi_{12} + \Pi_{12}^*G + \Pi_{22} < 0 \text{ on } \mathbb{C}^0. \]

Clearly there exists some small \( \varepsilon > 0 \) such that
\[ G^*(\Pi_{11} + \varepsilon I)G + G^*\Pi_{12} + \Pi_{12}^*G + \Pi_{22} < 0 \text{ on } \mathbb{C}^0 \]
(14) persists to hold. With \( \Pi_{11e} = \Pi_{11} + \varepsilon I \) we can then apply the Schur complement in order to obtain
\[ \begin{pmatrix} -\Pi_{11e} & \Pi_{11e}G \\ G^*\Pi_{11e}G^* + \Pi_{12}^*G + \Pi_{22} \end{pmatrix} < 0 \text{ on } \mathbb{C}^0, \]
and this can be expressed as
\[ \text{He}(\begin{pmatrix} -I & 2G^* \\ \Pi_{12} & \Pi_{22} \end{pmatrix}) < 0 \text{ on } \mathbb{C}^0, \]
which is (10). The reverse direction directly follows from pre- and post-multiplying (10) with \( \text{col}(G, I)^* \) and \( \text{col}(G, I) \) respectively.

Note that Theorem 2 is an auxiliary result of independent interest. For example, IQC-multipliers of the form \( \Pi = (\alpha, 0, 0) \) have emerged in a number of gain-scheduling and distributed controller synthesis problems [31], [35]. Since the augmented IQC-multiplier \( \Pi_e \) possesses the same structure, Theorem 2 might be useful for generalizations to larger classes of dynamic IQC-multipliers. Further, we will exploit the result in the sequel in order to proof of Theorem 1.

IV. A DISSIPATIVITY BASED PROOF OF THEOREM 1

As discussed in Section II-C, a key feature of the IQC-framework involves the computational search of a symmetric matrix \( P \in \mathbb{P} \) that satisfies the FDI in Corollary 1 in order to verify whether or not the feedback interconnection (1) is robustly stable for a certain class \( \Delta \). It hence makes sense to prove Theorem 1 by taking Corollary 1 as a starting point. For this reason, let us first assume that the first two conditions in Theorem 1 hold for all \( \Delta \in \Delta \), with \( \Pi \) replaced by \( \Pi_e \), and that there exists some \( P \in \mathbb{P} \) for which the FDI (5) is satisfied. Let us also assume, for notational simplicity, that \( \Pi_{11} > 0 \) on \( \mathbb{C}^0 \) such that \( \varepsilon \) in Theorem 2 can be set to zero. By Theorem 2 the FDI (5) is then equivalent to
\[ \begin{pmatrix} \Psi G_e \end{pmatrix}^* \begin{pmatrix} 0 \\ P \end{pmatrix} \begin{pmatrix} \Psi G_e \end{pmatrix} < 0 \text{ on } \mathbb{C}^0. \] (15)

If we assume \( \Psi \) and \( G_e \) to admit the realizations \( (A, B, C, D) \) and \( (A_e, B_e, C_e, D_e) \), with \( \text{eig}(A_e) \subset \mathbb{C}^- \) and \( \text{eig}(A_e) \subset \mathbb{C}^- \), we can define the realization
\[ \begin{pmatrix} \Psi G_e \end{pmatrix} = \begin{bmatrix} A & 0 & B & D_e \\ 0 & A_e & 0 & B_e \\ 0 & 0 & A & B_e \\ C_e & 0 & D_e & D_e \end{bmatrix}. \]

By the KYP-lemma (15) is equivalent to the existence of a matrix \( V_e = V_e^* \) for which the following LMI is satisfied [34]:
\[ \begin{pmatrix} \Psi^*P\Psi \end{pmatrix} = \begin{bmatrix} A & 0 & B & D_e \\ 0 & A_e & 0 & B_e \\ 0 & 0 & A & B_e \\ C_e & 0 & D_e & D_e \end{bmatrix}. \]

Here \( V_e \) is assumed to be partitioned as
\[ V_e = \begin{pmatrix} V_{11} & V_{12} & V_{13} \\ V_{21}^T & V_{22} & V_{23} \\ V_{31}^T & V_{32} & V_{33} \end{pmatrix}, \]
were \( V_{11}, V_{22} \) and \( V_{33} \) have compatible dimensions with \( A, A_e, B_e \) respectively.

Now observe that \( V_e \) is in general an indefinite matrix. As discussed in Section II-D, this is one of the main problems that prevented us from concluding stability of the feedback interconnection (1). In order to overcome this difficulty we need to factorize the IQC-multiplier \( \Psi^*P\Psi \) as \( \Psi^*P^*\Psi \), with \( P = P^T, \Psi, \Psi^{-1} \in \mathbb{H}_\infty \). For this reason, let us collect some existing insights from the literature [31].

Lemma 1: Let \( Q \in \mathbb{H}_\infty \). Then \( Q \) satisfies \( Q = Q^* \) if it admits a realization with the structure
\[ Q = \begin{pmatrix} -A & E_Q & C_Q^* \\ 0 & A & B_Q \\ -B_Q^* & C_Q & D_Q \end{pmatrix}, \] (17)
with \( D_Q = D_Q^T, E_Q = E_Q^T \) and \( \text{eig}(A_Q) \subset \mathbb{C}^- \).

Lemma 2: Let \( S \in \mathbb{H}_\infty \) and \( Q = Q^* \in \mathbb{H}_\infty \) satisfy \( \text{He}(Q) < 0 \) on \( \mathbb{C}^0 \). Then there exists a \( T_Q \in \mathbb{H}_\infty \) with \( T_Q^* = T_Q \in \mathbb{H}_\infty \) such that \( Q = T_Q Q^* \). If \( Q \) is realized as in (17), then \( D_Q \) is invertible and the ARE
\[ A_QZ_Q + Z_Q A_Q + E_Q - (\Psi^* P\Psi)^{-1} D_Q^{-1}(B_Q^T Z_Q + C_Q) = 0 \] (18)
has a unique stabilizing solution \( Z_Q = Z_Q^* \), i.e. \( A_Q - B_Q \tilde{C}_Q \) is Hurwitz for \( \tilde{C}_Q := D_Q^{-1}(B_Q^T Z_Q + C_Q) \). A symmetric canonical Wiener-Hopf factorization is given by \( Q = T_Q Q^* \).

Now observe that
\[ \Psi^*P\Psi = \begin{pmatrix} -A & C_Q^* & PC_Q & C_Q^* & PD_Q \\ 0 & A & B_Q \\ -B_Q^* & D_Q^* & PC_Q & D_Q^* & PD_Q \end{pmatrix} \] (19)
has precisely the structure of (17). Hence, by Lemma 2, feasibility of (16) implies the existence of the stabilizing solution \( Z \) of the ARE (18) corresponding to realization (19) which can be expressed as
\[ (\Psi^* P\Psi)^{-1} \begin{pmatrix} 0 & Z \\ Z & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} I & 0 \\ 0 & P \end{pmatrix} = \begin{pmatrix} \tilde{C}_Q^* \\ \Psi \end{pmatrix} \begin{pmatrix} I \\ \tilde{P} \end{pmatrix} \] (20)

We can thus identify \( S \) and \( Q \) in Lemma 2 with \( G_e \) and \( \Psi^* P\Psi \) respectively, and define the realization \( \tilde{\Psi} = (A, B, \tilde{C}, I) \) such that \( \Psi^* P\Psi = \tilde{\Psi}^* P \tilde{\Psi} \), with \( \tilde{P} \) \( = D_Q^* PC_Q \), \( \tilde{C}_Q = (D_Q^* PD_Q)^{-1}(B_Q Z + D_Q^* PC_Q) \) and \( \tilde{\Psi}, \Psi^{-1} \in \mathbb{H}_\infty \).

It is now possible to merge (20) and (16) by applying the gluing-lemma [33, Theorem 3]. This yields the inequality
for
\[
\tilde{V}_e := \begin{pmatrix}
V_{11} & V_{12} - Z & V_{13}^T & V_{13} \\
V_{12}^T - Z & V_{22} & V_{23} & V_{23}^T \\
V_{13} & V_{23} & V_{33} & V_{33}^T
\end{pmatrix}.
\]
Subsequently, a simple congruence transformation reveals that (21) is equivalent to
\[
\text{He}(\tilde{V}_e \Delta \tilde{V}_e) < 0,
\]
where
\[
\begin{pmatrix}
\tilde{A}_e & \tilde{B}_e \\
\tilde{C}_e & \tilde{D}_e
\end{pmatrix} = \begin{pmatrix}
A_e & -B_e D_e C_e & B_e C_e & B_e D_e \\
0 & A_e & -B_e C_e & B_e \\
-C_e & D_e C_e & C_e & D_e
\end{pmatrix} = \begin{pmatrix}
G_e & \tilde{\Psi} e^{-1} := \tilde{G}_e,
\end{pmatrix}
\]
Clearly, since \(A_e, A_e \neq B_e C_e\) and \(A_e\) are all Hurwitz we can conclude the same for \(A_e\). Moreover, the left-upper block of (22) reads as \(\text{He}(\tilde{V}_e \Delta_e) < 0\) which implies that \(\tilde{V}_e > 0\).

Now recall from Theorem 2 that \(\Sigma(\Pi, q, \Delta(q)) \geq 0\) is equivalent to \(\Sigma(\Pi, q_e, \Delta_e(q_e)) \geq 0\). With \(\Pi = \tilde{\Psi} e\tilde{\Psi}\), \(\tilde{q}_e = \tilde{\Psi} e\tilde{q}\), and \(\tilde{p}_e = \tilde{\Psi} e\tilde{p}\), we obtain \(\Sigma(\tilde{\Theta}, \tilde{q}_e, \Delta_e(q_e)) > 0\), where \(\Delta_e = \tilde{\Psi} e\tilde{\Psi} - \tilde{\Theta}\) defines a bounded and causal operator on \(L_2 e\). We conclude that
\[
\int_0^\infty \tilde{q}_e(t) \tilde{\Delta}_e \tilde{q}_e(t) dt \geq 0 \quad \forall \tilde{q}_e \in \mathbb{L}_2 e.
\]
Moreover, for any \(\tilde{q}_e \in \mathbb{L}_2 e\), we infer that the truncated signal (i.e. \(\tilde{q}_e e\tilde{q}_e = \tilde{q}_e\) on \([0, T]\) and \(\tilde{q}_e e\tilde{q}_e = 0\) on \((T, \infty)\)) satisfies \(\tilde{q}_e \in \mathbb{L}_2 e\). By causality of \(\tilde{\Delta}_e\), we conclude
\[
\int_0^\infty \tilde{q}_e(t) \tilde{\Delta}_e \tilde{q}_e(t) dt \geq 0
\]
\[
\Rightarrow \int_0^\infty \tilde{q}_e(t) \tilde{\Delta}_e \tilde{q}_e(t) dt \geq 0
\]
\[
\Rightarrow \int_0^\infty \tilde{q}_e(t) \tilde{\Delta}_e \tilde{q}_e(t) dt \geq 0
\]
\[
\Rightarrow \int_0^T \tilde{q}_e(t) \tilde{\Delta}_e \tilde{q}_e(t) dt \geq 0 \quad \forall \quad T \geq 0.
\]
In contrast to what we saw in Section II-D, it is now nice to see that (24) does satisfy the nonnegativity property for all \(T \geq 0\). This resolves the last obstacle for proving stability of (1). Indeed, all previous steps were a preparation in order to transform the augmented feedback interconnection (9) into
\[
\begin{cases}
\tilde{q}_e = \tilde{G}_e \tilde{p}_e + \tilde{\mu}_e \\
\tilde{p}_e = \tilde{\Delta}_e \tilde{q}_e + \tilde{\eta}_e,
\end{cases}
\]
with the external disturbances related by \(\tilde{\mu}_e = \tilde{\Psi} \tilde{\mu}_e\) and \(\tilde{\eta}_e = \tilde{\Psi} \tilde{\eta}_e\). As illustrated in Figure 1, observe that we proceeded in a very classical fashion [2], [29]. The key twist is to perform all arguments in the state-space, starting with the original parametrization of the IQC-multiplier \(\Pi = \tilde{\Psi} e\tilde{\Psi}\).
Since the right-hand size is bounded for $T \to \infty$ we can infer the same (due to $\tilde{V}_e > 0$) for the left-hand side. This shows that $\tilde{q}_e, \tilde{p}_e, x, \Delta(\tilde{q}_e) \in L_2$. Robust stability of the feedback interconnection (1) is finally proven by observing that for all $\tilde{\mu}_e, \tilde{\eta}_e \in L_2$ we have
\[
\frac{1}{2} \int_0^\infty \|\tilde{q}_e(t)\|^2 + \|\tilde{p}_e(t)\|^2 \, dt \leq \gamma \int_0^\infty \|\tilde{\mu}_e(t)\|^2 + \|\tilde{\eta}_e(t)\|^2 \, dt.
\]
Clearly this shows that $\|\mathcal{I}(\tilde{G}_e, \Delta_e)^{-1}\|_{\mathcal{L}_2} \leq \gamma$ for all $\Delta \in \Delta$.

V. A BRIEF SKETCH OF POSSIBLE GENERALIZATIONS AND CONSEQUENCES OF THE MAIN RESULT

Let us conclude the paper by briefly discussing the main results. We have established a connection between the IQC-framework and the dissipativity approach. This not only shows that IQCs can be interpreted as dynamic supply functions, but also opens the way to merge frequency-domain techniques with time-domain conditions known from Lyapunov-theory. For example, one could think of incorporation hard time-domain constraints, such as $L_1$ or generalized $H_\infty$ constraints, which could only be applied if the IQCs were valid on all finite horizons. This might lead to nice applications in the field of anti-windup compensator design (see e.g. [10] and references therein). Further, it might also be possible to establish a connection with parameter dependent Lyapunov functions. We refer the reader to [26], [27] for some more concrete applications. However, note that the inertia assumption on the storage function in [26, Theorem 3] is by no means true in general. As shown in Section IV it is required to perform a shifting operation on the storage function through the solution of an algebraic Riccati equation (ARE). Therefore, the storage function does not solely consist of original data, which causes difficulties in finding any of the listed generalizations. This could be resolved by means of an LMI characterization corresponding to the ARE [21].

VI. CONCLUSIONS

In this paper we have given a dissipativity based proof of the well-known integral quadratic constraint (IQC) theorem under mild assumptions. This opens the way to merge frequency-domain techniques with time-domain conditions known from Lyapunov-theory.

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