Funnel control with disturbance observer for two-mass systems

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Abstract—Funnel control in combination with a (simplified) disturbance observer is applied to speed control of elastic two-mass systems where solely motor side speed measurement is available for feedback. The disturbance observer increases damping of the closed-loop system and ensures that the tracking error vanishes asymptotically (i.e. steady state accuracy). Comparative measurements results illustrate improved damping and control performance compared to PI-funnel control (i.e. funnel control in combination with a PI controller).

I. INTRODUCTION

Funnel control — developed by Ilchmann et al. (see e.g. [1], [2], [3] and references therein) — is a (high-gain based) time-varying control strategy which guarantees ‘tracking with prescribed transient accuracy’, i.e. the absolute value of the tracking error (difference between reference and regulated output) is limited by a prescribed (possibly non-increasing) function of time. It is applicable for systems with known upper bound on the relative degree, bounded-input bounded-output (BIBO) stable zero-dynamics, and known sign of the high-frequency gain. Hence, only ‘structural system properties’ must be known for controller implementation yielding robust performance.

Funnel control has been successfully applied to speed and position control of stiffly and elastically coupled mechatronic systems (see e.g. [6], [7], [8], [9], [10] and the survey [11] for a comparison with classical PI(D) control).

Since funnel control is a proportional control strategy, it should be used in combination with a proportional-integral (PI) controller (i.e. PI-funnel control) to achieve ‘steady state accuracy’ (see e.g. [6], [10] or [12]). In [6], PI-funnel control in combination with state feedback and a high-pass filter is applied to speed control of elastic two-mass systems. To achieve active damping of the closed-loop system, motor and load speed must be available for feedback. In most industrial applications, solely a single speed sensor is installed — usually on motor side (lacking load side measurement).

The purpose of this paper: (i) to show that funnel control is admissible for speed control of elastic two-mass systems with (simplified) disturbance observer (as in [13]) when only motor speed is available for feedback (see Proposition 3.1), (ii) to show that funnel control with observer achieves steady state accuracy without additional PI controller (see Corollary 3.3) and (iii) to demonstrate (by measurements) that the closed-loop system with funnel control and observer yields improved damping and control performance compared to PI-funnel control (see Fig. 4 and 5 in Section III).

The following notation is used throughout the paper:

\[ \mathbb{N}, \mathbb{R}, \mathbb{C} \]

\[ [a, b) \]

\[ \mathbb{R}_{>0}, \mathbb{R}_{>0} \]

\[ \mathbb{R}(s), \mathbb{Z}(s) \]

\[ x \in \mathbb{R}^n, o_n \in \mathbb{R}^n \]

\[ \det(A), \text{spec}(A) \]

\[ I_n \in \mathbb{R}^{n \times n} \]

\[ f(:): f(t) \]

\[ C(I; Y) \]

\[ \mathcal{L}_\infty^{\text{(loc)}}(I; Y) \]

\[ \|f\|_\infty \]

\[ \mathcal{W}^k, \infty(I; Y) \]

\[ a, b \]

\[ \mathbb{R} \]

\[ \mathbb{R}^n \]

\[ n \in \mathbb{N} \]

The paper is organized as follows: Section II re-visits funnel control and PI-funnel control for class \( S_1 \) of systems with relative degree one, BIBO stable zero-dynamics and known sign of the high-frequency gain (see Definition 2.1). Section III covers application: funnel speed control of elastic two-mass systems (2MS) which only allow for feedback of the motor speed. The (simplified) disturbance observer from [13] is adopted and it is shown that the two-mass system with this observer is element of system class \( S_1 \). Finally, measurement results at the laboratory setup underpin the theoretical results.

II. FUNNEL CONTROL (RELATIVE-DEGREE-ONE CASE)

For relative-degree-one systems, funnel control is a simple time-varying, proportional output feedback control strategy which guarantees ‘tracking with prescribed transient behavior/accuracy’ (see e.g. [1], [14]) and, in contrast to high-gain adaptive control strategies (e.g. adaptive \( \lambda \)-tracking control), it allows for gain increase and decrease (see e.g. [15]). Measurement noise is tolerated.

A. System class

In view of the speed control application in Section III, we consider the following ‘tailored’ system class (similar to...
that in [16, Section 1.6.2]) consisting of single-input single-output (SISO) systems with relative degree one, BIBO stable zero-dynamics (the unperturbed system is minimum-phase), known sign of the high-frequency gain, bounded disturbances and bounded functional perturbation.

**Definition 2.1 (System class \(S_1\))**: Let \(n, m \in \mathbb{N}, h \geq 0\), \((A, b, c) \in \mathbb{R}^{n \times n} \times \mathbb{R}^n \times \mathbb{R}^n\) and \(B_\Sigma \in \mathbb{R}^{n \times m}\). A system, given by the functional differential equation

\[
\dot{x}(t) = Ax(t) + b(u(t) + u_d(t)) + B_\Sigma \begin{pmatrix} d(t) + (\Sigma x)(t) \end{pmatrix}, \tag{1}
\]

with initial trajectory \(x(-h) = x^0 \in C([-h, 0]; \mathbb{R}^{n+1})\), disturbances \(u_d: [-h, \infty) \rightarrow \mathbb{R}\) and \(d: [-h, \infty) \rightarrow \mathbb{R}^m\), causal operator \(\Sigma: C([-h, \infty); \mathbb{R}^n) \rightarrow L_{\infty}^\text{loc}(\mathbb{R}_0^+; \mathbb{R}^m)\), control input \(u: \mathbb{R}_0^+ \rightarrow \mathbb{R}\) and regulated output \(y(\cdot)\), is of Class \(S_1\) if, and only if, the following ‘system properties’ hold:

\((sp_1)\) the relative degree is one, i.e. \(\gamma_0 := c^T b \neq 0\) and \(\text{sign}(\gamma_0)\) is known;

\((sp_2)\) the unperturbed system is minimum-phase, i.e.

\[
\forall s \in \mathbb{C} \text{ with } \Re(s) \geq 0: \text{ det } \left[ sI_n - A \right] b \neq 0; \tag{2}
\]

\((sp_3)\) operator \(\Sigma\) is of class\(^4\) \(\mathcal{T}\) (see Def. 1 in [1]) and globally bounded, i.e. \(\Sigma \in \mathcal{T}\) and \(M_{\Sigma} := \sup \left\{ \|\Sigma x\| \right\} \| x \| \geq 0\), \(\Sigma(x) \in C([-h, \infty), \mathbb{R}^n)\) is bounded and essentially bounded.

The constant \(h \geq 0\) quantifies the ‘memory’ of the functional perturbation modeled by operator \(\Sigma \in \mathcal{T}\). Operator class \(\mathcal{T}\) comprises e.g. relay, backlash, elasto-plastic & Preisach hysteresis, nonlinear delay systems (see [1] and references therein) and nonlinear, dynamic friction (see e.g. [3]).

**Remark 2.2**: Although funnel control is feasible for systems with actuator (input) saturation (see [17] or [18]), in this paper, input saturation is not considered as it severely increases the effort to achieve active damping.

### B. Control objective

The control objective of funnel control (for system class \(S_1\)) is to assure that the tracking error

\[
e(t) := y_{\text{ref}}(t) - y(t), \quad y_{\text{ref}}(\cdot) \in \mathcal{W}^{1, \infty}(\mathbb{R}_0^+; \mathbb{R}) \tag{2}
\]

(with reference \(y_{\text{ref}}\)) evolves within the prescribed ‘performance funnel’ \(\mathcal{F}_\psi := \{ (t, e) \in \mathbb{R}_0^+ \times \mathbb{R} \mid |e| < \psi(t) \}\) with funnel boundary \(\partial \mathcal{F}_\psi(t) = \psi(t)\) (see Fig. 1), i.e. \((t, e(t)) \in \mathcal{F}_\psi\) for all \(t \geq 0\). More precisely, we want ‘tracking with prescribed transient accuracy’, i.e.

\[
\exists \varepsilon > 0 \forall t \geq 0: |e(t)| \leq \psi(t) - \varepsilon. \tag{3}
\]

The boundary function \(\psi(\cdot)\) is chosen from the set

\[
\psi(0); \quad e(0) = \psi(t); \quad |\dot{\psi}(t)| = \partial \mathcal{F}_\psi(t) = \psi(t) \quad \text{time } t \text{ [s]}
\]

![Fig. 1. Illustration of performance funnel \(\mathcal{F}_\psi\) with boundary \(\psi(\cdot)\), asymptotic accuracy \(\lambda > 0\) and exemplary error \(e(\cdot)\).](image)

Any \(\psi(\cdot) \in \mathcal{B}_1\) is bounded and has essentially bounded derivative. Moreover, for any \(\psi(\cdot) \in \mathcal{B}_1\), the (prescribed) asymptotic accuracy is given by \(\lambda := \lim_{t \to \infty} \psi(t) > 0\). Note that boundary set \(\mathcal{B}_1\) as in (4) also allows for (temporarily) increasing boundaries to e.g. avoid unacceptable large control actions due to (a priori known) rapid changes in reference (or disturbance).

**Example 2.3**: Let \(T_E > 0 [s]\) and \(\lambda \geq 0 > 0\). Then an admissible funnel boundary is given by

\[
\psi_E: \mathbb{R}_0^+ \rightarrow (\lambda, \Lambda], \quad t \mapsto (\lambda - \lambda) \exp \left(-t/T_E\right) + \lambda. \tag{5}
\]

\(\psi_E(\cdot)\) is non-increasing, has asymptotic accuracy \(\lambda > 0\), starts at \(\Lambda = \psi_E(0) = \|\psi_E\|_\infty\) and \(\|\psi_E\|_\infty = (\lambda - \lambda)/T_E\).

### C. Funnel controller

A funnel controller for system class \(S_1\) is given by (see [1], [14] or [16, Theorem 4.4] with identical notation as here)

\[
u_{\text{FC}}(t) = \text{sign}(\gamma_0) k(t) e(t) \tag{FC}
\]

where \(\zeta(\cdot) \in \mathcal{B}_1\) is a scaling function (e.g. to fix a minimal gain since \(k(t) \geq \zeta(t) / \psi(t)\) for all \(t \geq 0\)). Gain scaling has been introduced in [14], (FC) is a proportional but time-varying controller. Integral control action is missing. Gain ‘adaptation’ in (FC) precludes boundary contact and is as follows: gain \(k(\cdot)\) increases, if error \(e(\cdot)\) draws close to \(\psi(\cdot)\) (more aggressive control) and decreases, if error \(e(\cdot)\) becomes small (more relaxed control).

For \(u(t) = u_{\text{FC}}(t)\), funnel controller (FC) assures for systems of class \(S_1\): (i) ‘tracking with prescribed transient accuracy’, i.e. (3), and (ii) boundedness of state \(x(\cdot)\), gain \(k(\cdot)\) and control action \(u(\cdot)\) (see e.g. [16, Theorem 4.4]).

### D. PI-funnel control

It is well-known that proportional controllers might not achieve asymptotic tracking and/or disturbance rejection (of piecewise constant references and/or loads, see e.g. [19]). To allow for steady state accuracy, i.e. \(\lim_{t \to \infty} e(t) = 0\) (FC) must be combined with a PI controller (see e.g. [12], [10] unbounded funnel boundaries with unbounded derivative are also admissible (see [1]).
or [6]) given by
\[ \dot{x}_I(t) = k_I u_{FC}(t), \quad x_I(0) = 0 \]
\[ u_{PI}(t) = x_I(t) + k_P u_{FC}(t) \]
where \( k_P, k_I > 0 \).

For the following, we denote the serial interconnection of (FC) and (PI) as PI-
funnel control and write (FC)+(PI). In [16, Corollary 5.2] it has been shown that (FC)+(PI) is admissible for systems of class \( S_1 \) and, for constant references, that \( \lim_{t \to \infty} e(t) = 0 \) is achieved, if \( \lim_{t \to \infty} \frac{d}{dt} (x(t), x_I(t)) = 0 \) (‘steady state’)
exists.

### III. Application: Speed Control of 2MS

In this section, the speed funnel control problem for elastic two-mass systems (2MS), as discussed in [6], is re-examined for a more realistic scenario: We consider two-mass systems which solely provide measurement of the motor speed. To achieve active damping without full state feedback (required in [6]), the idea of introducing an (simplified) disturbance observer (see [13] and references therein) is adopted. In addition to the mathematical analysis, also measurement results at the laboratory setup (see Fig. 2) are presented.

#### A. Model of elastic two-mass system (2MS)

![Fig. 2. Laboratory setup: elastically coupled electrical drive system.](image)

A 2MS with gear having ratio \( g_r \) [1] (backlash is neglected) consists of two inertias \( \Theta_1 \) [kg m\(^2\)] (on motor side) and \( \Theta_2 \) [kg m\(^2\)] (on load side), which are coupled by an elastic shaft with stiffness \( c_S \) [Nm/rad] and damping \( d_S \) [Nm/rad/s] (see Fig. 2 and Fig. 3). The state variable

\[ x_S(t) := (\omega_1(t), \phi_S(t), \omega_2(t))^\top \in \mathbb{R}^3 \]

represents motor speed in [rad/s], angle of twist in [rad] and load speed in [rad/s] at time \( t \geq 0 \) [s], resp. The mechanical system (see Fig. 3) is accelerated by drive torque \( u \) [Nm] and is subject to actuator disturbance \( u_A \) [Nm], load torque \( m_L \) [Nm] and motor & load side friction, modeled by \( \omega_1(\cdot) \to v_1 \omega_1(\cdot) + (\Phi_1 \omega_1)(\cdot) \) [Nm] and \( \omega_2(\cdot) \to v_2 \omega_2(\cdot) + (\Phi_2 \omega_2)(\cdot) \) [Nm], resp. The friction operators \( \Phi_1, \Phi_2 \) are generally bounded. The mathematical model of a 2MS is given by (see e.g. [6] or [8])

\[
\begin{align*}
\frac{d}{dt} x_S(t) &= A_S x_S(t) + b_S(u(t) + u_A(t)) + B_S \begin{pmatrix} m_L(t) + (\Phi_1 \omega_1(t)) \end{pmatrix} \\
y(t) &= c_S^\top x_S(t), \quad x_S(0) = x_0 \in \mathbb{R}^3
\end{align*}
\]

where

### TABLE I

<table>
<thead>
<tr>
<th>System, implementation and controller data.</th>
</tr>
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<tbody>
<tr>
<td>( A_S = \begin{pmatrix} d_S + g_r^2 v_1 \Theta_1 &amp; -c_S \Theta_1 &amp; d_S \Theta_1 \ g_r \Theta_1 &amp; -1 &amp; 0 \ -g_r \Theta_1 &amp; 0 &amp; -1 \end{pmatrix} ),</td>
</tr>
<tr>
<td>( b_S = \begin{pmatrix} k_d \Theta_1 \ 0 \ 0 \end{pmatrix} ), ( B_S = \begin{pmatrix} 0 \ -1/\Theta_1 \ 0 \end{pmatrix} ), ( c_S = \begin{pmatrix} 1 \ 0 \ 0 \end{pmatrix} ),</td>
</tr>
<tr>
<td>( \Theta_1, \Theta_2 &gt; 0, c_S, d_S &gt; 0, g_r \in \mathbb{R} \setminus {0}, v_1, v_2 &gt; 0, k_A &gt; 0, u_A(\cdot), m_L(\cdot) \in \mathcal{L}^\infty(\mathbb{R}_0^\infty; \mathbb{R}) ),</td>
</tr>
<tr>
<td>and ( \forall i \in {1, 2}: \Phi_i \in \mathcal{T} ) (see Def. 1 in [11]) &amp; ( M_{\Phi_i} := \sup {</td>
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In contrast to the model in [6], the 2MS (6), (7) includes nonlinear, dynamic friction on motor and load side and a gear with ratio \( g_r \neq 0 \). Moreover, it is assumed that only motor side speed measurement is available for feedback (i.e. \( c_S^\top = (1, 0, 0) \)) in contrast to full state-feedback (i.e. \( c_S = (c_1, c_2, c_3) \in \mathbb{R} \setminus \{0\} \)) as in [6]. In most industrial applications a single sensor is installed on motor side.

#### B. Simplified disturbance observer

We adopt the idea of Hori et al. (see [13] and references therein) to implement a (simplified) disturbance observer to increase damping of the closed-loop system. First we re-introduce the observer in the frequency domain (following [13]). Then, we give a state space representation of the observer for the later analysis.

\[ \text{Gain scaling } c(\cdot) \text{ is chosen, such that } k(t) = c(t)/\psi(t) \geq 1 \text{ for all } t \geq 0. \]
1) Analysis in frequency domain: The (simplified) disturbance observer, as proposed in [13], is depicted in Fig. 3 (note the abuse of notation: frequency and time domain are mixed). For the following, we introduce $\hat{k}_A > 0$ (estimate of actuator gain), $\Theta_1 > 0$ (estimate of motor inertia) and $T_{DO} > 0$ (filter time constant) and assume that the Laplace transforms of $\hat{\omega}_1(t)$, $m_{DO}(t)$ and $u(t)$ exist. From the block diagram in Fig. 3 we observe that

$$m_{DO}(s) = (1 - k_{DO}) \left( \frac{u(s)}{1 + s T_{DO}} - \frac{s \hat{\Theta}_1}{k_A \hat{k}_A \hat{\omega}_1(s)} \right).$$

The observer output $m_{DO}(t)$ is the weighted sum of low-pass filtered input $u(t)$ and high-pass filtered speed measurement $\hat{\omega}_1(t)$. The high-pass filter is used to approximate the time derivative of $\omega_1(t)$. If speed measurement is very noisy, large filter constants $T_{DO}$ might be necessary and will corrupt the observer performance. For the ideal case, i.e., $\hat{k}_A = k_A, \Theta_1 = \Theta_1, T_{DO} = 0$ and $m_{DO}(t) = 0$ (hence, $\hat{\omega}_1(t) = \omega_1(t)$), it follows from (8) with $\omega_1(s) = \hat{\Theta}_1 u(s) + u_A(s) - \hat{\omega}_1(s)$ that (see Fig. 3)

$$m_{DO}(s) = (1 - k_{DO}) \left( \frac{\hat{\Theta}_1}{k_A T_{DO}} - \frac{u(s)}{1 + s T_{DO}} \right).$$

Hence, for $k_{DO} = 0$, the observer torque after the actuator, i.e., $m_{DO}/k_A$, corresponds to the disturbance torque $\tilde{m} - k_A u_A$ acting on the motor drive (see Fig. 3).

2) Analysis in state space: To introduce the state space representation of (8), note that a high-pass filter can be represented as low-pass filter with direct feed-through, i.e.

$$\frac{s \hat{\Theta}_1}{k_A T_{DO}} + \frac{1}{1 + s T_{DO}} = \frac{1 - \hat{k}_A}{1 + s T_{DO}} \left( \frac{u(s)}{1 + s T_{DO}} - \frac{\hat{\Theta}_1}{k_A T_{DO}} \right),$$

which yields

$$m_{DO}(s) = \left( \frac{1 - \hat{k}_A}{1 + s T_{DO}} \right) \left( \frac{u(s)}{1 + s T_{DO}} - \frac{\hat{\Theta}_1}{k_A T_{DO}} \right)$$

$$= \left( \frac{1 - k_{DO}}{k_A T_{DO}} \right) \frac{\hat{\Theta}_1}{k_A T_{DO}} \hat{\omega}_1(s).$$

Now, by introducing the state variable $x_{DO}$, invoking (11) and $u(t) = u^*(t) + m_{DO}(t)$, we arrive at (noise is neglected)

$$\dot{x}_{DO}(t) = \frac{1}{T_{DO}} \left( -x_{DO}(t) + u^*(t) + m_{DO}(t) \right) + \frac{\hat{\Theta}_1}{k_A T_{DO}} \hat{\omega}_1(t).$$

Inserting $m_{DO}(t) = (1 - k_{DO}) x_{DO}(t) - \frac{\hat{\Theta}_1}{k_A T_{DO}} \hat{\omega}_1(t)$ above yields the disturbance observer in state space

$$\dot{x}_{DO}(t) = \frac{1}{T_{DO}} \left( -k_{DO} x_{DO}(t) + u^*(t) + \frac{\hat{\Theta}_1}{k_A T_{DO}} \hat{\omega}_1(t) \right)$$

$$m_{DO}(t) = \left( \frac{1 - k_{DO}}{k_A T_{DO}} \right) \left( x_{DO}(t) - \frac{\hat{\Theta}_1}{k_A T_{DO}} \hat{\omega}_1(t) \right),$$

where $x_{DO}(0) = 0$. Note that, for $k_{DO} = 0$, the first line in (DO) gives simple integration of $u^*(t)$ and $m_{DO}(t)$.

The Laplace transform of some $f(\cdot) \in C^1_b([0, \infty); \mathbb{R})$ is given by

$$f(s) := \mathcal{L}(f(t)) := \int_0^\infty f(t) \exp(-st) \, dt$$

for $\mathbb{R}(s) \geq \alpha$ if there exists $\alpha \in \mathbb{R}$ such that $[t \mapsto \exp(-\alpha t)f(t)] \in C^1([0, \infty); \mathbb{R})$ [20, p. 742].

C. 2MS with disturbance observer is element of class $\mathcal{S}_1$

Introduce the ‘new input’ $u^*(t)$ and note that

$$u(t) = u^*(t) + (1 - k_{DO}) \left( x_{DO}(t) - \frac{\hat{\Theta}_1 \omega_1(t)}{k_A T_{DO}} \right).$$

Inserting (12) into (6) and defining the extended state $x(t) := (x_2(t)^T, x_{DO}(t))^T \in \mathbb{R}^4$ yields the extended 2MS with disturbance observer as follows

$$\frac{d}{dt} x(t) = A x(t) + B (u^*(t) + u_A(t))$$

$$+ B_L \left( \left( \hat{\Theta}_1 \omega_1(t) \right), m_{DO}(t) + \left( \tilde{\Theta}_2 \omega_2(t) \right) \right), \quad y(t) = e^T x(t), \quad x(0) = (x_2^0, 0)^T \in \mathbb{R}^4$$

where $A = \begin{bmatrix} -\frac{d_1 + d_2}{\tau_1} & -\frac{1}{\tau_1} & -\frac{c_1}{\tau_1} & 0 \\ \frac{d_1}{\tau_2} & 0 & \frac{c_2}{\tau_2} & 0 \\ \frac{\hat{k}_A k_1}{\hat{k}_A T_{DO}} & 0 & 0 & 0 \\ \frac{1}{\tau_1} & 0 & 0 & 0 \end{bmatrix}$, $B = \begin{bmatrix} \frac{1}{\tau_1} & 0 \\ 0 & 0 \\ 0 & -\frac{1}{\tau_2} \\ 0 & 0 \end{bmatrix}$, $c = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$.

Now the following result can be formulated.

Proposition 3.1: For $T_{DO} > 0$, $k_{DO} \in \mathbb{R}$, $\tilde{u}_A > 0$ and $\hat{\Theta}_1 > 0$, the extended two-mass system (13) with (14) is element of system class $\mathcal{S}_1$.

Proof: Define $h := 0, B_{\hat{\Theta}_1} \hat{\Theta}_1 = B$, $d(t) := (0, m_{DO}(t))^T$ and $f(x)(t) := ((\tilde{\Theta}_1 \omega_1(t)), (\tilde{\Theta}_2 \omega_2(t)))^T$. Then, system (13) can be written in form (1). $e^T b = \frac{\hat{k}_A}{\tau_1} \tilde{u}_A > 0$ yields $(s \tilde{u}_1, s \tilde{u}_2)$. Applying Laplace’s formula to det $[\begin{bmatrix} \frac{1}{\tau_1} & 0 \\ 0 & 0 \\ 0 & -\frac{1}{\tau_2} \\ 0 & 0 \end{bmatrix}]$ several times yields

$$-k_A \Theta_1 - \frac{c_s}{\tau_2} - \frac{d_s + d_2}{\tau_2} \left( s + \frac{k_{DO}}{T_{DO}} + \frac{1 - k_{DO}}{k_A T_{DO}} \right) \neq 0$$

for all $s \in \mathbb{C}$ with $\mathbb{R}(s) \geq 0$, since, for $T_{DO} > 0$ and the data in (7), the polynomials $s^1 + 1/T_{DO}$ and $s^2 + s(d_s + d_2)/\Theta_2 + c_s/\Theta_2$ are both Hurwitz. This completes the proof. ■

Remark 3.2: For application, $k_{DO} \in [0, 1]$ is adequate: (i) $k_{DO} = 1$ disables the disturbance observer, (ii) $k_{DO} = 0$ gives “full” feedback of the disturbance torque and (iii) $0 < k_{DO} < 1$ gives “partial” feedback of the disturbance torque (e.g. useful in presence of actuator saturation, not considered in this paper). Taking noise sensitivity into account, time constant $T_{DO}$ should be chosen as small as possible to approximate the time derivative of $\omega_1(\cdot)$ as good as possible.

D. Steady state accuracy with disturbance observer

In view of the functional perturbation in (1) (or nonlinear, dynamic friction in (6) modeled by operators $\tilde{\Theta}_1, \tilde{\Theta}_2$), it...
cannot be shown that steady state accuracy will be reached in general. But, the following corollary holds.

**Corollary 3.3:** Let \( k_{DO} = 0 \) and \( y_{ref}(t) = g_1 \omega_{2,ref}(t) = y_0 \in \mathbb{R} \) for all \( t \geq 0 \). If steady state of closed-loop system (FC), (13) exists, i.e. \( \lim_{t \to \infty} \frac{d}{dt}(x_2(t), x_{DO}(t)) = 0_1 \), then \( \lim_{t \to \infty} e(t) = 0 \) and \( \lim_{t \to \infty} (\omega_{2,ref}(t) - \omega_2(t)) = 0 \).

**Proof:** In view of [16, Theorem 4.4] and Proposition 3.1, the closed-loop initial-value problem (FC), (13) has a continuous and bounded solution \((x_2(t), x_{DO}(t))\) on \( \mathbb{R}_{\geq 0} \) and (3) holds with \( e(t) = y_{ref}(t) - \omega_1(t) \).

Moreover, \( \lim_{t \to \infty} e(t) = y_{ref}(t) - \omega_1(t) \) implies \( \lim_{t \to \infty} (\omega_{2,ref}(t) - \omega_2(t)) = 0_1 \), which with \( u^*(t) = u_{FC}(t) = \frac{\zeta(t)}{\psi(t) - \hat{e}(t)} \) and \( \frac{\zeta(t)}{\psi(t) - \hat{e}(t)} > 0 \) for all \( t \geq 0 \) yields \( \lim_{t \to \infty} e(t) = \lim_{t \to \infty} (y_{ref}(t) - \omega_1(t)) = 0 \). Hence, \( \lim_{t \to \infty} (\omega_{2,ref}(t) - \omega_2(t)) = 0 \).

Note that Corollary 3.3 holds for any \( \hat{k}_A, \Theta_1 > 0 \). Hence, for \( k_{DO} = 0 \), the observer design is robust and, moreover, funnel control (FC) with disturbance observer (DO) makes PI-funnel control (FC)+(PI) obsolete. Steady state accuracy is achievable even without PI controller.

### E. Implementation and measurements

In view of Proposition 3.1 and [16, Theorem 4.4], funnel control (FC) is admissible for elastic two-mass systems with disturbance observer, i.e. \( u^*(t) = u_{FC}(t) \) in (13), (14). In brief, we write (FC)+(DO) for funnel control with disturbance observer. Moreover, [16, Proposition 5.7 and Corollary 5.2] show that PI-funnel control (FC)+(PI) is admissible for elastic two-mass systems, i.e. \( u(t) = u_{PI}(t) \) in (6), (7). Both controllers, i.e.

- PI-funnel control: (FC)+(PI) and
- Funnel control with dist. observer: (FC)+(DO).

are implemented for speed control of the laboratory setup (see Fig. 2; colored lines are as in Fig. 4, 5). Control objective is tracking with ‘prescribed transient accuracy’ of the constant speed reference \( y_{ref}(\cdot) = g_1 \omega_{2,ref}(\cdot) = 5 \) [rad/s] under time-varying load \( m_L(\cdot) \in L_\infty([\mathbb{R}_{\geq 0}; \mathbb{R}) \) (see bottom of Fig. 4).

Implementation is performed with Matlab/Simulink on a xPC target real-time system running with sampling time \( T_s = 1 \cdot 10^{-3} [s] \). For the disturbance observer design, we choose the estimates \( \hat{k}_A = 0.8 k_A \) and \( \hat{\Theta}_1 = 0.5 \Theta_1 \) (better parameters may be obtained by simple measurements or from data sheets). Further parameter estimation or friction identification/compensation is not required for controller implementation. Implementation data is collected in Tab. I. Note that, the funnel controller (FC) is identical for both experiments. The comparative measurement results are shown in Fig. 4. The experiment is designed such that the available range of the motor torque is not exceeded, i.e. \( \max_{t \geq 0} |m_L(t)| = 22 [\text{Nm}] \). An analysis including actuator saturation is part of future work. Both controller implementations achieve (i) ‘tracking with prescribed transient accuracy’ (on motor side by adequate gain adaption), i.e. (3) with \( e(t) = y_{ref}(t) - \omega_1(t) \) (see second & third row in Fig. 4), and (ii) steady state accuracy on motor and load side, i.e. \( \lim_{t \to \infty} (y_{ref}(t) - \omega_1(t)) = 0 \) and \( \lim_{t \to \infty} (\omega_{2,ref}(t) - \omega_2(t)) = 0 \) (see first & second row in Fig. 4), resp. Important to note, that \( |e(t)| < \psi(t) \) for all \( t \geq 0 \) does not imply \( |\omega_{2,ref}(t) - \omega_2(t)| < \psi(t) \) for all \( t \geq 0 \) (even if measurements might indicate this, see first row of Fig. 4). (FC)+(DO) achieves better damping than (FC)+(PI). For (FC)+(DO) the oscillations in angle of twist \( \phi_S(\cdot) \) are drastically reduced (see Fig. 5). Whereas, for (FC)+(PI), oscillations in \( \phi_S(\cdot) \) are (nearly) undamped and have the resonance frequency 9.7 [Hz] of the laboratory setup (see [16, p. 187]). However, to achieve active damping, (FC)+(DO) initially requires twice as much motor torque.
as (FC)+(PI) (see fourth row in Fig. 4). Finally, control performance is evaluated by means of relative overshoot and the integral time (weighted) absolute error (ITAE) criterion, i.e.,

$$ITAE(\omega_{2,ref}(\cdot) - \omega_2(\cdot), t_0, t_{end}) = \int_{t_0}^{t_{end}} |\omega_{2,ref}(\tau) - \omega_2(\tau)| \, d\tau,$$

where $0 \, [s] = t_0 < t_{end} = 20 \, [s]$ and $\omega_{2,ref}(\cdot) - \omega_2(\cdot) \in L^\infty([0, 20]; \mathbb{R})$. In both categories, (FC)+(PI) shows better performance than (FC)+(DO) (see Tab. II).

IV. OUTLOOK

Future work will concentrate on (i) the extension of the simplified observer (e.g., by including internal models as proposed in [10]), (ii) the analysis of the effects of actuator (input) saturation on the closed-loop system with disturbance observer, (iii) a statement about the evolution of load speed error $\omega_{2,ref}(\cdot) - \omega_2(\cdot)$ and (iv) the use of the disturbance observer for the position funnel control problem of elastic two-mass systems (as discussed in [8]).

REFERENCES


