Quantum Control Model for Spatial Propagation of Electromagnetic Fields in Dielectrics

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Abstract

The design of many quantum optical devices is attributed to the determination of spatially distributed parameters (e.g., refraction index in photonic crystal), which calls for a theoretical description for the spatial propagation of quantum fields. This paper discusses the spatial evolution of quantized electromagnetic fields in waveguide with spatially variant dielectric parameters, which is in parallel with the time evolution in the standard input-output formalism [C. W. Gardiner and M. J. Collett, Phys.Rev.A 31, 3761 (1985)]. This formalism is non-trivial to quantum optics in nonlinear and dispersive medium, and provides a new perspective of modeling and analyzing the control of quantum fields by matter.

1. Introduction

During the past 20 years, quantum technologies which are motivated by quantum information processing [1–3] have been greatly advanced. To manipulate the underlying quantum states, quantum control [4, 5] plays an important role when highly precise and robust operations are demanded. With systematic introduction of control theory, the modeling, analysis and optimization of quantum systems have been broadly studied, and many of the results have been applied to real systems in the laboratory [6, 7].

In the literature, almost all quantum control systems are modeled as time-evolving, and the control parameters are usually taken as time-dependent functions [8, 9], e.g. laser pulses. Such treatment often ignores the spatial distribution of the systems, which is valid for point-like systems (e.g. a charge point, a mass point and an atom) or systems propagating in homogeneous spacetime. However, there are many situations where such assumption is improper. For example, spatial distribution properties have to be considered for electromagnetic fields propagating in dispersive dielectric medium (e.g., fiber-based quantum devices [10–12] and nonlinear optical crystals [13]) or distributed-parameter nanoresonators (e.g., beams or membranes [14]).

In this paper, we will study the modeling of quantum control systems propagating in spatially varying medium (see Fig. 1), while the temporal evolution is decomposed into separate modes by Fourier transform. This is partially inspired by the spirit of Einstein’s relativity theory [15], in which the time and the space are equivalently treated in a covariant manner [16]. We will show that the broadly adopted Langevin equation formalism [17, 18] for quantum control can be translated in parallel to the spatial propagation, but the physics behind are essentially distinct [19].

Roughly speaking, corresponding to the well-known Hiesenberg equation for time evolution of an observable $\hat{O}(t)$

$$i\hbar \frac{d\hat{O}(t)}{dt} = [\hat{O}(t), \hat{H}\{C(t)\}],$$

(1)

where $\hat{H}\{\cdot\}$ is the system’s Hamiltonian involving a time-variant control function $C(t)$, the one-dimensional spatial propagation equation in $x$-direction can be described as follows

$$-i\hbar \frac{\partial \hat{O}(x)}{\partial x} = [\hat{O}(x), \hat{G}\{C(x)\}],$$

(2)

where $\hat{G}\{\cdot\}$ involving a space-variant control function $C(x)$ denotes the momentum operator that boosts the spatial propagation. The minus sign on the left hand side comes from the Minkowski metric.

In addition to the fundamental importance, a practically important motivation for this study is that the spatial variable parameters are naturally used in the design
of optical waveguides. For example, in many applications, the dielectric parameters $\varepsilon_i, i = 1, 2, 3 \cdots$ can be taken as classical control variables that modulate the optical signals. They can be changed with external sources, e.g. electric field [20], acoustic wave [21], magnetic field [22].

Compared with time evolution studies, there are much fewer approaches suitable for modeling the spatial evolutions of electromagnetic fields in dielectrics. One approach is proposed by Huttner [23], where a spatial coordinate dependent operator $\mathcal{C}(x)$ (or $\mathcal{C}_i(x)$) was introduced to represent the wave traveling forward (or backward) in direction $x$. Based on this method, a spatial langevin equation can be derived for traveling electromagnetic fields in optical devices. However, this model is only applicable to linear medium [24]. Another scheme is stemmed from the Maxwell equations [25], where quantum noises are phenomenologically introduced to preserve the commutation relations. The spatial evolution of traveling waves in dielectrics is solved by Green-function method. This approach is convenient for describing nonlinear optical phenomena and multi-slab devices, particularly for the periodic waveguides (e.g. photon crystalline [26]).

In this paper, we will apply the momentum operator to derive the quantum control models for boosting the spatial propagation [27, 28] of electromagnetic waves. This formalism will be useful for calculating photon statistics in linear dielectrics [19]. These equations also provide an alternative way to describe spatially distributed quantum open systems.

This paper is organized as follows. In Section 2, we overview the classical theory of electromagnetic fields described by Maxwell equations. In Section 3 the quantized description of electromagnetic fields is given. Then, in section 3.2 we use the momentum operator methods [28] to establish the spatial langevin evolution equations for one dimension. Conclusions are drawn in Section 4.

2. Classical Description of Electromagnetic Fields in Dielectrics

The dynamics equations of classical electromagnetic fields in dielectrics can be described by the following Maxwell equations

\[
\begin{align*}
\nabla \cdot \mathbf{D} &= \rho, \\
\nabla \times \mathbf{E} &= -\frac{\partial \mathbf{B}}{\partial t}, \\
\nabla \cdot \mathbf{B} &= 0, \\
\n\nabla \times \mathbf{H} &= \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t},
\end{align*}
\]

(3)

where $\mathbf{B}, \mathbf{E}, \mathbf{D}, \mathbf{H}$ are the vectors of magnetic induction, electric field, electric displacement and magnetic field strength; $\rho$ and $\mathbf{J}$ are the charge density and current density distributed in space. The vector

\[
\nabla = \begin{bmatrix}
\frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}
\end{bmatrix}
\]

(4)

denotes the gradient operator. The signs "\times" and ",," mean the cross product and the dot product respectively. The relationship between the electric displacement vector $\mathbf{D}$ and electric field vector $\mathbf{E}$ is

\[
\mathbf{D}(\mathbf{r}, t) = \varepsilon_0 \mathbf{E}(\mathbf{r}, t) + \mathbf{P}(\mathbf{r}, t),
\]

(5)

where $\varepsilon_0$ is the dielectric constant in vacuum. The electric polarization field $\mathbf{P}(\mathbf{r}, t)$ is a linear or nonlinear function of $\mathbf{E}(\mathbf{r}, t)$, which describes the reaction of the dielectrics against the external field $\mathbf{E}(\mathbf{r}, t)$.

The electromagnetic field can be equivalently described by the magnetic vector potential $\mathbf{A}(\mathbf{r}, t)$ and the scalar potential $\varphi(\mathbf{r}, t)$ of the field ($\varphi(\mathbf{r}, t) = 0$ when charge density $\rho$ is zero in the dielectrics), by which the magnetic and electric fields can be expressed as

\[
\mathbf{B}(\mathbf{r}, t) = \nabla \times \mathbf{A}(\mathbf{r}, t),
\]

(6)

\[
\mathbf{E}(\mathbf{r}, t) = -\nabla \varphi(\mathbf{r}, t) - \frac{\partial}{\partial t} \mathbf{A}(\mathbf{r}, t).
\]

(7)

In the electrical insulator, it can be assumed $\rho = 0, \mathbf{J} = 0$ and $\mathbf{B}(\mathbf{r}, t) = \mu_0 \mathbf{H}(\mathbf{r}, t)$. Under the so called Coulomb gauge $\nabla \cdot \mathbf{A} = 0$ [15], the Maxwell equations (3) become

\[
\Delta \mathbf{A}(\mathbf{r}, t) = -\frac{1}{c^2} \frac{\partial^2}{\partial t^2} \mathbf{A}(\mathbf{r}, t) + \mu_0 \frac{\partial}{\partial t} \mathbf{P}(\mathbf{r}, t) = 0,
\]

(8)

where the Laplacian

\[
\Delta = \nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}
\]

(9)
In the linear dielectrics, the electric polarization field \( \mathbf{P}(\mathbf{r},t) \) is a linear functional of \( \mathbf{E}(\mathbf{r},t) \) [29], i.e.

\[
\mathbf{P}(\mathbf{r},t) = \varepsilon_0 \int_{-\infty}^{t} \mathbf{R}^{(1)}(t - \tau) \cdot \mathbf{E}(\mathbf{r},\tau) \, d\tau
\]

\[
= -\varepsilon_0 \int_{-\infty}^{t} \mathbf{R}^{(1)}(t - \tau) \cdot \frac{\partial}{\partial \tau} \mathbf{A}(\mathbf{r},\tau) \, d\tau,
\]

(10)

where \( \mathbf{R}^{(1)}(t) \) is the first-order polarization response function whose Fourier transform \( \chi^{(1)}(\omega) \) is called linear susceptibility.

3. Quantum Spatial Langevin Equation

In this paper, we adopt the macroscopic quantization methods from Maxwell equations [30, 31], from which the quantum spatial Langevin models will be derived for electromagnetic fields traveling in continuous dielectrics.

3.1. Second quantization of electromagnetic fields

Electromagnetic fields vary in the spacetime. In most quantum optics textbooks, an electromagnetic field is decomposed into a group of spatial modes according to the conditions on the boundary (e.g., harmonics in an optical cavity), and the temporal evolution of each mode is treated as a harmonic oscillator. Alternatively, one can decompose the fields into separated temporal modes, and quantize the spatial propagation as harmonic waves as well. Mathematically, this is done by firstly Fourier transforming the Maxwell equations with respect to the temporal (spatial) variables, and then quantizing the resulting equations where the field quantities are only dependent on the spatial (temporal) variables.

From Eqs. (8) and (10), one can see that it is much more difficult to decompose the fields into spatial modes than into temporal modes, which motivates us to study the spatial properties in this paper. To do so, we first Fourier transform Eq. (8) to obtain the Helmholtz equation

\[
\nabla^2 \hat{\mathbf{A}}(\mathbf{r},\omega) + \frac{\omega^2}{c^2} \varepsilon(\omega) \hat{\mathbf{A}}(\mathbf{r},\omega) = 0,
\]

(11)

where \( \hat{\mathbf{A}}(\mathbf{r},\omega) \) is the Fourier transform of \( \mathbf{A}(\mathbf{r},t) \) with respect to the time variable \( t \) and the permittivity \( \varepsilon(\omega) = 1 + \chi^{(1)}(\omega) \) is generally a complex number. Denote that

\[
\sqrt{\varepsilon(\omega)} = n(\omega) = \beta(\omega) + i\gamma(\omega),
\]

(12)

where \( n(\omega) \) is the refractive index. The parameter \( \beta(\omega) > 0 \) is the dispersion coefficient. The absorption coefficient \( \gamma(\omega) \) can be negative in active medium (e.g. Erbium-doped optical fiber amplifier) and be positive when dealing with passive medium (e.g. fused silica optical fiber). The complex permittivity \( \varepsilon(\omega) \) in passive linear dielectrics satisfies the following Kramers-Kronig relation

\[
\varepsilon_R(\omega) = 1 + \frac{P}{\pi} \int_{-\infty}^{+\infty} \frac{d\omega'}{\omega'^2 - \omega^2},
\]

\[
\varepsilon_I(\omega) = -\frac{P}{\pi} \int_{-\infty}^{+\infty} \frac{d\omega'}{\omega'^2 - \omega^2},
\]

(13)

where \( \varepsilon_R(\omega) \) is the real part of \( \varepsilon(\omega) \), \( \varepsilon_I(\omega) \) is the imaginary part of \( \varepsilon(\omega) \) and \( P \) denotes the Cauchy principal value.

Next, to quantize the electromagnetic fields, one need to replace \( \mathbf{A}(\mathbf{r},t) \) and \( -\mathbf{D}(\mathbf{r},t) \) by the Hermitian operators \( \hat{\mathbf{A}}(\mathbf{r},t) \) and \( -\hat{\mathbf{D}}(\mathbf{r},t) \) on a proper Hilbert space, which satisfy the spatial non-commutative relations [32]

\[
[\hat{A}_i(\mathbf{r},t), -\hat{D}_j(\mathbf{r}',t')] = i\hbar \delta_{ij} \delta(t-t'),
\]

(15)

and the temporal non-commutative relations

\[
[\hat{A}_i(\mathbf{r},t), -\hat{D}_j(\mathbf{r},t')] = i\hbar \delta_{ij} \delta(t-t'),
\]

(16)

where \( i, j = x, y, z \) is the direction in space, \( \delta_{ij}(t-t') \) is delta function and \( \delta_{ij}^\perp(\mathbf{r}-\mathbf{r}') \) is the transverse delta function [33] that has the property \( \nabla \cdot \delta_{ij}^\perp(\mathbf{r}-\mathbf{r}') = 0 \), i.e.

\[
\delta_{ij}^\perp(\mathbf{r}-\mathbf{r}') = \int \frac{d^3k}{(2\pi)^3} (\delta_{ij} - \frac{k_ik_j}{k^2}) e^{i\mathbf{k}(\mathbf{r}-\mathbf{r}')}. \]

(17)

Because \( \varepsilon(\omega) \) is a complex number and \( \hat{\mathbf{A}}(\mathbf{r},\omega) \) is Hermitian, a noise operator \( \hat{\mathbf{j}}_n(\mathbf{r},\omega) \) has to be introduced on the right hand side of Eq.(11) [25] for consistency with non-commutative relation (15), i.e.,

\[
\nabla^2 \hat{\mathbf{A}}(\mathbf{r},\omega) + \frac{\omega^2}{c^2} \varepsilon(\omega) \hat{\mathbf{A}}(\mathbf{r},\omega) = \hat{\mathbf{j}}_n(\mathbf{r},\omega).
\]

(18)

Express \( \hat{\mathbf{j}}_n(\mathbf{r},\omega) \) as

\[
\hat{\mathbf{j}}_n(\mathbf{r},\omega) = \frac{\omega}{c^2} \sqrt{\frac{\hbar}{\pi \varepsilon_0}} \varepsilon_I(\omega) \hat{\mathbf{f}}(\mathbf{r},\omega),
\]

(19)

one can verify that \( \hat{\mathbf{f}}(\mathbf{r},\omega) \) satisfies the following normalized non-commutation relations

\[
[\hat{f}_i(\mathbf{r},\omega), \hat{f}_j^\dagger(\mathbf{r}',\omega')] = \delta_{ij}^\perp(\mathbf{r}-\mathbf{r}') \delta(\omega-\omega'),
\]

(20)

\[
[\hat{f}_i(\mathbf{r},\omega), \hat{f}_j(\mathbf{r}',\omega')] = [\hat{f}_i^\dagger(\mathbf{r},\omega), \hat{f}_j^\dagger(\mathbf{r}',\omega')] = 0.
\]

(21)

The noise operator \( \hat{\mathbf{j}}_n(\mathbf{r},\omega) \) or \( \hat{\mathbf{f}}(\mathbf{r},\omega) \) can be taken as environment noise bath as in [17].
As mentioned in introduction, the coefficient $\varepsilon(\omega)$ can change with the position $r$ like the frequency $\omega$, i.e. $\varepsilon = \varepsilon(r, \omega)$, which can be predesigned in optical devices. In this regard, $\varepsilon(r, \omega)$ can also be taken as a classical control input. This forms an important class of quantum control problems where field-matter interactions are used to manipulate the quantized field, while in many other circumstances, the field-matter interaction is used to control the matter.

3.2. Spatial Langevin Equations Based on Momentum Operator Description

Momentum operator method comes from the theory of quantum electrodynamics [19, 34, 35]. Being simple [28, 36–39] and parallel with the cavity quantization method in standard quantum optics, it can facilitate the description of the spatial evolution for the traveling electromagnetic fields.

The spatial evolution of quantum electromagnetic fields is determined by Eq. (2) while the time evolution is determined by the Heisenberg equation (1). As a matter of fact, Eq. (2) can be equivalently translated into the Heisenberg equation (1) in linear and non-dispersive optical systems, but not for nonlinear quantum optical processes. Under general circumstances, we need to introduce the linear momentum operator.

The Minkowski linear momentum [40] of an electromagnetic field distributed in the volume $V$ is defined as [15]

$$G = \iiint_V \mathbf{D} \times \mathbf{B} \, d\tau.$$  \hspace{1cm} (22)

In the following, from Eqs. (5)-(7), we will derive a Langevin equation driven by noise operator $\hat{J}_n(r, \omega)$ or $\hat{F}(r, \omega)$ (the noise may originate from coherent pump resources, thermal lights, vacuum background and other mechanical or electrical sources).

Generally, the Helmholtz equation (18) is hard to solve. However, in many optical devices, electromagnetic beams can be treated as propagating in only one dimension and $\varepsilon(r, \omega)$ varies only in this direction. For example, we can suppose electromagnetic fields traveling in $x$ direction as in Fig. 1 and Eq. (18) becomes [25]

$$\frac{\hbar^2}{\varepsilon} \frac{\partial^2}{\partial x^2} \hat{A}(x, \omega) + \frac{\omega^2}{c^2} \varepsilon(x, \omega) \hat{A}(x, \omega) = \frac{1}{\varepsilon \sqrt{\varepsilon}} \hat{J}_n(x, \omega), \hspace{1cm} (23)$$

where $\varepsilon$ is the cross-sectional area of the beam. In terms of $\hat{A}(x, \omega)$, the electric displacement operator $\hat{D}(x, \omega)$ can be expressed as [41]

$$\hat{D}(x, \omega) = -i \omega \varepsilon_0 \varepsilon(x, \omega) \hat{A}(x, \omega) - \frac{i}{\mu_0 \omega \sqrt{\varepsilon}} \hat{J}_n(x, \omega), \hspace{1cm} (24)$$

and the magnetic induction is

$$\hat{B}(x, \omega) = \frac{\partial}{\partial x} \hat{A}(x, \omega). \hspace{1cm} (25)$$

Then, from Eq. (22) and Eq. (35) in [42], the momentum operator of the whole system at time $t = 0$ reduces to

$$\hat{G} \{ \varepsilon(x, \omega) \} = 2 \frac{\sqrt{\varepsilon}}{\mu_0} \iiint_0^\infty D^{(-)}(x) \hat{B}^{(+)}(x) \, dx, \hspace{1cm} (26)$$

where

$$D^{(-)}(x) = \int_0^\infty d\omega D(x, \omega) \hspace{1cm} (27)$$

and

$$\hat{B}^{(+)}(x) = \int_0^\infty d\omega \hat{B}(x, -\omega) = \int_0^\infty d\omega \hat{B}^\dagger(x, \omega) \hspace{1cm} (28)$$

are the negative and positive frequency parts of the operator $\hat{D}(x)$ and $\hat{B}(x)$ respectively. From the above equations (24)-(27), the linear momentum operator is

$$\hat{G} \{ \varepsilon(x, \omega) \} = \hat{G}_{sys} \{ \varepsilon(x, \omega) \} + \hat{G}_{int} \hspace{1cm} (29)$$

where $\hat{G}_{sys} \{ \varepsilon(x, \omega) \}$ and $\hat{G}_{int}$ are

$$\hat{G}_{sys} \{ \varepsilon(x, \omega) \} = -2 \frac{i \varepsilon_0}{\sqrt{\varepsilon}} \int_0^\infty \int_0^\infty d\omega d\omega' \varepsilon(x, \omega) \hat{A}(x, \omega) \hat{B}^\dagger(x, \omega') \hspace{1cm} (30)$$

and

$$\hat{G}_{int} = -2 \frac{i \sqrt{\varepsilon} \varepsilon_0}{\mu_0} \int_0^\infty \int_0^\infty d\omega d\omega' \frac{1}{\omega} \hspace{1cm} (31)$$

Then, from Eqs. (2), (29) – (31), for arbitrary electromagnetic operator $\hat{Y}(x, \omega)$, the spatial Langevin equation can be established as

$$\frac{\partial}{\partial x} \hat{Y}(x, \omega) = i \hbar \left[ \hat{H}(x, \omega), \hat{G}_{sys} \{ \varepsilon(x, \omega) \} + \hat{G}_{int} \right]. \hspace{1cm} (32)$$

For example, $\hat{Y}(x, \omega) = \hat{A}^2(x, \omega)$ when $\omega > 0$, we can derive that

$$\frac{\partial}{\partial x} \hat{A}^2(x, \omega) = 2 \frac{\varepsilon_0}{\mu_0} \int_0^\infty \int_0^\infty d\omega d\omega' \omega' \left[ \hat{A}(x, \omega), \hat{B}^\dagger(x', \omega') \right] + 2 \frac{\sqrt{\varepsilon}}{\mu_0} \int_0^\infty \int_0^\infty d\omega d\omega' \frac{1}{\omega} \int_0^\infty d\omega'' \cdot \hat{J}_n(x', \omega'') \left[ \hat{A}(x, \omega), \hat{B}^\dagger(x', \omega') \right], \hspace{1cm} (33)$$

1274
where we have used the property
\[ \hat{D}; \hat{E} \hat{F} = [\hat{D}; \hat{E}] \hat{F} + [\hat{E}; \hat{D}] \hat{F} + \hat{E}[\hat{D}; \hat{F}] \] (34)
for arbitrary three operators and
\[ [\hat{A}(x, \omega), \hat{j}_n(x', \omega')] = 0. \] (35)
The latter property is a little tedious to prove, and we give the key steps in the appendix.

It is noted that the advantage of the spatial Langevin equations is the relation to the optical dielectric permittivity \( \varepsilon(\omega) \) in equations. The relation (35) is a classical physical variable and is in common use in optical experiments. If the permittivity changes with time \( t \) or position \( \mathbf{r} \), this method is also effective. This property will be very useful in dealing with spatial periodic structure and designing control structures for optical devices.

An example of two-slab dielectric is given in [25].

4. Conclusion

This paper derives a control model for manipulation quantum electromagnetic fields in spatially designed waveguides. The formalism derived shows that the open quantum control systems can be described by spatial Langevin equations, which is similar to the standard input-output formalism. These equations are even more suitable for modeling traveling electromagnetic fields in general dielectrics, which allows for inhomogeneity and nonlinearity that are to be studied in the future. These methods provide a new approach for evaluating and designing the control strategies that are easier to understand in experiments.

Note that the spatial variable \( \mathbf{r} \) is not the same as the time variable \( t \), because spatial propagations can be non-causal. This essential difference will also be studied in the future. Starting from this formalism, the models in quantum networks such as the series product and the feedback will be explored.

Appendix

In this section, we prove the property in Eq. (35). Suppose the electromagnetic field is traveling in medium with dielectric parameter \( \sqrt{\varepsilon(\omega)} = \beta(\omega) + iy(\omega) \), then using the Green-function method and from equation (20) in [25] and the relation Eq. (19), the operator \( \hat{A}(x, \omega) \) is
\[ \hat{A}(x, \omega) = \int_{-\infty}^{+\infty} dx' G(x, x', \omega) \hat{j}_n(x', \omega) \] (36)
\[ = \frac{\omega}{c^2} \sqrt{\frac{\hbar \varepsilon_i(\omega)}{\pi \varepsilon_0}} \int_{-\infty}^{+\infty} dx' G(x, x', \omega) \hat{f}(x', \omega). \]

Then use the Eq. (21) when \( \omega, \omega' > 0 \), we can get the following relation
\[ [\hat{A}(x, \omega), \hat{j}_n(x', \omega')] = 0. \] (37)

References


