Abstract—We present a method for the accurate approximation of the largest null-controllable set $\mathcal{N}_\infty$ for constrained bilinear systems. It is central to the presented approach that a simple quantitative measure of the accuracy of approximation can be determined. This measure can be used as a termination criterion for an iterative approximation of $\mathcal{N}_\infty$ with step sets. If the termination criterion is met, the proposed method results in an inner approximation of $\mathcal{N}_\infty$ that covers a requested percentage of $\mathcal{N}_\infty$.

I. INTRODUCTION

Consider a nonlinear discrete time system of the form

$$x(k + 1) = f(x(k), u(k)), \quad x(0) = x_0,$$  

(1)

with input and state constraints

$$u(k) \in \mathcal{U} \subset \mathbb{R}^m, \quad x(k) \in \mathcal{X} \subset \mathbb{R}^n, \quad \forall k \in \mathbb{N},$$  

(2)

where $\mathcal{U}$ and $\mathcal{X}$ are solid convex polytopes that contain the origin. Assume that the origin is an equilibrium point of the system, i.e., $f(0,0) = 0$. We call a sequence of inputs admissible if all its elements and the resulting trajectory $x(k)$ respect the constraints $u(k) \in \mathcal{U}$ and $x(k) \in \mathcal{X}$, respectively.

It is a recurring and important problem to calculate or approximate the largest null-controllable set $\mathcal{N}_\infty$, i.e., the set of all states $x_0 \in \mathcal{X}$ for which there exists an admissible input sequence that steers the system to the origin in a finite number of steps (see, e.g., [7], [9] or [11]). The set $\mathcal{N}_\infty$ can be approximated by the set of all states $x_0 \in \mathcal{X}$ that can be steered to the origin with an admissible input sequence with at most $i$ steps. We call this set the $i$-step null-controllable set and denote it by $\mathcal{N}_i$. More precisely, $\mathcal{N}_i$ is iteratively defined by

$$\mathcal{N}_{i+1} = Q(\mathcal{N}_i) \quad \text{with} \quad \mathcal{N}_0 = \{0\},$$  

(3)

where $Q(T)$ refers to the so-called one-step-set

$$Q(T) := \{x \in \mathcal{X} : \exists u \in \mathcal{U} : f(x, u) \in T\}.$$  

(4)

The sequence $\{\mathcal{N}_i\}_{i=0}^\infty$ is known to tend towards the largest null-controllable set, i.e., $\mathcal{N}_i \rightarrow \mathcal{N}_\infty$ as $i \rightarrow \infty$. Moreover, $\{\mathcal{N}_i\}_{i=0}^\infty$ is non-decreasing, i.e., $\mathcal{N}_i \subseteq \mathcal{N}_{i+1} \subseteq \mathcal{N}_\infty$ for all $i \in \mathbb{N}$ [8], and therefore approaches $\mathcal{N}_\infty$ from its interior. The volume fraction

$$\eta_i := \frac{\text{vol}(\mathcal{N}_i)}{\text{vol}(\mathcal{N}_\infty)} \in [0,1]$$  

(5)

indicates how well $\mathcal{N}_i$ approximates $\mathcal{N}_\infty$. We call $\mathcal{N}_i$ accurate approximation of the largest null-controllable set $\mathcal{N}_\infty$, if $\eta_i = \frac{\text{vol}(\mathcal{N}_i)}{\text{vol}(\mathcal{N}_\infty)} \geq \eta^*$ for a given accuracy $\eta^*$.

Unfortunately, $\eta_i$ can in general not be determined, since $\text{vol}(\mathcal{N}_\infty)$ is unknown.

As a remedy we can derive and work with an underesti-
II. EXISTING METHODS

Section II-A applies to the general system class (1). The techniques summarized in Sects. II-B and II-C are restricted to linear and bilinear systems, respectively.

A. The Largest Controlled Invariant Set

We introduced (3) to determine the sequence of null-controllable sets \( \{N_i\}^\infty_{i=0} \). In order to approximate the largest controlled invariant set \( C_{\infty} \), a related sequence \( \{C_i\}^\infty_{i=0} \) can be constructed with

\[
C_{i+1} = Q(C_i) \quad \text{with} \quad C_0 = \mathcal{X},
\]

which only differs from (3) with respect to the initial set \( C_0 \) [3]. The sequence \( \{C_i\}^\infty_{i=0} \) defined by (8) tends to \( C_{\infty} \) for \( i \to \infty \) [2]. Some other basic properties, which follow from the definitions (3) and (8), are as follows. While \( \{N_i\}^\infty_{i=0} \) is non-decreasing, i.e. \( N_i \subseteq N_{i+1} \subseteq N_{\infty} \), \( \{C_i\}^\infty_{i=0} \) is non-increasing, i.e. \( C_i \supseteq C_{i+1} \supseteq C_{\infty} \). By definition, the largest null-controllable set \( N_{\infty} \) is a controlled invariant set, which implies \( N_{\infty} \subseteq C_{\infty} \). By collecting all inclusion properties we obtain

\[
N_i \subseteq N_{\infty} \subseteq C_{\infty} \subseteq C_i
\]

from which we infer \( \text{vol}(N_{\infty}) \leq \text{vol}(C_i) \) for all \( i \in \mathbb{N} \). Thus,

\[
\hat{\eta}_i \leq \frac{\text{vol}(N_{\infty})}{\text{vol}(C_i)} = \eta_i
\]

for all \( i \in \mathbb{N} \), which implies that \( \hat{\eta}_i \) constitutes an underestimator for the accuracy \( \eta_i \) of \( N_i \) defined in (5).

B. A Tailored Underestimator for Linear Systems

The sets \( N_i \) and \( C_i \) need to be known to calculate \( \hat{\eta}_i \) according to (10). Obviously, \( N_i \) and \( C_i \) can be computed by (3) and (8), respectively, if the one-step-set \( Q(T) \) defined in (4) can be evaluated. For linear systems, where \( f(x,u) = Ax + Bu \) for some \( A \in \mathbb{R}^{n \times n}, b \in \mathbb{R}^n \), Keerthis and Gilbert [9] showed how to evaluate (4). It is convenient to introduce the following extended state \( z \) and the associated constraints \( Z \) to summarize the results due to Keerthis and Gilbert [9]:

\[
z := \begin{bmatrix} x \\ u \end{bmatrix} \quad \text{and} \quad Z := \{ z \in \mathbb{R}^{n+1} \mid x \in \mathcal{X}, u \in \mathcal{U} \}
\]

Furthermore, define the matrices \( P := [I_n \ 0] \) and \( S := [A \ b] \) with \( P, S \in \mathbb{R}^{n \times (n+1)} \). Then, according to [9], the one-step-set can be determined with

\[
Q(T) = \begin{bmatrix} S^{-1}T \cap Z \end{bmatrix}
\]

for an arbitrary set \( T \subseteq \mathbb{R}^n \).

Note that (11) results in a convex polytope, if both \( T \) and \( Z \) are convex polytopes (see, e.g., Prop. 3.2 in [8]). Since \( Z, N_0 \) and \( C_0 \) are convex polytopes for linear systems, the sets \( N_i \) and \( C_i \) are convex polytopes for all \( i \in \mathbb{N} \) in this case.

Hence, in the linear case, the evaluation of (10) requires the computation of the volume of two polytopes. Since the volume computation for polytopes is computationally expensive, it is advisable to use another, more efficient underestimator. By \( \lambda_i \in [0,1] \) denote the largest scaling factor such that \( \lambda_i C_i \subseteq N_i \), i.e.

\[
\lambda_i := \max \lambda \quad \text{s.t.} \quad \lambda C_i \subseteq N_i
\]

and note that (12) is a linear optimization problem. Then, an underestimator \( \hat{\eta}_i \leq \eta_i \) is given by

\[
\hat{\eta}_i = \lambda^n,
\]

since \( \lambda_i C_i \subseteq N_i \) implies \( \text{vol}(\lambda_i C_i) \leq \text{vol}(N_i) \) and since \( \text{vol}(\lambda_i C_i) = \text{vol}(\lambda_i I_n C_i) = \det(\lambda_i I_n) \text{vol}(C_i) = \lambda^n \text{vol}(C_i) \).

In summary we have

\[
\lambda^n \leq \frac{\text{vol}(N_i)}{\text{vol}(C_i)} \leq \frac{\text{vol}(N_i)}{\text{vol}(N_{\infty})} = \eta_i
\]

for \( \lambda_i \) from (12).

C. One-Step-Set Computation for Bilinear Systems

We briefly summarize how to compute exact null-controllable sets for bilinear systems (7) subject to polytopic constraints (2). The approach summarized here was introduced in [12]. Lemma 1 states conditions under which the bilinear system (7) can be transformed into a linear one.

Lemma 1 (Lemma 2 in [12]): Let \( c \in \mathbb{R}^n \) be such that

\[
c^T A^k(b + N x) = 0 \quad \text{for all} \ x \in \mathbb{R}^n \ 	ext{and} \ k \in \mathbb{N}_0^2
\]

(14) \( e^T A_{n+1}^{-1}(b + N x^*) \neq 0 \) for all \( x^* \in \{ x \mid b + N x \neq 0 \} \), (15) and let \( \tilde{A} := M^{-1} \tilde{A} M \) and \( \tilde{b} := M^{-1} b \), where

\[
\tilde{A} := \begin{bmatrix} 0 & I_{n-1} \\ 0 & 0 \end{bmatrix}, \quad \tilde{b} := e_n \quad \text{and} \quad M := \begin{bmatrix} e^T A^0 \\ \vdots \\ e^T A^{n-1} \end{bmatrix}
\]

Then, the relation

\[
\bar{A} x + (b + N x) u = \hat{A} x + \hat{u} \varphi(x,u)
\]

holds for all \( x \in \mathbb{R}^n \) and \( u \in \mathcal{U} \), where \( \varphi(x,u) := c^T A_{n+1}^{-1}(b + N x) u + e^T A_{n} x \).

Specifically, we have \( Q(T) = P(S^{-1}T \cap Z) \), where \( S := [\bar{A} \ \tilde{b}] \) and \( \tilde{Z} \) is as in (18). Obviously, the system dynamics are simplified by applying Lem. 1, but \( \tilde{Z} \) must be replaced by the more complicated set \( \tilde{Z} \). Essentially, the bilinearity was removed from the system dynamics, but now appears in \( \tilde{Z} \). In particular, \( \tilde{Z} \) is in general not convex, while \( \tilde{Z} \) is. However, according to the following lemma, the set \( \tilde{Z} \) can be written as the union of two convex polytopes.

Lemma 2 (Lemma 4 in [12]): Let \( \mathcal{U} \) and \( \mathcal{X} \) be convex polytopes of the form

\[
\mathcal{U} = \{ u \in \mathbb{R}^q \mid h_u u \leq d_u \} \quad \text{and} \quad \mathcal{X} = \{ x \in \mathbb{R}^n \mid H x \leq d_x \}
\]

where \( h_u, d_u \in \mathbb{R}^2 \) and \( H, d_x \in \mathbb{R}^{n \times n} \), \( d_x \in \mathbb{R}^n \) with \( q \in \mathbb{N} \). Then, the set \( \tilde{Z} \) as defined in (18) can be expressed as the union \( \tilde{Z} = \tilde{Z}_1 \cup \tilde{Z}_2 \) of the two convex polytopes

\[
\tilde{Z}_1 = \{ \tilde{z} \in \mathbb{R}^{n+1} \mid H_{\tilde{z}}^{(+)} \tilde{z} \leq d_{\tilde{z}}^{(+)} \}\quad \text{and} \quad \tilde{Z}_2 = \{ \tilde{z} \in \mathbb{R}^{n+1} \mid H_{\tilde{z}}^{(-)} \tilde{z} \leq d_{\tilde{z}}^{(-)} \}
\]

where

\[
H_{\tilde{z}}^{(\pm)} = \begin{bmatrix} H_x \\ \mp c^T A^{n-1} N \\ \mp h_u c^T A^n + d_x c^T A^{n-1} N \pm h_u \end{bmatrix},
\]

and

\[
d_{\tilde{z}}^{(\pm)} = \begin{bmatrix} d_x \\ \pm c^T A^{n-1} b \\ \pm d_x c^T A^{n-1} b \end{bmatrix}.\]
Using the decomposition from Lem. 2, null-controllable sets \( N_i \) of bilinear systems can be calculated as follows. Essentially, Lemma 3 implements the iteration (3) and exploits the special structure \( \hat{Z} \) of the transformed bilinear system.

**Lemma 3 (Lem. 5 in [12]):** Let \( \hat{Z}^1 \) and \( \hat{Z}^2 \) be defined as in Eqs. (19)–(22). Assume there exist convex sets \( N_1^0, \ldots, N_2^n \) such that \( N_i = \bigcup_{j=1}^{n} N_i^j \). Define the sets
\[
N_{i+1}^{j+1} = P(\hat{S}^{-1}N_i^j \cap \hat{Z}^1), \quad N_{i+1}^{j+1} = P(\hat{S}^{-1}N_i^j \cap \hat{Z}^2)
\]
for every \( j \in N_i^1 \). Then, for every \( j \in N_i^1, N_{i+1}^{j+1} \) as well as \( N_{i+1}^{j+1} \) is a convex set and
\[
N_{i+1} = \bigcup_{j=1}^{n} N_{i+1}^{j+1} \cup N_{i+1}^{2j+1}
\]
for \( N_{i+1} \) as specified in (3).

The set \( C_{i+1} \) can be calculated analogously to \( N_{i+1} \). Assume \( C_i \) is given as the union \( C_i = \bigcup_{j=1}^{n} C_i^j \) of \( l \) convex sets \( C_i^j \). The expressions (23) and (24) can be replaced by
\[
C_{i+1} = \bigcup_{j=1}^{n} C_{i+1}^{j+1} \cup C_{i+1}^{2j+1}
\]
to calculate \( C_{i+1} \) specified in (8).

Note that the union of convex regions may be convex or non-convex. In general, we obtain non-convex null-controllable sets for bilinear systems (see Ex. 2 in Sec. IV).

**III. ACCURATE APPROXIMATION OF THE LARGEST NULL-CONTROLLABLE SET FOR BILINEAR SYSTEMS**

This section contains the main results of the paper. Section III-A explains why a naive extension of the underestimators (10) and (13) from linear to bilinear systems is not appropriate. This motivates the approach explained in the remainder of Sect. III, which introduces a convenient, tree-based representation of \( N_i = \bigcup_{j=1}^{n} N_i^j \) and \( C_i = \bigcup_{j=1}^{n} C_i^j \) from Lem. 3 (Sect. III-B), the actual calculation of the underestimator \( \hat{N}_i \) (Sect. III-C), and an algorithm for its computation (Sect. III-D).

**A. Naive extension of (10), (13) to the bilinear case fails**

Assume \( N_i = \bigcup_{j=1}^{n} N_i^j \) and \( C_i = \bigcup_{j=1}^{n} C_i^j \) have been determined. In order to evaluate the underestimator \( \hat{N}_i \) from (10), it remains to calculate the volumes \( \text{vol}(N_i) \) and \( \text{vol}(C_i) \).

Unfortunately, this is computationally expensive, since the sets \( N_i^j \) (resp. \( C_i^j \)) are in general not pairwise disjoint. Loosely speaking, we must not just sum over the volumes \( \text{vol}(N_i^j) \) (resp. \( \text{vol}(C_i^j) \)), but need to subtract the volumes of nonempty intersections. More precisely,
\[
\text{vol}(N_i) = \sum_{j \neq 0} (-1)^{|j|+1} \text{vol} \left( \bigcap_{j \neq 0} N_i^j \right)
\]
results with the inclusion-exclusion principle [1, p. 61]. In order to evaluate (27) the volume of \( \sum_{j=1}^{n} \frac{1}{j!} (t-j) \) polytopes must be calculated. If the set \( N_i \) consists of \( l = 16 \) subsets, for example, \( 2l - 1 = 65535 \) polytopes result. Since the computational effort is high for a single polytope, this extension of (10) from the linear to the bilinear case is not attractive from a computational point of view.

Similarly, it is not straightforward to extend the underestimator (13) to the bilinear case. In contrast to the linear case, the sets \( C_i \) may be non-convex (see Ex. 2 in Sect. IV). It is easy to prove and illustrated in Fig. 1b that \( \lambda_i C_i \not\subset C_i \) may result for some or all \( \lambda_i \in [0,1] \) if \( C_i \) is not convex. This implies that \( \lambda_i C_i \not\subset N_i \) cannot hold, since \( N_i \subseteq C_i \) according to (9). Consequently, the optimization problem (12) is in general not meaningful in the bilinear case.

\[\text{Fig. 1.}\] Let \( \lambda < 1 \). (a) Scaled convex set \( T \) with \( \lambda T \not\subseteq T \). (b) Scaled non-convex set \( T \) with \( T \not\subseteq \lambda T \). (c) Scaled convex set \( T \) with \( 0 \not\subseteq T \), \( \mu \not\subseteq T \) and \( \lambda (T - \mu) + \mu \not\subseteq T \).

**B. Null-controllable set representation with binary tree**

It proves to be convenient to describe \( N_i \) (resp. \( C_i \)) by a binary tree, where each node corresponds to exactly one of the subsets \( N_i^j \) of the union \( N_i = \bigcup_{j=1}^{n} N_i^j \). A node is uniquely characterized by the tuple \((i, j)\) of its subset \( N_i^j \), where \( i \) and \( j \) are the depth of the node and the number of the branch counted from left to right, respectively (see Fig. 2). The root node \((0, 1)\) corresponds to \( N_0 = N_0^0 = \{0\} \), which obviously is a convex set. Recall \( N_i^j \) associated with the node \((i, j)\) (where \( j \in N_i^1 \)) is recursively defined by (28). The following lemma essentially states that the set \( N_i \) is given by the union of all subsets \( N_i^j \) associated with the nodes on the level \( i \) of the binary tree (cf. Fig. 2). Sets \( C_i = \bigcup_{j=1}^{n} C_i^j \) can be represented by a binary tree correspondingly.
The proof of Lem. 4 is given in the appendix.

C. An underestimator for the current accuracy

The following proposition shows how to calculate \( \hat{\eta}_i \) for \( N_i \) from the subsets \( N_i^j \) and \( C_i^j \).

Proposition 1: Let \( i \in \mathbb{N} \) be arbitrary and, for all \( j \in \mathbb{N}^J_i \), define \( N_i^j \) and \( C_i^j \) as in Lemma 4. Let \( \mathcal{J}_i \subseteq \mathbb{N}^J_i \) be arbitrary such that \( N_i^j \neq \emptyset \) for all \( j \in \mathcal{J}_i \) and assume

\[
\bigcup_{j=1}^{\mathcal{J}_i} N_i^j = \bigcup_{j \in \mathcal{J}_i} N_i^j \quad \text{and} \quad \mathcal{Q}(\mathcal{B}) \subseteq \mathcal{B},
\]

where \( \mathcal{B} := \bigcup_{j \in \mathcal{J}_i} C_i^j \). Then there exist, for every \( j \in \mathcal{J}_i \), \( \mu_{ij} \in N_i^j \) and \( \lambda_{ij} \in [0, 1] \) such that

\[
\lambda_{ij} := \max_\lambda \left\{ \lambda : \lambda (C_i^j - \mu_{ij}) + \mu_{ij} \subseteq N_i^j \right\}
\]

which yields \( \lambda_{ij} \in [0, 1] \), since

\[
0 \cdot (C_i^j - \mu_{ij}) + \mu_{ij} = \mu_{ij} \subseteq N_i^j \subseteq 1 \cdot (C_i^j - \mu_{ij}) + \mu_{ij} = C_i^j.
\]

It remains to prove (33). Assumption (32) implies

\[
\lambda_{ij} (C_i^j - \mu_{ij}) + \mu_{ij} = \lambda_{ij} \lambda_{ij} (C_i^j - \mu_{ij}) = \lambda_{ij} \lambda_{ij} \lambda_{ij} (C_i^j - \mu_{ij}) 
\]

since \( \lambda_{ij} \lambda_{ij} \lambda_{ij} (C_i^j - \mu_{ij}) = \lambda_{ij} \lambda_{ij} \lambda_{ij} (C_i^j - \mu_{ij}) \). We have \( \lambda_{ij} \lambda_{ij} \lambda_{ij} (C_i^j - \mu_{ij}) \leq \lambda_{ij} \lambda_{ij} \lambda_{ij} (C_i^j - \mu_{ij}) \) for every \( j \in \mathcal{J}_i \). Now, according to Lemma 5,

\[
\hat{\eta}_i \leq \frac{\text{vol}(\mathcal{N}_i)}{\text{vol}(\mathcal{J}_i)} \quad \text{with} \quad T := \bigcup_{j \in \mathcal{J}_i} N_i^j,
\]

where \( \hat{\eta}_i \) is as defined in (33). We obviously have \( T = \bigcup_{j \in \mathcal{J}_i} N_i^j = \bigcup_{j=1}^{\mathcal{J}_i} N_i^j = \mathcal{N}_i \), due to the first condition in (31) and according to Lemma 4. Moreover, the second condition in (31) in combination with Lemma 6 guarantees that \( \mathcal{N}_\infty \subseteq \mathcal{B} \). Thus, we have \( \text{vol}(\mathcal{N}_\infty) \leq \text{vol}(\mathcal{B}) \) and finally

\[
\hat{\eta}_i \leq \frac{\text{vol}(\mathcal{J}_i)}{\text{vol}(\mathcal{B})} \leq \frac{\text{vol}(\mathcal{N}_\infty)}{\text{vol}(\mathcal{B})}.
\]

D. An algorithm for the accurate approximation of \( N_\infty \)

Algorithm 1 implements the iterative computation of the null-controllable sets \( \mathcal{N}_i \) according to (23)–(24). In order to provide a lower bound for the current accuracy of the approximation of \( N_\infty \), the underestimator \( \hat{\eta}_i \) as introduced in Prop. 1 is considered. Obviously, if the termination criterion \( \hat{\eta}_i \geq \eta^* \) is met, Alg. 1 returns an accurate approximation of the largest null-controllable set in terms of \( \mathcal{N}_i \). Otherwise, the algorithm stops unsuccessfully after a finite, user-defined number of steps \( i^* \in \mathbb{N} \). Nevertheless, in the last case, a measure of the achieved accuracy \( \hat{\eta}_i \) is still returned.

Algorithm 1: Accurate approximation of the largest null-controllable set for bilinear systems.

1. Set \( N_0 = \{0\}, C_0 = \mathcal{X}, \eta_0 = 0 \).
2. While \( \eta^* > \hat{\eta}_i \) and \( i < i^* \) do
3. For each \( j \in \mathcal{J}_i \) do
4. Compute \( N_{i+1}^j \) and \( C_{i+1}^j \) according to (23).
5. Compute \( C_{i+1}^j \) and \( C_{i+1}^j \) according to (25).
6. Set \( \mathcal{J}_{i+1} := \{j \in \mathcal{J}_i \} \cup \{j \in \mathcal{J}_i \} \).
7. Set \( i := i^* + 1 \) and compute \( \hat{\eta}_i \) according to (34).
8. Set \( \mathcal{N}_i := \bigcup_{j \in \mathcal{J}_i} N_i^j \) and \( B := \bigcup_{j \in \mathcal{J}_i} C_i^j \).
9. If \( \mathcal{Q}(\mathcal{B}) \subseteq \mathcal{B} \) then
10. For each \( j \in \mathcal{J}_i \) do
11. Choose \( \mu_{ij} \in N_i^j \) and compute \( \lambda_{ij} \) according to (37).
12. Set \( \hat{\eta}_i := \max(\hat{\eta}_i, 1 + \sum_{j \in \mathcal{J}_i} (\lambda_{ij} - 1)^{-1}) \).
13. Else \( \hat{\eta}_i := \hat{\eta}_i - 1 \).
14. Return set \( \mathcal{N}_i \), accuracy \( \hat{\eta}_i \) and terminate.

IV. Numerical examples

We first demonstrate the stepwise approximation of \( N_\infty \) with a one-dimensional example. We then present a two-
dimensional example and illustrate the non-convexity of $N_\infty$. For both examples, we try to achieve the accuracy $\eta^* = 0.99$ and consider the (polytopic) constraints $\mathcal{X} = \{ x \in \mathbb{R}^n \mid \|x\|_\infty \leq 2 \}$ and $\mathcal{U} = \{ u \in \mathbb{R} \mid \|u\|_\infty \leq 1 \}$. Note that $\eta^* = 0.99$ essentially means that the resulting approximation covers 99% of the exact largest null-controllable set.

**Example 1:** Consider the bilinear system with $A = 1.2,$ $b = 0.4$ and $N = 0.8.$ Without giving details, we claim that $c = 1.0$ fulfills the conditions (14)–(15) and that the exact linearization reads $\hat{A} = 0.0,$ $\hat{b} = 1.0$ with $M = c^T = 1.0.$ The sets $\hat{Z}^1$ and $\hat{Z}^2$ defined in (19)–(22) are visualized in Fig. 3. We refer to [12] for more details on the representation of $\hat{Z}.$

![Fig. 3. Sets $\hat{Z}^1$ and $\hat{Z}^2$ for Ex. 1. Sets $\hat{S}^{-1}N_0 \cap \hat{Z}^1,$ $\hat{S}^{-1}C_0 \cap \hat{Z}^1$ and $\hat{S}^{-1}C_0 \cap \hat{Z}^2,$ which are required in step $i = 0$ of Alg. 1, are marked in green and yellow, respectively.](image)

We analyze the first step of Alg. 1 in detail. Lines 3 and 4 are carried out with the sets $\hat{S}^{-1}N_0^i = \{ 0.0 \ 1.0 \}^{-1}\{ 0 \}$ and $\hat{S}^{-1}C_0^i = \{ 0.0 \ 1.0 \}^{-1}\mathcal{X} = \{ \hat{x} \in \mathbb{R}^2 \mid -2 \leq \hat{x}_2 \leq 2 \},$ respectively. Subsequently, the intersections of $\hat{S}^{-1}N_0^i$ and $\hat{S}^{-1}C_0^i$ with the sets $\hat{Z}^1$ and $\hat{Z}^2$ are calculated. The resulting sets are visualized in Fig. 3. Obviously, the set $\hat{S}^{-1}N_0^i \cap \hat{Z}^2$ is empty. Finally, evaluating the projections in (23) and (25) yields

$$\begin{align*}
N_1^i &= \{-0.2, 1.0\}, & N_2^i &= \emptyset \quad \text{and} \\
C_1^i &= \{-0.5, 2.0\}, & C_2^i &= \{-2.0, -0.5\}.
\end{align*}$$

We make two interesting observations. First, the set $N_2^i$ is empty. Thus, we have $N_2^i \subseteq N_1^i$ and the evaluation of the index set $J_i$ in line 7 of Alg. 1 will result in $J_i = \{ 1 \}.$ Second, the set $C_1^i$ reads $C_1^i \cup C_2^i = \{-2.0, 2.0\} = C_0.$ Thus, the largest controlled invariant set equals the state constraints, i.e., $C_\infty = \mathcal{X}.$ An analysis of the following steps reveals that $N_i^i = N_1^i,$ thus $J_i = \{ 1 \},$ for every $i \in \mathbb{N}.$ Moreover, we find that the set $C_1^i$ tends towards $N_1^i$ for $i \to \infty$ (see Tab. IV). In fact, we meet the termination criterion $\hat{\eta}_i \geq \eta^*$ of Alg. 1 for $i = 6.$ Thus, the set $\hat{N}_6 = \{-0.393750, 2.0\}$ approximates $N_\infty$ within the chosen accuracy $\eta^* = 0.99.$ Note that it is easy to prove that the largest null-controllable set reads $N_\infty = \{-0.4, 2.0\}.$ The proper termination of Alg. 1 is remarkable, since $C_\infty = \{-2.0, 2.0\}.$ Thus, the direct evaluation of the underestimator (10) (as proposed for linear systems) yields

$$\hat{\eta}_i = \frac{\text{vol}(N_i^i)}{\text{vol}(C_i^i)} \leq \frac{\text{vol}(N_\infty)}{\text{vol}(C_\infty)} = \frac{2.4}{4.0} = 0.6 \ll \eta^* = 0.99.$$ 

Obviously, it is not possible to meet the termination criterion $\hat{\eta}_i \geq \eta^*$ by calculating $\frac{\text{vol}(N_i^i)}{\text{vol}(C_i^i)}$ for this example.

### Table I

**Numerical results for Ex. 1.**

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<thead>
<tr>
<th>$i$</th>
<th>$N_i = N_i^1$</th>
<th>$C_i^1$</th>
<th>$\hat{\eta}_i$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>$[0.0000000, 0.0]$</td>
<td>$[0.0000000, 2.0]$</td>
<td>$0.000000$</td>
</tr>
<tr>
<td>1</td>
<td>$[-0.2000000, 1.0]$</td>
<td>$[-0.5000000, 2.0]$</td>
<td>$0.3750012$</td>
</tr>
<tr>
<td>2</td>
<td>$[-0.3000000, 2.0]$</td>
<td>$[-0.4500000, 2.0]$</td>
<td>$0.8846156$</td>
</tr>
<tr>
<td>3</td>
<td>$[-0.3500000, 2.0]$</td>
<td>$[-0.4250000, 2.0]$</td>
<td>$0.9400001$</td>
</tr>
<tr>
<td>4</td>
<td>$[-0.3875000, 2.0]$</td>
<td>$[-0.4062500, 2.0]$</td>
<td>$0.9845301$</td>
</tr>
<tr>
<td>5</td>
<td>$[-0.3997500, 2.0]$</td>
<td>$[-0.4031250, 2.0]$</td>
<td>$0.9922279$</td>
</tr>
<tr>
<td>6</td>
<td>$[-0.3996094, 2.0]$</td>
<td>$[-0.4001563, 2.0]$</td>
<td>$0.9955118$</td>
</tr>
</tbody>
</table>

**Example 2:** Consider the bilinear system with $A = \begin{bmatrix} 1.12 & 0.54 \\ 0.76 & 0.92 \end{bmatrix},$ $b = \begin{bmatrix} 0.5 \\ -1.0 \end{bmatrix},$ $N = \begin{bmatrix} 0.4 \\ -0.8 \end{bmatrix}.$ We claim without proof that $c^T = \begin{bmatrix} 2.0 & 1.0 \end{bmatrix}$ fulfills (14)–(15) and that the exact linearization is given in terms of $\hat{A} = \begin{bmatrix} 6.0 & 4.0 \\ -9.0 & -6.0 \end{bmatrix},$ $\hat{b} = \begin{bmatrix} -1.0 \\ 2.0 \end{bmatrix},$ $M = \begin{bmatrix} 2.0 & 1.0 \\ 3.0 & 2.0 \end{bmatrix}.$ Algorithm 1 terminates after the 27th step (see Tab. IV). The set $\hat{N}_{27}$ as well as some intermediate results for the steps $i = \{ 2, 5, 10 \}$ are visualized in Fig. 4. Obviously, the largest null-controllable set is not convex.

![Fig. 4. Null-controllable sets $N_i$ (green) and corresponding overestimations $B$ (yellow) evaluated for Ex. 2. Red areas visualize $\hat{X} \setminus B.$ Sets $N_i$ and $B$ are intermediate results of Alg. 1 for the steps $i \in \{ 2, 5, 10, 27 \}.$](image)

It is apparent from Fig. 4 that the proposed underestimator is conservative. While the approximations obtained after 10 steps is found to be very close to that after 27 steps by visual inspection, the value of the underestimator after 10 steps,
\( \hat{\eta}_{10} = 0.0485 \), suggests that the approximation is still far from accurate. Table IV provides another interesting result. Assume that the set \( \mathcal{J}_i \) (as evaluated in line 7 of Alg. 1) reads \( \mathcal{J}_i = N_{i+1}^2 \) for each step. Then, the number of subsets describing \( \mathcal{N}_i \) would evaluate to \( l = 2^i \). Thus, the number of subsets would increase exponential with the number of steps (see fourth column in Tab. IV). Fortunately, Ex. 2 (as well as Ex. 1) illustrates that this dramatic increase does not necessarily occur (see third column in Tab. IV). In contrast, the number of subsets seem to stagnate for \( i \to \infty \) for both examples.

### TABLE II

| \( \hat{\eta}_i \) | \( |\mathcal{J}_i| \) | \( l = 2^i \) |
|---|---|---|
| 0 | 0.0000000 | 1 |
| 2 | 0.0010133 | 4 |
| 5 | 0.0065739 | 32 |
| 10 | 0.0184508 | 1024 |
| 20 | 0.8650666 | 25 |
| 27 | 0.9919346 | 134217728 |
| 30 | 0.9964801 | 1073741824 |

**V. Conclusions**

We presented a method for the accurate approximation of the largest null-controllable set \( N_{\infty} \) for bilinear systems with input and state constraints. The proposed approach builds on the computation of the \( i \)-step null-controllable set \( \mathcal{N}_i \) as introduced in [12]. It is the main contribution of the present paper to derive a measure for the accuracy of the approximation \( \mathcal{N}_i \) of the largest null-controllable set \( N_{\infty} \). This measure can be used to state a termination criterion for the iterative approximation of \( N_{\infty} \) with \( \mathcal{N}_i \).

We illustrated the resulting method with two examples. For both examples, the proposed algorithm returned an accurate approximation of the largest null-controllable set. In fact, in both cases, the approximation includes more than 99% of all null-controllable states. Future work has to address the extension to multi-input systems.

### References