Abstract—We propose a novel vibrational control law for the Stephenson-Kapitza pendulum. The considered control law achieves practical asymptotic stabilization of the upper pendulum position and the region of attraction is large enough to swing up the pendulum. The control law includes a state dependent term that allows to approximate an $L_v^g V$ control law in a time averaged sense. The control law is still of vibrational type and, in a neighborhood of the upper pendulum position, it coincides with the open-loop vibrational input as it is used in the Stephenson-Kapitza pendulum.

I. INTRODUCTION

The problem of stabilizing the upper equilibrium of a pendulum whose suspension point can only be moved vertically has a long history. A. Stephenson discovered in 1908 that the inverted position of such a pendulum might be stabilized by periodically moving the suspension point up and down with high frequency [1] (see Fig. 1).

![Schematic drawing of a vertically driven pendulum](image)

Fig. 1. Schematic drawing of a vertically driven pendulum

This vibrational stabilization phenomenon is highly nonlinear and it took nearly 50 years until P. Kapitza provided a first solid theoretical study for this very interesting phenomenon [2]. Since then, vibrational stabilization phenomena have been studied extensively and have found many applications in mechanics, control or physics ([3],[4],[5],[6]).

The motivation and goal of this work is to develop a vibrational control law which does not only stabilize locally the upper pendulum position but also enlarges the region of attraction and even allows in principle to swing up the pendulum. We wish to design a control law which is compatible with the classical Stephenson-Kapitza pendulum in the sense that for the upper pendulum position the proposed control law coincides with the open-loop vibrational control input as used for the Stephenson-Kapitza pendulum.

The control law proposed in this paper achieves these goals by introducing to the open-loop vibrational control input an additional feedback term that mimics an $L_v^g V$ control law in a time averaged sense. This additional $L_v^g V$-term is motivated by well-known Lyapunov and energy-based controllers used to swing up a horizontally moved pendulum. Moreover, it vanishes at the upper equilibrium position and thus guarantees compatibility with the open-loop approach.

To the best of our knowledge, no similar control laws for the Stephenson-Kapitza pendulum have been proposed so far in the literature.

The construction of the control law is based on Lie bracket averaging techniques ([7], [8], [9], [10]). Specifically, as observed in [11], an $L_v^g V$ control law can be approximated via Lie bracket averaging techniques. The proposed control law utilizes such an approximation. Somehow surprising and in contrast to many Lie bracket averaging results [10], we show that the approximated (vibrational) $L_v^g V$ control law is able to stabilize the upper equilibrium (in an averaged sense), while this is not possible with the actual (non-vibrational) $L_v^g V$ control law. This underlines the need of a vibrational control law in the proposed constructions despite the use of state feedback information.

Finally, we would like to mention that this work is mainly motivated by theoretical curiosity, but it can also deal as a guiding example for enlarging the stability region of systems stabilized by vibrational mechanisms.

The remainder of this paper is structured as follows. In Section II, we introduce the necessary preliminaries. In Section III, we present the proposed control law and the main stability results. In Section IV, we present simulation results and finally, in Section V, we give a summary and an outlook.

II. PRELIMINARIES

A. Stephenson-Kapitza pendulum

We consider a pendulum as depicted in Fig. 1 whose suspension point can be moved along the y-axis, also known as the Stephenson-Kapitza pendulum. The equations of motion are given by ([12])

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_2 \\ -\frac{g}{l} \sin(x_1) - \lambda x_2 + \frac{0}{-\frac{1}{2} \sin(x_1)} \end{bmatrix} u \quad (1)$$

with states $x_1 := \phi, x_2 := \dot{\phi}$ and an input $u := \dot{z}$, where $z$ is the vertical position of the suspension point. Moreover, $g$ is the earth’s gravitational constant, $l$ is the pendulum’s length and $\lambda > 0$ is a damping coefficient. For the sake of simplicity we will restrict the angle $x_1$ to the interval $[0, 2\pi]$ in the following in order to not to have to deal with periodicity. It is a well-known fact that the unforced system has equilibria at $\bar{x}_l = [0, 0]^T$ and $\bar{x}_u = [\pi, 0]^T$ corresponding to the lower and the upper pendulum position. A linearization argument...
reveals that the lower equilibrium $\bar{x}_l$ is asymptotically stable and the upper one $\bar{x}_u$ is unstable what also matches the physical intuition.

As first experimentally discovered by Stephenson [1] and later theoretically verified by Kapitza [2], the upper pendulum position can be locally stabilized by moving the suspension point up and down periodically with high frequency. To be precise, the stabilizing input is a sinusoidal movement (vibrational open-loop input) of the form

$$z(t) = -A\sin(\omega_0 t),$$  \hspace{1cm} (2)

or equivalently, $u(t) = z(t) = A\omega_0^2\sin(\omega_0 t)$. Notice that no state information is required such that this is a pure open loop approach. There exist several works (e.g. [12],[13],[14]) treating the stability analysis, mainly relying on averaging or Floquet theory. We briefly sum up the existing results now. First, a necessary condition for stabilization is that $\frac{g}{\omega_0^2 A^2} < \frac{1}{2}$, i.e. both $\omega_0$ and $A$ have to be quite large compared to $g$ and $l$ such that the applied input clearly is of high-frequency high-amplitude type. However, this condition only ensures stability in a small neighborhood of $\bar{x}_0$. An estimate of the region of attraction is given in [14] by

$$\{(x_1,x_2) \in \mathbb{R}^2 : |x_1 - \pi| < \arccos\left(\frac{2gl}{\omega_0^2 A^2}\right), x_2 = 0\}. \hspace{1cm} (3)$$

**B. Lie bracket approximation and practical stability**

For completeness we repeat some results from [10] that we utilize later. Consider an input affine system with drift of the form

$$\dot{x} = f(x) + \sum_{i=1}^{m} g_i(x)\omega u_i(\omega t), \hspace{1cm} (4)$$

where $\omega > 0$. The following assumptions are imposed:

A1. $f, g_i \in \mathcal{C}^2 : \mathbb{R}^n \to \mathbb{R}^n, i = 1, \ldots, m$,

A2. $u_i \in \mathcal{C}^\infty : \mathbb{R} \to \mathbb{R}, i = 1,2, \ldots, m$ and for every $i = 1,2, \ldots, m$ there exist constants $M_i > 0$ such that $\sup_{\Theta \in \mathbb{R}} |u_i(\Theta)| \leq M_i$, .

A3. $u_i(\cdot) = T = \frac{2\pi}{m}$-periodic and has zero average, i.e. $u_i(\Theta + T) = u_i(\Theta)$ and $\int_{0}^{T} u_i(\Theta) d\Theta$ for all $\Theta \in \mathbb{R}, i = 1,2, \ldots, m$.

Moreover, associate to (4) a so-called *Lie bracket system* or extended system given by

$$\dot{z} = f(z) + \sum_{i,j=1}^{m} \sum_{i+j+1=1}^{m} [g_i(z), g_j(z)] \omega v_{ij} \hspace{1cm} (5)$$

with $v_{ij} = \frac{1}{T} \int_{0}^{T} u_j(\Theta) \left( \int_{0}^{\Theta} u_i(\tau) d\tau \right) d\Theta$ and where $[g_1(z), g_2(z)] := \frac{\partial}{\partial z} g_2(z) - \frac{\partial}{\partial z} g_1(z)$ is the Lie bracket between the two vector fields $g_1, g_2$. Both systems (4) and (5) are related in such a way that the solutions of (4) approximate those of (5) and for $\omega \to \infty$ the approximation error converges to zero. In order to establish a link between the stability properties of the system (4) and its Lie bracket approximation system (5), we need the notion of practical stability as summarized below.

**Definition 1:** (i) A compact set $E \subseteq \mathbb{R}^n$ is said to be **practically uniformly stable** for (4) if for every $\epsilon > 0$ there exists a $\delta > 0$ and an $\omega_0 > 0$ such that for all $t_0 \in \mathbb{R}$ and for all $\omega > \omega_0$

$$x(t_0) \in U_\delta^E \Rightarrow x(t) \in U_\epsilon^E, \hspace{1cm} t \geq t_0 \hspace{1cm} (6)$$

where $U_\delta^E$ denotes a $\delta$-neighborhood of the set $E$.

(ii) A compact set $E \subseteq \mathbb{R}^n$ is said to be **practically uniformly attractive** for (4) if there exists a $\delta > 0$ such that for every $\epsilon > 0$ there exists a $t_f \geq 0$ and an $\omega_0 > 0$ such that for all $t_0 \in \mathbb{R}$ and for all $\omega > \omega_0$

$$x(t_0) \in U_\delta^E \Rightarrow x(t) \in U_\epsilon^E, \hspace{1cm} t \geq t_0 + t_f \hspace{1cm} (7)$$

(iii) A compact set $E \subseteq \mathbb{R}^n$ is said to be **practically uniformly asymptotically stable** for (4) if it is practically uniformly stable and practically uniformly attractive.

(iv) Let $E \subseteq \mathbb{R}^n$. A compact set $E \subseteq \mathbb{R}^n$ is said to be $S$-**practically uniformly asymptotically stable** for (4) if it is practically uniformly stable and for every $\delta, \epsilon > 0$ there exists a $t_f \geq 0$, a $c > 0$ and an $\omega_0 > 0$ such that for all $t_0 \in \mathbb{R}$ and all $\omega > \omega_0$

$$x(t_0) \in S \cap U_\delta^E \Rightarrow x(t) \in U_\epsilon^E, \hspace{1cm} t \geq t_0 + t_f \hspace{1cm} (8)$$

Using for example the results established in [10], one can prove the following statement:

**Theorem 1:** Let Assumptions A1 - A3 be satisfied and let $S \subseteq \mathbb{R}^n$. Assume that a compact connected set $E_c$ is $S$-asymptotically stable for (5). Then, $E_c$ is $S$-practically uniformly asymptotically stable with respect to (4).

Loosely speaking, the theorem states that if the set $E_c$ is asymptotically stable for (5), then $E_c$ is practically asymptotically stable for (4) and if $S$ is a set which belongs to the region of attraction of $E_c$ for (5), then $S$ also belongs to the region of attraction of $E_c$ for (4).

The above stated theorem cannot be exactly found in [10] but it can be rather easily established based on the results in [10]. Since the theorem will also appear somewhere else, the proof is omitted here. Moreover, for this work, the inputs $u_i$ will be sines and cosines and $m = 2$, which is a setup that has been extensively studied in the literature (see e.g. [9, Example 2]).

**III. MAIN RESULTS**

In the present section we derive our main results. We motivate the chosen control law in the first part. In the second part we analyze the closed loop system.

**A. Vibrational Control Law**

As already mentioned in the introduction, our goal is to design a control law that stabilizes the upper pendulum position and enlarges the region of attraction in comparison to the usual open-loop approach to stabilize the Stephenson-Kapitza pendulum.

In the following, we introduce the so-called vibrational $L_2V$ control law, similarly to [11]. Consider a general nonlinear input-affine system

$$\dot{x} = f(x) + g(x)u, \hspace{1cm} (9)$$

In the present section we derive our main results. We motivate the chosen control law in the first part. In the second part we analyze the closed loop system.
where \( x \in \mathbb{R}^n \) and \( u \in \mathbb{R} \). Suppose \( \bar{x} \) denotes an equilibrium and assume we know a Control Lyapunov Function (CLF) [15] \( V : \mathbb{R}^n \to \mathbb{R}^+ \), i.e. a function that fulfills \( V(\bar{x}) = 0 \) and
\[
V(x) > 0 \quad \forall x : x \neq \bar{x} \\
L_f V(x) < 0 \quad \forall x : L_f V(0) = 0, x \neq \bar{x},
\]
(10)
where \( L_f V(x) = \frac{\partial V}{\partial x}(x)f(x) \), \( L_g V(x) = \frac{\partial V}{\partial x}(x)g(x) \). If the second inequality does not strictly hold (\( \leq \)), then we speak of a weak CLF. Given a (weak) CLF, we can construct a (potentially) stabilizing feedback law by choosing
\[
u(x) = -\alpha L_g V(x), \quad \alpha > 0
\]
(11)
since then the closed loop satisfies \( \dot{V}(x) = L_f V(x) - \alpha L_g V(x)^2 \) which might become negative semidefinite given \( \alpha \) is large enough. The closed-loop is given by
\[
\dot{x} = f(x) - \alpha L_g V(x)g(x).
\]
(12)
A feedback of the form (11) is known as a feedback of the form (11) is known as a feedback (or damping controller). Now consider again (9) and let \( u \) be of the form
\[
u_{cl}(x,t) = V(x)\sqrt{\omega} \cos(\omega t) + 2\alpha \sqrt{\omega} \sin(\omega t),
\]
(13)
where \( V \) is a (weak) CLF. We call a control law of the form (13) an vibrational \( L_g V \) control law. Since \( V \) vanishes at the desired equilibrium, the control law (13) resembles the open-loop control (2) in a small neighborhood around that equilibrium. We will discuss this point in more detail in Section III-B. Now, with \( u \) defined by (13), the closed loop is given by
\[
\dot{x} = f(x) + g(x)V(x)\sqrt{\omega} \cos(\omega t) + g(x)2\alpha \sqrt{\omega} \sin(\omega t)
\]
(14)
which is of the form (4) when identifying \( g_1(x) = g(x)V(x), g_2(x) = \alpha g(x) \). The chosen periodic sine and cosine inputs \( u_i(\omega t) \) were already considered in [7] and are special in that way that they lead to \( V_1 = 1 \) in (5). Note that it is possible to derive different control laws by choosing other input signals fulfilling assumptions A2 and A3. Now, for the associated Lie bracket system as defined in (5) we obtain
\[
\dot{x} = f(x) + [g_1(x), g_2(x)] = f(x) - \alpha L_g V(x)g(x).
\]
(15)
This is exactly the closed loop (12), i.e. the vibrational \( L_g V \) control law approximates the original \( L_g V \) control law in a Lie bracket approximation sense. Moreover, as stated in Theorem 1, if the \( L_g V \) control law renders the desired equilibrium point asymptotically stable, then the vibrational \( L_g V \) control law renders the desired equilibrium point practically asymptotically stable.

Remark 1: It is also possible to use non-constant gain \( \alpha(x,t) \) in (11). For example, the well-known Sontag’s formula is of this type. However, if \( \alpha(x,t) \) is state dependent, then a control law of the form (13) would not result in a closed loop of the form (15).

As a result of Theorem 1 and the above observations, the natural idea of how to find the desired control law for the Stephenson-Kapitza pendulum would be to find a CLF for (1), compute the associated \( L_g V \) control law that renders the upper equilibrium asymptotically stable and conclude practical asymptotic stability using Theorem 1. As shown next, this is not possible here.

Lemma 1: There exists no finite \( \alpha > 0 \) and no twice continuously differentiable CLF \( V \) such that an \( L_g V \) control law (11) renders the upper equilibrium \( \bar{x}_u \) of (1) asymptotically stable.

Proof: Using an \( L_g V \) control law for (1) yields
\[
\frac{\partial f_{cl}}{\partial x}(\bar{x}_u) = \left[ \begin{array}{c} 0 \\ 1 \\ -\lambda \end{array} \right],
\]
(17)
and consequently the Jacobian has always an eigenvalue with positive real part.

Remark 2: The result above can be generalized to nonlinear input affine systems of the form (9) with \( f(\bar{x}) = g(\bar{x}) = 0 \) and control laws with \( u(x,t) = 0 \) at the equilibrium \( \bar{x} \) that is to be stabilized.

Nevertheless, it is possible to achieve asymptotic stability (in an averaged sense) using the vibrational \( L_g V \) control law as we will see in Section III-B. This somehow contradicts the intuition since the approximation, i.e. the vibrational \( L_g V \) control law, achieves what the exact version cannot achieve. Before we start our closed loop analysis we propose a CLF for (1) which will be used in the \( L_g V \) control law.

Lemma 2: \( V : \mathbb{R}^2 \to \mathbb{R}^+ \) defined by
\[
V(x_1,x_2) := c_1(\cos(x_1) + 1) + \frac{1}{2}(c_2x_2 - \sin(x_1))^2,
\]
(18)
c_1 > 0, c_2 > 1 is a weak CLF for (1) and the desired equilibrium \( \bar{x}_u \).

Proof: Since \( \cos(x_1) > -1 \) we have \( V(x_1,x_2) \geq 0 \) and \( V(x_1,x_2) = 0 \) if and only if \( \cos(x_1) = -1 \) and \( c_2x_2 - \sin(x_1) = 0 \), i.e. \( x_1 = (2k + 1)\pi, x_2 = 0, k \in \mathbb{Z} \), which describes exactly the upper pendulum position \( \bar{x}_u \). Hence \( V \) fulfills the first part of (10). Furthermore we calculate
\[
L_g V(x) = -\frac{c_2}{I} \sin(x_1)(c_2x_2 - \sin(x_1))
\]
(19)
\[
L_f V(x) = -c_1x_2 \sin(x_1)
\]
(20)
\[
- (c_2x_2 - \sin(x_1))(\cos(x_1)x_2 + \frac{c_2g}{I} \sin(x_1) + c_2\lambda x_2)
\]
such that \( L_g V(x) = 0 \) if \( x_1 = k\pi \) and arbitrary \( x_2 \) or if \( x_2 = \frac{1}{c_2} \sin(x_1) \). In the first case we obtain
\[
L_f V(k\pi,x_2) = -c_2x_2^2((-1)^k + c_2\lambda) < 0 \quad \text{if} \quad c_2\lambda > 1.
\]
(21)
In the second case we have
\[
L_f V(x_1, \frac{1}{c_2} \sin(x_1)) = -\frac{c_1}{c_2} \sin^2(x_1) < 0
\]
(22)
which proves the second condition from (10).

The proposed CLF is a modification of the pendulum’s
an averaged energy. With this CLF the $L_\varphi V$ control reads as

$$u(x) = \frac{c_2 \alpha}{l} \sin(x_1)(c_2 x_2 - \sin(x_1))$$  \hspace{1cm} (23)$$

and the vibrational $L_\varphi V$ control law is given by

$$u_{vib}(x,t) = \left[\frac{c_1}{l} \left( \cos(x_1) + 1 \right) + \frac{1}{2} \left( c_2 x_2 - \sin(x_1) \right)^2 \right] \cdot \sqrt{\omega} \cos(\omega t) + 2\alpha \sqrt{\omega} \sin(\omega t).$$  \hspace{1cm} (24)$$

For $x = \bar{x}_u$, we obtain a pure open-loop control law $u_{vib}(x,t) = 2\alpha \sqrt{\omega} \sin(\omega t)$ similar to (2).

**B. Closed Loop Stability Analysis**

In this section we analyze the closed loop system consisting of (1) together with a vibrational control law as defined by (24). Before we begin we briefly sketch the main steps in the following argumentation. We start with an analysis of the non-vibrational closed loop, i.e. the Lie bracket approximation of the original closed loop, showing that additional equilibria can occur. We show that those additional equilibria are locally asymptotically stable and can be pushed arbitrary close to the desired equilibrium $\bar{x}_u$ by letting the parameters $c_2, \alpha$ in the $L_\varphi V$ control law (11) tend to infinity. Afterwards, we characterize a region in the state space where solutions of the closed loop converge to a desired target set. That set contains the desired equilibrium $\bar{x}_u$ and it can be made arbitrary small by letting the parameter $c_2$ in the CLF tend to infinity. Then we utilize Theorem 1 to obtain a stability statement for the closed loop system from the analysis of the Lie bracket approximation. Finally, we show that the desired equilibrium $\bar{x}_u$ of the closed loop is asymptotically stable (in an averaged sense), by showing that the linearization of the closed loop after an appropriate change of coordinates coincides with the Stephenson-Kapitza pendulum (1) and (2). Using the control law as defined by (23), the closed loop system is given by

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = \frac{g}{l} \sin(x_1) - \lambda x_2 - \frac{c_2 \alpha}{l^2} \sin^2(x_1)(c_2 x_2 - \sin(x_1))$$  \hspace{1cm} (25)$$

where we denote the right-hand side of (25) with $f_{cl}(x_1, x_2)$. An interesting point to observe first is that additional equilibria can occur. The next lemma states results about the existence, the position and the stability properties of additional equilibria.

**Lemma 3:** The closed loop defined by (25) has additional equilibria if and only if $\frac{g}{c_2 \alpha} \leq 1$ and they are explicitly given by

$$\bar{x}_{1A} = \arcsin(\sqrt{\frac{gl}{c_2 \alpha}}), \ \bar{x}_{1B} = \pi - \arcsin(\sqrt{\frac{gl}{c_2 \alpha}}),$$

$$\bar{x}_{1C} = \pi + \arcsin(\sqrt{\frac{gl}{c_2 \alpha}}), \ \bar{x}_{1D} = 2\pi - \arcsin(\sqrt{\frac{gl}{c_2 \alpha}}),$$

and $\bar{x}_{2A/B/C/D} = 0$, where $\arcsin(\cdot) : [0, 1] \to [0, \frac{\pi}{2}]$. Moreover, $\bar{x}_B$ and $\bar{x}_C$ are asymptotically stable and $\bar{x}_A$ and $\bar{x}_D$ are unstable.

**Proof:** From $f_{cl}(\bar{x}_1, \bar{x}_2) = 0$ we obtain $\bar{x}_2 = 0$ from the first component and putting that into the second one yields

$$0 = \sin(\bar{x}_1)(-\frac{g}{l} + \frac{c_2 \alpha}{l^2} \sin^2(\bar{x}_1)).$$  \hspace{1cm} (27)$$

Thus, either $\sin(\bar{x}_1) = 0$ or $\sin^2(\bar{x}_1) = \frac{gl}{c_2 \alpha}$. The first equation corresponds to the original equilibrium whereas the second one gives the additional ones that, due to $0 \leq \sin^2(x_1) \leq 1$, can exist if and only if $\frac{g}{c_2 \alpha} \leq 1$ what proves the existence condition. We analyze the stability by considering the linearization of the closed loop (25) about the equilibria. The eigenvalues of the Jacobian of $f_{cl}$ are given by

$$s_{l}^{(1/2)} = -\frac{1}{2} \left( \lambda + \frac{c_2 \alpha}{l} \right) \pm \sqrt{\left( \lambda + \frac{c_2 \alpha}{l} \right)^2 + \frac{8g}{l} \cos(\bar{x}_1)}.$$

where $I = \{A, B, C, D\}$ such that

$$\text{Re}(s_{l}^{(1/2)}) < 0 \iff \cos(\bar{x}_1) < 0 \iff \bar{x}_1 \in \left(\frac{\pi}{2}, \frac{3\pi}{2}\right).$$  \hspace{1cm} (29)$$

This proves the stability proposition. 

There is another important point following from Lemma 3: If $c_2, \alpha \to \infty$, then the additional equilibria approach the original ones, i.e.

$$\lim_{c_2, \alpha \to \infty} \bar{x}_{B/C} = \bar{x}_u, \ \lim_{c_2, \alpha \to \infty} \bar{x}_{A/D} = \bar{x}_l.$$  \hspace{1cm} (30)$$

The situation is depicted in Fig. 2. Consequently, for any given $\varepsilon > 0$ we can find $c_2, \alpha$ such that the stable equilibria $\bar{x}_{B/C}$ are in an $\varepsilon$-neighborhood of the desired equilibrium $\bar{x}_u$.

The following key lemma gives an estimate of a region $\mathcal{S}$ in the state space such that, when solutions are initialized in this region, they converge towards a desired target set $E (E_{init})$ which contains the desired equilibrium point $\bar{x}_u$ and this target set can be made arbitrary small when the parameter $c_2$ tends to infinity.

**Lemma 4:** Let

$$\mathcal{E}_E := \{x_1, x_2 \in \mathbb{R}^2 : |x_1 - \pi| \leq \bar{x}_{1A}, |x_2| \leq \varepsilon_2\}$$  \hspace{1cm} (31)$$

$$\mathcal{E}_S := \{x_1, x_2 \in \mathbb{R}^2 : |x_1 - \pi| > \bar{x}_{1A}, |x_1 - \pi| > \bar{x}_{1B} - \varepsilon_1\}$$

$$\cup \{x_1, x_2 \in \mathbb{R}^2 : |x_1 - \pi| \leq \bar{x}_{1A}, |x_2| > \varepsilon_2\}$$  \hspace{1cm} (32)$$

with $\varepsilon_{1/2}$ defined by (42) and (46), respectively. Let $E$ be the smallest level set of $V_2$ defined by (33) fully containing $\mathcal{E}_E$, let $S$ be the largest level set of $V_2$ fully contained in $\mathcal{E}_S$. 

![Fig. 2. An illustration of the equilibria of the closed loop (25). The solid ones are asymptotically stable whereas the dashed ones are unstable. For $c_2, \alpha \to \infty$ both upper stable ones approach the upper pendulum position and both lower unstable ones approach the lower pendulum position.](image-url)
and let $E_{inv}$ be the largest invariant set in $E$. Then there exist $c_2, \alpha$ such that $E_{inv}$ is asymptotically stable for the closed loop (25) and any solution with $x(t_0) \in \mathcal{S}$ converges to $E_{inv}$.

Moreover, $\epsilon_1/2 \to 0$ for $c_2 \to \infty$ and $E_{inv}$ reduces to $\hat{x}_n$.

**Proof:** Consider

$$V_2(x_1, x_2) = c_3 \left( \cos(x_1) + 1 \right) + \frac{1}{2} \left( c_2 x_2 - \sin(x_1) \right)^2 \tag{33}$$

where $c_2, c_3 > 0$ and will be specified later. The idea of the proof is as follows: we first show that $V_2 < 0$ for all $x \in \mathcal{S}$. From that we can conclude that $E$ is attractive for all trajectories starting in $S$ (see also Fig. 3).

The derivative of $V_2$ along the solutions of (25) is given by

$$V_2(x_1, x_2) = -c_2 \cos^2(x_1) \left( \frac{c_2 \alpha}{l_2} \sin^2(x_1) - \frac{g}{l} \right)$$

$$+ \sin(x_1) x_2 \left[ -c_3 + \cos(x_1) + c_2 \left( \frac{2 \alpha}{l_2} \sin^2(x_1) - \frac{g}{l} \right) \right]$$

$$+ c_2 \lambda + c_2 \frac{c_2 \alpha}{l_2} \sin^2(x_1)$$

$$\left[ \begin{array}{c} x_1 \\ x_2 \end{array} \right]^T \Delta(x_1) \left[ \begin{array}{c} x_1 \\ x_2 \end{array} \right], \tag{34}$$

where we introduced

$$a(x_1) := \frac{c_2 \alpha}{l_2} \sin^2(x_1) - \frac{g}{l}$$

$$b(x_1) := \cos(x_1) + c_2 \lambda + \frac{c_2 \alpha}{l_2} \sin^2(x_1)$$

$$\Delta(x_1) := \left[ \begin{array}{c} -c_2 a(x_1) \\ \frac{1}{2} \left( b(x_1) + c_2 a(x_1) - c_3 \right) \\ -c_2 b(x_1) \end{array} \right]. \tag{35}$$

Let $\mathcal{M}_{S_1}, \mathcal{M}_{S_2}$ denote the two components of $\mathcal{M}_S$ defined by (32). We first show that $V_2$ is negative definite on $\mathcal{M}_{S_1}$. This is implied if $\Delta(x_1)$ is negative definite for all $(x_1, x_2) \in \mathcal{M}_{S_1}$. This is the case if and only if

$$b(x_1) > 0 \quad \text{and} \quad \det \left( -\Delta(x_1) \right) = \det \left( \Delta(x_1) \right) > 0 \quad \text{for all} \quad (x_1, x_2) \in \mathcal{M}_{S_1}. \tag{36}$$

From the first requirement we obtain the inequality $c_2 \geq \frac{1}{\lambda} \sqrt{1 - \frac{g^2}{c_2 \alpha}}$ which implies for $c_2$ that

$$c_2 \frac{c_2 \alpha}{l_2} \lambda - c_2 \alpha + g l \geq 0. \tag{37}$$

For the determinant we compute

$$\det \left( \Delta(x_1) \right) = \frac{c_2^2 a(x_1) b(x_1) - \left( b(x_1) + c_2 a(x_1) - c_3 \right)^2}{4}. \tag{38}$$

Since $a(x_1) = 0$ on the boundary of $\mathcal{M}_{S_1}$, we choose $c_3$ such that $\det \left( \Delta(\bar{x}_{1B}) \right) = \det \left( \Delta(\bar{x}_{1C}) \right) = 0$, i.e.

$$c_3 = b(\bar{x}_{1B}) = -\sqrt{1 - \frac{g l}{c_2 \alpha}} + c_2 \lambda + \frac{c_2 g}{l} > 0, \tag{39}$$

where the positivity of $c_3$ is ensured by (37). To show that the determinant is greater than zero for all $(x_1, x_2) \in \mathcal{M}_{S_1}$ we now show that the boundaries of $\mathcal{M}_{S_1}$ are the only zeros of $\det \left( \Delta(x_1) \right)$ and that $\det \left( \Delta(x_1) \right) < 0$ for some $(x_1, x_2) \in \mathcal{M}_{S_1}$. Replacing all the terms in $\det \left( \Delta(x_1) \right)$ and after a cumbersome calculation we obtain

$$\det \left( \Delta(x_1) \right) = -c_2^2 \left( \frac{3 c_2 \alpha}{l^2} b(\bar{x}_B) \right) + \cos(x_1) \cos(\bar{x}_B)$$

$$\left( \frac{c_2^2}{l^2} (c_2 \alpha - g) b(\bar{x}_B) - \frac{1}{4} \cos^2(\bar{x}_B) \right) \tag{40}$$

such that the zeros $\dot{x}_{1B}$ are given by

$$\cos(\bar{x}_{1B}) = \frac{1}{1 + c_2 \alpha b(\bar{x}_B) \left( \cos(\bar{x}_B) \pm \left( \cos^2(\bar{x}_B) + \left( \frac{c_2^2}{l^2} (c_2 \alpha - g) b(\bar{x}_B) - \frac{1}{4} \cos^2(\bar{x}_B) \right) \right)^2 \right)^{1/2}}. \tag{41}$$

From (38) we see that $\det \left( \Delta(\bar{x}_{1B}) \right) = 0$, $\det \left( \Delta(\bar{x}_{1A}) \right) < 0$ such that

$$\dot{x}_{1A} = \bar{x}_{1B}, \quad \dot{x}_{1B} = \bar{x}_{1A} + \epsilon_1 \tag{42}$$

which implicitly defines $\epsilon_1 > 0$. Since the coefficient of the quadratic term in $\cos(x_1)$ in (40) is negative we know that $\det \left( \Delta(x_1) \right) > 0$ for all $(x_1, x_2)$ in between the two zeros, i.e.

$$\det \left( \Delta(x_1) \right) > 0 \quad \text{for all} \quad (x_1, x_2) \in \mathcal{M}_{S_1}. \tag{43}$$

Hence

$$\dot{V}_2 \leq \sqrt{\frac{g l}{c_2 \alpha}} |x_1| |x_2| \left( 2 + 3 c_2 \frac{g}{l} \right) - c_2 |x_2|^2 (c_2 \lambda - 1) + c_2 \frac{g}{l} \tag{44}$$

and the right hand side is zero if

$$|x_2| = \frac{-\sqrt{\frac{g l}{c_2 \alpha}} (2 + 3 c_2 \frac{g}{l}) - \sqrt{\frac{g l}{c_2 \alpha} (2 + 3 c_2 \frac{g}{l})^2 + 4 c_2^2 (c_2 \lambda - 1)} - 2 c_2 (c_2 \lambda - 1)}{2 c_2^2 (c_2 \lambda - 1) - 1} =: \epsilon_2. \tag{45}$$

Since the coefficient of the quadratic terms in (45) is negative we conclude that $\dot{V}_2(x_1, x_2) < 0$ for all $|x_2| < \epsilon_2, (x_1, x_2) \in \mathcal{M}_{S_2}$. Equation (46) implies that $\epsilon_2$ decreases monotonically.
if $c_2$ increases and $\varepsilon_2 \to 0$ as $c_2 \to \infty$.

So far we showed that $\dot{V}_2 < 0$ for all $x \in \mathcal{M}_S$. Consequently, solutions starting in the largest level set of $V_2$ fully contained in $\mathcal{M}_S$, i.e. $x(t_0) \in S$, cannot leave $S$. The situation is illustrated in Fig. 3. Moreover, the solutions enter into the smallest level set of $V_2$ fully containing $\mathcal{M}_E$, i.e. into $E$, in finite time which they cannot leave, too. However, once they entered into $E$, LaSalle’s invariance principle implies that these solutions converge to the largest invariant set in $E$, i.e. $E_{inv}$, which is therefore asymptotically stable. The equilibrium $\bar{x}_u$ is trivially contained in $E_{inv}$. To see that $E$ (and therefore also $E_{inv}$) become infinitely small when $\mathcal{M}_E$ does, i.e. the level sets do not degenerate, consider the following quadratic approximation of $V_2$ about $x_1 = \pi$:

$$V_2(x_1, x_2) \approx c_3(x_1 - \pi)^2 + \frac{1}{2}c_2(x_2 + x_1 - \pi)^2 \quad (47)$$

$$= c_3^2((x_1 - \pi)^2 + \frac{1}{2}c_2^2 + \frac{1}{2}c_2 + \frac{1}{2}c_2(x_1 - \pi)).$$

By (39) $c_3$ is quadratic in $c_2$ such that we can conclude that no degenerated level sets can occur when $c_2 \to \infty$. Hence, when $\mathcal{M}_E$ becomes infinitely small, $E$ also does and $E_{inv}$ reduces to $\bar{x}_u$.

Remark 3: We did not discuss the existence of the sets $S$ and $E$. However, given $c_2$ is sufficiently large, the existence is guaranteed by the convergence properties of the sets.

In Lemma 4 we introduced the level sets $S$ and $E$ but did not calculate them. We will discuss this issue now assuming that $c_2$ is sufficiently large to guarantee existence of $S$ and $E$. To calculate $S$ we have to find the minimal value of $V_2(x_1, x_2)$ on the boundaries of $\mathcal{M}_S$ that then defines the appropriate level set. More precisely, $S$ is given by

$$S = \{ (x_1, x_2) \in \mathbb{R}^2 : V_2(x_1, x_2) < k_S \}, \quad (48)$$

where

$$k_S = \min_{x_1 = \frac{\pi}{2} + \varepsilon_1, x_1 = \sin(\bar{x}_1 + \varepsilon_1), c_2 = \sin(\bar{x}_1 + \varepsilon_1)} V_2(x_1, x_2) = \min \{ V_2\left(\bar{x}_1 + \varepsilon_1, \frac{1}{c_2} \sin(\bar{x}_1 + \varepsilon_1)\right), V_2\left(\bar{x}_1 + \varepsilon_1, \frac{1}{c_2} \sin(\bar{x}_1 + \varepsilon_1)\right) \} = V_2\left(\bar{x}_1 + \varepsilon_1, \frac{1}{c_2} \sin(\bar{x}_1 + \varepsilon_1)\right).$$

After some calculations omitted here due to space constraints and letting $c_2 \to \infty$, we can give the largest possible $S$

$$S_\infty = \{ (x_1, x_2) \in \mathbb{R}^2 : |x_2| < \sqrt{\frac{2g}{1 - \cos(x_1)} \frac{1}{c_2}} \} \quad (50)$$

Similarly, $E$ is given by $E = \{ (x_1, x_2) \in \mathbb{R}^2 : V_2(x_1, x_2) < k_E \}$, where

$$k_E = \max_{x_1 \in \mathcal{M}_E} V_2(x_1, x_2) = \max_{x_1 \in \mathcal{M}_E, |x_2| = \varepsilon_2} V_2(x_1, x_2). \quad (51)$$

Summarizing, the last lemma showed that solutions of the closed loop system with the $L_2V$ control law starting in $S$ will asymptotically converge to $E_{inv}$. As a direct consequence of Theorem 1 we can now transfer those results to the vibrational case.

Theorem 2: Consider the closed loop (1) together with the vibrational control input (24) and assume that $c_2$ fulfills (37). Let $E_{inv}$ and $S$ be defined as in Lemma 4. Then the set $E_{inv}$ is $S$-practically uniformly asymptotically stable for the closed loop and $\bar{x}_u$ is contained in $E_{inv}$.

We demonstrate that the estimates given in Lemma 4 and Theorem 2 are good enough to swing up the pendulum in Section IV.

Let us now come back to the Stephenson-Kapitza approach. One of our goals was to design the control law in such a way that, at the upper equilibrium, the closed loop system with the vibrational $L_2V$ control law behaves qualitatively similar to the classical Stephenson-Kapitza pendulum. We show that this is the case in the next theorem using a linearization argument.

Theorem 3: The linearization of the closed loop system consisting of (1) and the vibrational control law (24) is equal to (1) and (2) after a local state transformation and if

$$\omega_k = \omega, A = 2\alpha \omega^{-\frac{3}{2}}. \quad (52)$$

Proof: Consider the closed loop system consisting of the vibrational control law (24) and (1). Let $\tilde{\xi}_1 = \sin(x_1), \tilde{\xi}_2 = x_2$ which is a local diffeomorphism for $x_1 \in \left(\frac{\pi}{2}, \frac{3\pi}{2}\right)$. With this transformation the closed loop is given by

$$\begin{bmatrix} \tilde{\xi}_1 \\ \tilde{\xi}_2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ -\frac{\varepsilon}{\xi} & -\lambda \end{bmatrix} \begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix} - \frac{\varepsilon}{\xi} \sqrt{\xi_2^2 + \alpha^2 \xi_2^2} \begin{bmatrix} \xi_2 \\ \xi_1 \end{bmatrix} \quad (53)$$

A linearization about the upper pendulum position $\bar{\xi}_i = 0, \bar{\xi}_2 = \bar{x}_{u2} = 0$ gives

$$\begin{bmatrix} \Delta \tilde{\xi}_1 \\ \Delta \tilde{\xi}_2 \end{bmatrix} = \begin{bmatrix} 0 & -\frac{\varepsilon}{\xi} \omega \sin(\omega t) \pm \alpha \omega \sin(\omega t) \\ -\frac{\varepsilon}{\xi} & -\frac{\varepsilon}{\xi} \pm \alpha \omega \sin(\omega t) \end{bmatrix} \begin{bmatrix} \Delta \xi_1 \\ \Delta \xi_2 \end{bmatrix}. \quad (54)$$

Applying the same transformation and the linearization to (1) and (2) we obtain

$$\begin{bmatrix} \Delta \tilde{\xi}_1 \end{bmatrix} = \begin{bmatrix} -\frac{\varepsilon}{\xi} \omega \sin(\omega t) \pm \alpha \omega \sin(\omega t) \\ -\frac{\varepsilon}{\xi} \pm \alpha \omega \sin(\omega t) \end{bmatrix}. \quad (55)$$

Equations (54) and (55) are identical if (52) holds.
Consequently, since the upper equilibrium position for (1) and (2) is asymptotically stable (in an averaged sense) (see [16, Example 10.10]), we can conclude that (1) driven by the proposed vibrational control law (24) asymptotically stabilizes the upper equilibrium position (in an averaged sense).

IV. SIMULATION

The following simulation illustrates the theoretical results. The model parameters are chosen as $l = 0.3 \text{m}, g = 9.81 \text{m/s}^2, \lambda = 11/\text{s}$ and for the design parameters in the CLF from (18) it is

$$
c_1 = 1, \quad c_2 = 1.5, \quad \alpha = 40, \quad \omega = 500 \text{rad/s}. \quad (56)
$$

In the Kapitza approach we choose the parameters according to (52). Typical plots of the angle are depicted in Fig. 4. Due to the initial conditions the Stephenson-Kapitza approach fails. The $L_g V$ controller asymptotically stabilizes the additional equilibrium at $\bar{x}_{B,1} = 2.9422 \text{rad}$. The vibrational $L_g V$ controller renders the upper equilibrium practically asymptotically stable.

Simulations with other parameters showed that the choice of the parameters $c_2$, $\alpha$ and $\omega$ plays a key role. First, $\omega$ has to be sufficiently large to make sure that the approximation error is sufficiently small. However, if it is chosen too large, the vibrational system converges to the additional equilibrium $\bar{x}_B$. In fact, all values between those two points can also be obtained.

The applied vibrational input clearly is of high amplitude and high frequency such that a practical implementation is likely to be difficult. Moreover, simulation results suggest that the position $z(t)$ is not ensured to remain bounded. An extension to the proposed control law guaranteeing boundedness is currently investigated.

V. CONCLUSIONS AND OUTLOOK

Utilizing ideas from Lyapunov-based feedback design and Lie bracket averaging techniques, we constructed a novel vibrational control law for the Stephenson-Kapitza pendulum which enlarges the region of attraction of the upper equilibrium position and which even allows to swing up the pendulum. The constructed control law is based on an $L_g V$ control law which has been approximated with Lie bracket averaging techniques (high-frequency, high-amplitude inputs). The approximation consists of an open-loop term and a feedback term, whereby the open-loop term is exactly the one used in the Stephenson-Kapitza pendulum and the feedback term vanishes at the upper equilibrium position. Thus, the proposed vibrational control law naturally extends the classical one. We showed in Theorem 3 that the proposed vibrational control law practically asymptotically stabilizes a region about the upper pendulum position of arbitrary small size and gave an estimate of its region of attraction including positions arbitrarily near to the lower equilibrium.

Possible future work includes the implementation in an experimental setup as well as the application to other areas where vibrational control is used. It is hoped that, based on the results for the specific system considered here, a more systematic design approach to vibrational stabilization techniques can be obtained.

REFERENCES