Consensus Control of Multi-agent System with Constraint - The Scalar Case

Chang Sun*, Chong Jin Ong* and Jacob K. White†

*National University of Singapore, Singapore-MIT Alliance, Singapore
†Massachusetts Institute of Technology, Singapore-MIT Alliance, U.S.A.

Abstract—This paper describes a new distributed algorithm for the consensus control of a group of agents connected via a communication network where the states of the agents lie within individually-defined constraints. It considers the case where the state of each agent is a scalar in an undirected network with no communication delays and uses the edge weights as the working variables to enforce satisfaction of the constraints. The convergence and consensus of the states under reasonably mild assumptions are provided.

The proposed algorithm has the advantage that it preserves many of the desirable properties of the standard unconstrained consensus algorithm: the consensus value is independent of the sequence of network changes; the average of the states remains a constant for all time steps; and the states of the agents, when all constraints are identical, follow the same trajectories as the standard consensus algorithm. Several examples are provided to illustrate the results.

I. INTRODUCTION

Decentralized consensus problem deals with the collective behavior of a group of agents connected via a communication network. The standard objective is to achieve consensus on the communicated variables in a dynamically changing network with possible time-delay or loss in communication. The study of such a system has been a focus of recent research attention with notable contributions on conditions for consensus in the presence of switching networks [1], [2] and time delays [3]. Collections of known properties of such systems can be found in [4], [5] while interesting applications are given in [6], [7] and others. Most of these works are for networks where the consensus variables are constraint-free. A significant contribution to the consensus problem where the variables are subjected to convex polyhedral constraints is made in the recent work of [8]. Their algorithm relies heavily on the projection operator to achieve consensus while satisfying the constraints. Other approaches for handling constraints have also appeared [9], [10]. These approaches introduce additional inputs to each agent in the form of barrier functions to keep the states within the feasible domain. However, the presence of such additional inputs also introduces new equilibrium points (local minima) for which the convergence to a consensus value is unclear.

Under appropriate conditions, an unconstrained symmetric switching network achieves consensus with a consensus value that is independent of the sequence/ordering of network switches. This switch-independence property is, unfortunately, not preserved in the algorithm proposed by [8]. Starting from the same initial states, the algorithm of [8] may converge to different consensus values for different switching sequences. In addition, the property that the average of the states is a constant for all time no longer holds for the algorithm of [8].

The proposed algorithm does not have such issues. It is a structurally different algorithm to [8] and uses the edge weights to ensure satisfaction of the constraints. When the assumptions are satisfied, the system converges, satisfy the average property at all time, achieves a consensus value that is independent of the switching sequences and dependent only on the initial values of the states.

The organization of this paper is as follows. This section ends with the notations used. Standard results on consensus problem are reviewed in Section II, together with the formulation of the problem. The proposed algorithm is stated in full in Section III while Section IV shows the convergence proof of the algorithm under the stated assumptions. Section V shows the consensus result under three additional assumptions. Examples and discussions are given in Section VI with the conclusions given in Section VII. Due to the page limit all proofs of the main theorems are given in the appendix while those of lemmas are omitted.

The notations used is standard. The sets of non-negative integers and real number are denoted by $\mathbb{Z}^+$ and $\mathbb{R}$ respectively. The $n$-vector $x$ has its $i$-th element denoted by $x_i$ and $\mathbb{Z}_n := \{1, 2, \cdots, n\}$. Given a set $S \subset \mathbb{R}^n$, $intS$, $\partial S$ and $|S|$ refer respectively to the interior, boundary and cardinality of $S$ (where $S$ is a discrete set). A diagonal matrix is denoted by $\text{diag}(a_1, \cdots, a_n)$. A matrix $M \in \mathbb{R}^{n \times n}$ is also referred to as $[m_{ij}]$. Vector and matrix norms are indicated by $\| \cdot \|_p$ where $p = 1, 2, \infty$.

II. PRELIMINARY AND PROBLEM FORMULATION

Consider the typical consensus problem with $n$ agents where the variable of the $i$-th agent at the $k$ instant is $x_i(k)$. The constraint on $x_i(k)$ is given by $X_i = \{x|\bar{x}_i \geq x \geq \underline{x}_i\} \subset \mathbb{R}$ where $\bar{x}_i, \underline{x}_i$ are respectively its upper and lower bounds. The intersection of the feasible domain is $X = \bigcap_{i=1}^{n} X_i$. The network is typically represented by a graph, $G(V, E)$, with vertex set $V$ and edge set $E$. The associated adjacency matrix $A = [a_{ij}]$ where $a_{ij} = 1$ if $(i, j) \in E$, $a_{ij} = 0$ otherwise and $a_{ii} = 0$ for all $i$. As only undirected graphs without communication delay are considered, $A$ is symmetric. For each
vertex \( i \), its neighbors are given by \( N_i(k) = \{j | a_{ij}(k) = 1\} \) with \( \bar{d}_i = \max_k |N_i(k)| \) being its maximal degree over time and \( \bar{d}_m := \max_{i \in \mathbb{Z}_m} d_i \) being the maximal degree of the network. The set of neighbors of \( i \) is further divided into \( N_i^+(k) = \{ j \in N_i(k) | x_i(k) - x_j(k) > 0 \} \) and \( N_i^-(k) = \{ j \in N_i(k) | x_i(k) - x_j(k) < 0 \} \).

The problem considered hereafter is

\[
x_i(k + 1) = x_i(k) + \sum_{j=1}^{n} c_{ij}(k) a_{ij}(k) (x_j(k) - x_i(k))
\]

(1)

\[
x_i(k) + \sum_{j=1}^{n} c_{ij}(k) a_{ij}(k) \delta_{ji}(k)
\]

(2)

\[
x_i(k) \in X_i \ \forall k \in \mathbb{Z}^+ \text{ and } \forall i \in \mathbb{Z}_n
\]

(3)

where \( a_{ij} \) refers to an edge determined by the true communication system and \( c_{ij} \) refers to the user-defined weight associated with \( a_{ij} \) and

\[
\delta_{ji}(k) := x_j(k) - x_i(k)
\]

(4)

Suppose \( c_{ij} \) for all \( i, j = 1, \ldots, n \) satisfies the conditions

\[
\sum_{j=1}^{n} c_{ij}(k) < 1, \quad c_{ij}(k) \geq 0 \ \forall k \in \mathbb{Z}^+
\]

(5)

and let

\[
f_{ij} := c_{ij} a_{ij}, \forall i \neq j, \quad f_{ii} := 1 - \sum_{j=1}^{n} f_{ij}.
\]

(6)

Equation (1) can be rearranged as

\[
x_i(k + 1) = f_{ii}(k)x_i(k) + \sum_{j=1}^{n} f_{ij}(k)x_j(k).
\]

(7)

With (5), it follows that

\[
f_{ii}(k) \geq 0, \quad f_{ii}(k) > 0 \quad \text{and} \quad \sum_{j=1}^{n} f_{ij}(k) = 1.
\]

(8)

These conditions show that \( F(k) := [f_{ij}(k)] \) is a row-stochastic, non-negative matrix with positive diagonal elements and has a spectral radius of value less than or equal to 1 [11]. Also, \( F(k) \) can be seen as the derived adjacency matrix of the network at time \( k \) following (7). Suppose the network does not change over time, contains a spanning tree and has no constraint, \( F \) has only one eigenvalue with unit magnitude and \( x \) achieve consensus asymptotically. This result is not necessarily true in the presence of constraints.

In order to enforce (3), additional information exchange is needed between agent \( i \) and its neighbors. For this purpose, it is assumed that \( \bar{x}_i, \bar{z}_i \) are known to agent \( i \) and are broadcasted to its neighbors in addition to the value of \( x_i(k) \). All results hereafter assumes a time-varying network. Hence, while the values of \( \bar{x}_j \) and \( \bar{z}_j \) for a given \( j \) are fixed, the inclusion of \( j \in N_i(k) \) depends on the configuration of the network and changes with \( k \).

III. THE UPDATE LAW AND ITS PROPERTIES

The approach uses the \( c_{ij} \)'s as the main working variables and their values are updated at each time instant. This section shows the details of the update by each agent and begins with the assumptions needed.

Assumption:

(A1) Upper bounds of \( \bar{d}_i \) and \( \bar{d}_m \), denoted by \( d_i \) and \( d_m \) respectively, are known to all agents.

(A2) \( x_i(0) \) is in \( X_i \).

Assumption (A1) is needed as the exact values of \( \bar{d}_i \) and \( \bar{d}_m \) may not be known. Since any upper bound (not necessarily tight) suffice, the assumption is mild and a convenience choice is \( d_i = d_m = n - 1 \). Assumption (A2) is needed because constraint (3) has to be satisfied at \( k = 0 \). For notational simplicity, let \( r_m := \frac{1}{d_m + 1} \).

Action to be taken by agent \( i \) at each instant \( k \)

(1) Broadcast the triplet \( \{x_i(k), \bar{x}_i, \bar{z}_i\} \) to and receive \( \{x_j(k), \bar{x}_j, \bar{z}_j\} \) from all \( j \in N_i(k) \).

(2) Compute the following intermediate variables:

\[
u_i(k) = \bar{x}_i - x_i(k), \quad \ell_i(k) = x_i(k) - \bar{z}_i,
\]

(9a)

\[
u_j(k) = \bar{x}_j - x_j(k), \quad \ell_j(k) = x_j(k) - \bar{z}_j, \quad \forall j \in N_i(k)
\]

(9b)

\[
\delta_{ji}(k) = x_j(k) - x_i(k), \quad \forall j \in N_i(k)
\]

(9c)

Update the values of \( c_{ij}(k) \) for all \( j \in N_i(k) \) according to

\[
c_{ij}(k) = \begin{cases} 
- r_m \min \{ \delta_{ji}(k), u_i(k), \ell_i(k) \} & \text{if } \delta_{ji}(k) > 0 \\
- r_m \min \{ -\delta_{ji}(k), \ell_i(k), u_j(k) \} & \text{if } \delta_{ji}(k) < 0 \\
r_m \delta_{ji}(k) & \text{if } \delta_{ji}(k) = 0
\end{cases}
\]

(10)

(3) The value of \( x_i(k + 1) \) is updated according to (2) using the values of \( c_{ij}(k) \) above.

The variables \( u_i(k) \) and \( \ell_i(k) \) \((u_j(k) \text{ and } \ell_j(k))\) refer to the deviations of \( x_i(k)(x_j(k)) \) from the upper and lower limits of its constraint. With (10), it is easy to see that

\[
c_{ij}(k)\delta_{ji}(k) = \begin{cases} 
- r_m \min \{ \delta_{ji}(k), u_i(k), \ell_i(k) \} & \text{if } \delta_{ji}(k) > 0 \\
- r_m \min \{ -\delta_{ji}(k), \ell_i(k), u_j(k) \} & \text{if } \delta_{ji}(k) < 0 \\
r_m \delta_{ji}(k) & \text{if } \delta_{ji}(k) = 0
\end{cases}
\]

(11)

This quantity, which is used in the update of \( x_i(k + 1) \) according to (2), plays an important role in the convergence of the states. Its properties as well as those of the network are stated in the following lemma whose proof is given in the Appendix.

Lemma 1. The system of (2) with \( c_{ij}(k), j \in N_i(k) \) \( \forall i \in \mathbb{Z}_n \) updated according to (10) with (A1)-(A2) satisfied has the following properties:

(i) \( c_{ij}(k) = c_{ji}(k) \) for all \( k \in \mathbb{Z}^+ \);

(ii) \( c_{ij}(k) \geq 0, \sum_{j \in \mathbb{Z}_n} c_{ij}(k) < 1 \) for all \( k \in \mathbb{Z}^+ \);

(iii) \( \sum_{i=1}^{n} x_i(k) = \sum_{i=1}^{n} x_i(0) \forall k \in \mathbb{Z}^+ \);

(iv) \( c_{ij}(k)\delta_{ji}(k) \leq r_m u_i(k) \) if \( \delta_{ji}(k) > 0 \) and \( c_{ij}(k)\delta_{ji}(k) \geq -r_m \ell_i(k) \) if \( \delta_{ji}(k) < 0 \);

(v) \( x_i(k) \in X_i \forall k \in \mathbb{Z}^+ \text{ and } \forall i \in \mathbb{Z}_n \);

(vi) The sequence \( \{x(k)\} \) has at least one converging subsequence;

(vii) \( x_{min}(k + 1) \geq x_{min}(k) \) where \( x_{min}(k) := \)
Property (i) shows that $F(k)$ of (6) keeps its symmetric structure for all $k$. Together with properties (ii), they show that (5) is satisfied and $F(k)$ preserves its properties as a row-stochastic, non-negative matrix with positive diagonal elements for all $k$. Property (iii) shows the average values of $x_i(k)$ remains a constant while property (v) shows the satisfaction of the constraints. Properties (iv), (vi) and (vii) are intermediate results needed for the development hereafter.

IV. CONVERGENCE

This section shows the convergence of the states of the agents under a time-varying network using the update law of (10). The basic idea is to show that any subsequence generated by the system converges to the same limit. Several additional notations are needed. From property (v) of Lemma 1, the existence of a converging subsequence is guaranteed. Without loss of generality, let there be two converging subsequences, $\{x_i(s_p^a)\}$ and $\{x_i(s_p^b)\}$, where $s_p^a : \mathbb{Z}^+ \to \mathbb{Z}^+$ is the index of the first subsequence and is a mapping from $p = 0, 1, \ldots$ to the time index. The same is true for $s_p^b$. Let the limits of the subsequences be

$$
\lim_{p \to \infty} x_i(s_p^a) = x^a := (x_1^a, x_2^a, \ldots, x_n^a)^T,
$$

$$
\lim_{p \to \infty} x_i(s_p^b) = x^b := (x_1^b, x_2^b, \ldots, x_n^b)^T.
$$

The corresponding sets of time instants of the subsequences are

$$
K^a = \{ k \in \mathbb{Z}^+ : s_p^a = k \text{ for some } p \in \mathbb{Z}^+ \},
$$

$$
K^b = \{ k \in \mathbb{Z}^+ : s_p^b = k \text{ for some } p \in \mathbb{Z}^+ \}.
$$

Lemma 2. Suppose $\{x_i(s_p^a)\}$ is a converging subsequence with $\lim_{p \to \infty} x_i(s_p^a) = x^a$. Then $\forall \epsilon > 0$, there exists a $p$, such that the following properties hold: (i) $|i(x_i(s_p^a) - x_i(s_p^b)| < 2\epsilon \forall p_1, p_2 > p$; (ii) $\delta_{ij}(s_p^a) > x_{ij} - x_{ij} - 2\epsilon \forall p > p$.

Theorem 1. Let assumptions (A1)-(A2) be satisfied. Suppose system (2) with update law (10) generates a sequence $\{x_i(k)\}$ having two converging subsequences $\{x_i(s_p^a)\}$ and $\{x_i(s_p^b)\}$ with $\lim_{p \to \infty} x_i(s_p^a) = x^a$ and $\lim_{p \to \infty} x_i(s_p^b) = x^b$, then $x^a = x^b$.

V. CONSENSUS OF STATES

Theorem 1 shows that system (2) with update law (10) converges under assumptions (A1)-(A2). Denote the converged values of $x_i$ by $x_i^\infty := \lim_{k \to \infty} x_i(k)$, $i \in \mathbb{Z}_n$, and define

$$
\tilde{I} := \{i|x_i^\infty = \bar{x}_i\}, \quad \bar{I} := \{i|x_i^\infty = \bar{x}_i\}, \quad \bar{I} := \{i|x_i > x_i^\infty < \bar{x}_i\}
$$

with $\tilde{I} \cup \bar{I} \subset \mathbb{Z}_n$. This section shows consensus among the agents under appropriate assumptions. To this end, additional notations and definitions associated with a time-varying network are needed. Given $k_1, k_2$ with $k_2 > k_1 \geq 0$, the unions of edges and graphs are $E_{k_1k_2} := \cup_{k=k_1}^{k_2} E(k)$ and $G_{k_1k_2} := (G, E_{k_1k_2})$, respectively. The edge set defined here is that associated with the adjacency matrix $A(k)$ and not the derived adjacent matrix $F(k)$. This distinction is needed as it is possible that $\lim_{k \to \infty} c_{ij}(k) = 0$ for some $(i,j)$, leading to $\lim_{k \to \infty} f_{ij}(k) = 0$ even though $a_{ij}(k) = 1$.

Definition 1. A graph $G$ is said to be jointly connected if for any $k \geq 0$, there exist a $\tau$, $\infty > \tau > 0$ such that $G_k^{k+\tau}$ is connected.

Definition 2. A node, $j$, is said to be a persistent neighbor (PN) of node $i$ and vice versa if for any $k \geq 0$, there exist a $\tau$, $\infty > \tau > 0$ such that $(i,j) \in E_k^{k+\tau}$. The collection of all persistent neighbors of $i$ is denoted as $N_i^p$.

Definition 3. A pair of nodes $(i,j)$ is said to be a persistent edge if $j \in N_i^p$ or $i \in N_j^p$.

Definition 4. A persistent path between nodes $i$ and $j$ is a sequence of persistent edges.

With the above definitions, additional assumptions are needed to achieve consensus of the network. These are

(A3) $\text{int}X \neq \emptyset$;

(A4) $\frac{1}{n} \sum_{i=1}^{n} x_i(0) \in X$;

(A5) The network $G$ (with adjacency matrix $A(k)$) is jointly connected $\forall k \in \mathbb{Z}^+$.

Clearly, (A3) eliminates unrealistic networks, it also implies that the sets $\tilde{I}$, $\bar{I}$ and $I$ are mutually exclusive in that $\tilde{I} \cap \bar{I} = \tilde{I} \cap I = \bar{I} \cap I = \emptyset$. The condition of (A4) may be hard to check in a large network having all different agents. However, when the agents belong to a limited number of classes such that $X$ is computable, (A4) can be satisfied by enforcing $x_i(0) \in X$. (A5) is a standard relaxed requirement for connectedness of the graph. The following lemma shows the relationship between (A5) and persistent paths.

Lemma 3. If (A5) holds, a persistent path exists between any pair of $(i,j) \in V \times V$.

Lemma 4. Suppose system (2) with update law (10) satisfy assumptions (A1)-(A3) and $j \in N_i^p$. The following properties hold: (i) $\lim_{k \to \infty} c_{ij}(k) = 0$ if $\lim_{k \to \infty} \delta_{ij}(k) \neq 0$; (ii) $\lim_{k \to \infty} c_{ij}(k) = r_m$ and $\lim_{k \to \infty} \delta_{ij}(k) = 0$ if $i \in \tilde{I}$ and $j \in \bar{I}$; (iii) $x_i^\infty \leq x_j^\infty$ if $i \in \tilde{I}$, $j \notin \bar{I}$ and $x_i^\infty \geq x_j^\infty$ if $i \in \bar{I}$, $j \notin \tilde{I}$.

The following lemma shows additional properties of the three sets of (14) under (A5).

Lemma 5. Suppose (A1)-(A3) and (A5) are satisfied. The sets $\tilde{I}$, $\bar{I}$ and $I$ defined by (14) satisfy the following: (i) if $\tilde{I} = \emptyset$ and $I \neq \emptyset$, then $\min_{i \in \tilde{I}} x_i^\infty = \min_{i \in \mathbb{Z}_n} x_i^\infty$. If $\tilde{I} = \emptyset$ and $I \neq \emptyset$, then $\max_{i \in \tilde{I}} x_i^\infty = \max_{i \in \mathbb{Z}_n} x_i^\infty$. (ii) if both $\tilde{I} = \emptyset$ and $I \neq \emptyset$, then $\min_{i \in \tilde{I}} x_i^\infty \leq \max_{i \in \tilde{I}} x_i^\infty$.

The main result of this section is now given.

Theorem 2. Suppose (A1)-(A5) are satisfied. Then $x_i^\infty = x_j^\infty = \frac{1}{n} \sum_{i=1}^{n} x_i(0)$ for all $i, j \in \mathbb{Z}_n$.

VI. DISCUSSION AND EXAMPLES

Numerical simulations of the proposed algorithm on several examples are given in this section. The first example is a system of 4 agents with constraints: $X_1 = [1, 5]$,
X_2 = [2, 6], X_3 = [3, 7] and X_4 = [4, 8] with d_{in} = 2. The network switches alternatively between two graphs of adjacency matrices,
\[
A_1 = \begin{bmatrix}
0 & 0 & 1 & 0 \\
0 & 0 & 1 & 1 \\
1 & 1 & 0 & 0 \\
0 & 1 & 0 & 0
\end{bmatrix}
\quad \text{and} \quad
A_2 = \begin{bmatrix}
0 & 1 & 0 & 0 \\
1 & 0 & 0 & 1 \\
0 & 1 & 0 & 1 \\
0 & 0 & 1 & 0
\end{bmatrix}
\] (15)
starting with A_1. The intersection of the constraint sets is X = \bigcap_{i=1}^{4} X_i = [4, 5] \neq \emptyset. Consider the case where x(0) = (3.0, 6.0, 6.0, 4.0), a choice that satisfies all assumptions with x_{ave}(0) = 4.75. Figures 1(a) and 1(b) show the time evolutions of x_i and c_{ij} for all (i,j), i \neq j except c_{14} since a_{14} = a_{41} = 0 in both A_1 and A_2. From 1(a), that the consensus value, x^\infty, is 4.75 \in \text{int} X and the average property holds for all time can be easily seen. In addition, \tilde{I} = I = \emptyset and |I| = 4. This fact together with \lim_{k \to \infty} \tilde{\theta}_j(k) = 0 imply that \lim_{k \to \infty} c_{ij}(k) = r_m = \frac{1}{3} for all (i,j) pairs according to property (ii) of Lemma 4 and this is shown in Figure 1(b). Figure 1(c) shows that the convergence rate of the agents, defined by R_t(k) := |x_i(k+1) - x_i^\infty|, approaches around 0.37 as k \to \infty, indicating that \{x_i\} converges geometrically. The last example is one given by (15). The time evolutions of \theta_i and r_i e^{j \theta_i} with \theta = (0, \pi, \frac{2\pi}{3}, \frac{\pi}{3}, \frac{2\pi}{3}, \frac{\pi}{3}, \frac{2\pi}{3}, \frac{\pi}{3}) are shown in Figure 2. Clearly, \theta_i reaches consensus for all i and all unicycles move along the same direction.

VII. CONCLUSIONS

This paper describes a new distributed algorithm for the consensus control of a group of agents whose scalar states are to lie within individually-defined constraints. It uses the edge weights of the network as the working variable to enforce the satisfaction of the constraints. The proofs of convergence and consensus of the states under reasonable assumptions are given. When the assumptions hold, the proposed algorithm converges to the consensus value independent of the switching sequences and preserves the average property of the network, just like the constrained-free algorithm. In addition, when the constraints of all agents are the same, the proposed algorithm becomes the standard consensus algorithm without constraint and recovers all the associated properties. Examples illustrating the working of the algorithm are provided.

REFERENCES


(a) Plots of \theta_i against k
(b) State Trajectories of the agents (x_i = r_i \cos \theta_i and y_i = r_i \sin \theta_i for all i \in \mathbb{Z}_5).

Fig. 2. Depiction of the \theta_i and the state trajectories for the 5 unicycles example.

Fig. 1. Plots of x_i(k), c_{ij}(k) and R_t(k) versus k for Example 1.
APPENDIX

Proof of Theorem 1

Proof: The proof is by induction. Since $s_p^a$ and $s_p^b$ are the indices of the converging subsequences, there exist indices $p_1$ and $p_2$ s.t. $s_p^a + \tau \in K^a \cup K^b$ and $s_p^b + \tau \in K^a \cup K^b$ for all $\tau \in \mathbb{Z}^+$. Since $\lim_{p \to \infty} x(s_p^a) = x^a$ and $\lim_{p \to \infty} x(s_p^b) = x^b$, for any $\epsilon > 0$, there exists $p_3$ and $p_4$ such that

$$-\epsilon < x_i(s_p^a) - x^a_i < \epsilon, \quad \forall i \in \mathbb{Z}_n \quad \text{and} \quad \forall p > p_3$$

(16)

$$-\epsilon < x_i(s_p^b) - x^b_i < \epsilon, \quad \forall i \in \mathbb{Z}_n \quad \text{and} \quad \forall p > p_4.$$  

(17)

Let $\bar{p} = \max\{p_1, p_2, p_3, p_4\}$. Hereafter, all references of the index $p$ is for the case where $p \geq \bar{p}$ unless otherwise stated. Without loss of generality, assume

$$x_i^a \leq x_2^a \leq \ldots \leq x_n^a$$  

(18)

and let $x_{min} = \min_{i \in \mathbb{Z}_n} \{x_i^a\}$. Note that no ordering is assumed for $x^b$ and $x_{min}$ can correspond to any $x_i^b$.

It is now shown by induction that $x_i^a = x_i^b$ for all $i \in \mathbb{Z}_n$ and that there exist positive constants, $C_1$ and $C_2$ such that

$$|c_{ij}(s_p^a)\delta_{ji}(s_p^a)| < C_1 \epsilon, \forall j \in N_i(s_p^a),$$

(19)

$$|c_{ij}(s_p^b)\delta_{ji}(s_p^b)| < C_2 \epsilon, \forall j \in N_i(s_p^b)$$

for all $p \geq \bar{p}$.

Induction base: prove by contradiction that $x_1^a = x_1^b$.

Consider subsequence $\{x(s_p^b)\}$. For each $p$ such that $s_p^a \in K^a$, there exists an index $\bar{p}(p)$, which depends on $p$, such that $s_p^{b}(\bar{p}(p)) > s_p^a$ and $s_p^{b}(\bar{p}(p)) \in K^b$. This is possible because $\{x(s_p^a)\}$ and $\{x(s_p^b)\}$ are subsequences. Then it follows from property (vii) of Lemma 1 that $\min_{i \in \mathbb{Z}_n} (s_p^{\bar{p}(p)}) \geq \min_{i \in \mathbb{Z}_n} (s_p^a)$ for any $p$ and the satisfaction of the following inequality:

$$x_1^b \geq x_{min} \lim_{p \to \infty} x_{min}(s_p^{\bar{p}(p)}) \geq \lim_{p \to \infty} x_{min}(s_p^a) = x_1^a.$$  

(20)

Now, suppose the equality condition of (20) does not hold, or equivalently, $\eta := x_1^b - x_1^a > 0$. The condition of (17) also implies that $\min_{i \in \mathbb{Z}_n}(s_p^{\bar{p}(p)}) > x_{min} - \epsilon$. Using this fact,

$$x_i(s_p^b) \geq \min_{i \in \mathbb{Z}_n}(s_p^b) > x_{min} - \epsilon \geq x_i^a - \epsilon \quad \forall i \in \mathbb{Z}_n$$  

(21)

where the last inequality follows from (20). In addition, using the update law of (10)

$$x_1(s_p^b + 1) = x_1(s_p^b) + \sum_{j \in N_i(s_p^b)} c_{ij}(s_p^b)(x_j(s_p^b) - x_1(s_p^b))$$

$$> x_1^b - \epsilon + \sum_{j \in N_i(s_p^b)} r_m(x_1^b - \epsilon - (x_1^b + \epsilon))$$

$$\geq x_1^b - \epsilon - d_1 r_m(\eta + 2\epsilon)$$

$$= x_1^b - r_m d_1 \eta - (1 + 2 r_m d_1) \epsilon$$

where the first strict inequality follows from (17) and (21) and that $c_{ij} \leq r_m$. It can be shown that choosing $\epsilon < \frac{-r_m d_1 \eta}{2 + 2 r_m d_1}$ leads to

$$x_1(s_p^b + 1) - x_1^a > \epsilon$$  

(22)

Hence, $s_p^b + 1 \notin K^a$ which implies that $s_p^b + 1 \in K^b$ since only two subsequences exist and $p \geq \bar{p}$. In addition, $s_p^b + 1 = s_p^{b+1}$ by definition. Repeating the above implies that $s_p^b + \tau \in K^b$ for all $\tau \in \mathbb{Z}^+$ which implies $\{x(s_p^a)\}$ is not a converging subsequence and contradicts the assumption. Therefore $x_1^a = x_1^b$.

The induction base is now established for the case of $i = 1$ for (19). Since $x_1^a = x_1^b$ and following (16) and (17), then for all $p$

$$|x_1(s_p^a + 1) - x_1^a| = \sum_{j \in N_1(s_p^a)} c_{1j}(s_p^a)\delta_{j1}(s_p^b)$$

$$+ x_1(s_p^a) - x_1^a < \epsilon.$$  

(23)

Since $|x_1(s_p^a) - x_1^a| < \epsilon$, using this with (23) imply that

$$\sum_{j \in N_1(s_p^a)} c_{1j}(s_p^a)\delta_{j1}(s_p^b) < 2\epsilon.$$  

(24)

Using property (ii) of Lemma 2 and $c_{ij}(s_p^a) \leq r_m$ in (24) leads to

$$2\epsilon > \sum_{j \in N_1(s_p^a)} c_{1j}(s_p^a)\delta_{j1}(s_p^b)$$

$$= \sum_{j \in N_1^+(s_p^a)} c_{1j}(s_p^a)\delta_{j1}(s_p^b) + \sum_{j \in N_1^-(s_p^a)} c_{1j}(s_p^a)\delta_{j1}(s_p^b)$$

$$\geq \sum_{j \in N_1^+(s_p^a)} c_{1j}(s_p^a)\delta_{j1}(s_p^b) + \sum_{j \in N_1^+(s_p^b)} r_m(-2\epsilon)$$

$$\geq \sum_{j \in N_1^+(s_p^b)} |c_{1j}(s_p^b)\delta_{j1}(s_p^b)| - 2d_1 r_m \epsilon$$

which implies $\sum_{j \in N_1^+(s_p^a)} |c_{1j}(s_p^a)\delta_{j1}(s_p^b)| < 2(1 + d_1 r_m) \epsilon$ and $|c_{1j}(s_p^a)\delta_{j1}(s_p^b)| < C_1 \epsilon$ for some finite constant $C_1$. A similar result can be obtained for the $s_p^b$ subsequence yielding $|c_{1j}(s_p^b)\delta_{j1}(s_p^b)| < C_2 \epsilon$ as stated in (19).

Induction hypothesis on $r$: Assume that the following 3 conditions hold for all $i \leq r$: $x_i^a = x_i^b$, $|c_{ij}(s_p^a)\delta_{ji}(s_p^b)| < C_1 \epsilon$, and $|c_{ij}(s_p^b)\delta_{ji}(s_p^b)| < C_2 \epsilon$, $\forall j \in N_i(s_p^b)$ for some finite constants $C_1$ and $C_2$. As a consequence of $x_i^a = x_i^b$, the following holds using a similar reasoning that leads to (24):

$$\sum_{j \in N_i(s_p^b)} c_{ij}(s_p^b)\delta_{ji}(s_p^b) = C_2 \epsilon,$$

(25)

for all $i < r$. Induction step: $x_r^a = x_r^b$ is proved by contradiction. Before the proof of the main induction step, the following intermediate result is needed.

Start of Intermediate result within Induction step

Claim: $\delta_{ji}(s_p^a) \geq -\eta - \alpha \epsilon \forall j \geq r$ for some $\alpha > 0$.

Proof: From (10),

$$x_i(s_p^a + 1) = x_i(s_p^a) + \sum_{j \in N_i(s_p^a)} c_{ij}(s_p^a)\delta_{ji}(s_p^a)$$

$$= x_i(s_p^a) + \sum_{j \in N_i(s_p^a), j = 1, \ldots, r-1} c_{ij}(s_p^a)\delta_{ji}(s_p^a)$$

$$+ \sum_{j \in N_i(s_p^a), j = r, \ldots} c_{ij}(s_p^a)\delta_{ji}(s_p^a)$$

(26)


From the fact that $p \in K^a \cup K^b$ it follows that $s^a_p + 1 = s^b_{\bar{p}}$ for some $\bar{p} \in \mathbb{Z}^+$. This means that the inequality above is equivalent to $x_r(s^b_{\bar{p}}) > x_r - (1 + d_mC_1 + 2r_m\alpha_\epsilon) \epsilon$, $\forall i \geq r$. Moreover by property (i) of Lemma 2, $x_i(s^b_{\bar{p}}) - x_i < -2\epsilon \forall i \in \mathbb{Z}_n$, so

$$x_i(s^b_{\bar{p}}) > x_i - (3 + d_mC_1 + 2r_m\alpha_\epsilon) \epsilon \quad \forall i \geq r$$

(26)

which, when $i = r$, implies that $\lim_{p \to \infty} x_r(s^b_{\bar{p}}) = x_r^b \geq x^a_r$. Now suppose that $x_r^b > x^a_r$ and let $\eta = x_r^b - x_r > 0$. From (17) and (26), it follows that

$$\delta_j(x_r) = x_r(s^b_{\bar{p}}) - x_r(s^b_{\bar{p}}) > x_r^b - (3 + d_mC_1 + 2r_m\alpha) \epsilon - (x_r^b + \epsilon) \quad = -\eta - (4 + d_mC_1 + 2r_m\alpha) \epsilon \quad =: -\eta - \alpha \forall j \geq r.$$  

(27)

End of Intermediate Result

Consider the subsequence $\{x(s^b_{\bar{p}})\}$ for any $p \geq \bar{p}$. From (10),

$$x_r(s^b_{\bar{p}} + 1) = x_r(s^b_{\bar{p}}) + \sum_{j \in N_r(s^b_{\bar{p}})} c_{rj}(s^b_{\bar{p}}) \delta_j(s^b_{\bar{p}})$$

$$= x_r(s^b_{\bar{p}}) + \sum_{j \in N_r(s^b_{\bar{p}}), j=1,...,r-1} c_{rj}(s^b_{\bar{p}}) \delta_j(s^b_{\bar{p}}) + \sum_{j \in N_r(s^b_{\bar{p}}), j=r+1,...,n} c_{rj}(s^b_{\bar{p}}) \delta_j(s^b_{\bar{p}})$$

$$> x_r^b - \epsilon + \sum_{j \in N_r(s^b_{\bar{p}}), j=1,...,r-1} C_1 \epsilon + \sum_{j \in N_r(s^b_{\bar{p}}), j=r+1,...,n} r_m(\eta + \alpha \epsilon)$$

$$\geq x_r^b - \epsilon - d_mC_1 \epsilon - d_mr_m(\eta + \alpha \epsilon)$$

$$= x_r^b - d_m(\eta + \alpha \epsilon) \quad \forall p \geq \bar{p}.$$  

(28)

Using the same argument as the paragraph following (22) for the base case, the above leads to a contradiction that $\{x(s^b_{\bar{p}})\}$ is not a subsequence and hence proves that $x_r^b = x_r^a$. Now the following is true for the same reasoning as (24),

$$\left| \sum_{j \in N_r(s^b_{\bar{p}})} c_{rj}(s^b_{\bar{p}}) \delta_j(s^b_{\bar{p}}) \right| < 2\epsilon.$$  

(29)

By induction hypothesis $\forall i < r$, $|c_{ij}(s^b_{\bar{p}})\delta_j(s^b_{\bar{p}})| < C_1 \epsilon$ for some finite constant $C_1$.

$$2\epsilon > | \sum_{j \in N_r(s^b_{\bar{p}})} c_{rj}(s^b_{\bar{p}}) \delta_j(s^b_{\bar{p}})|$$

$$\geq | \sum_{j \in N_r(s^b_{\bar{p}})} c_{rj}(s^b_{\bar{p}}) \delta_j(s^b_{\bar{p}})| - \sum_{j \in N_r(s^b_{\bar{p}})} |c_{rj}(s^b_{\bar{p}})\delta_j(s^b_{\bar{p}})|$$

$$\geq \sum_{j \in N_r(s^b_{\bar{p}})} |c_{rj}(s^b_{\bar{p}})\delta_j(s^b_{\bar{p}})| - d_rC_1 \epsilon$$

Therefore $\sum_{j \in N_r(s^b_{\bar{p}})} |c_{rj}(s^b_{\bar{p}})\delta_j(s^b_{\bar{p}})| < (2 + d_rC_1) \epsilon$, thus $|c_{rj}(s^b_{\bar{p}})\delta_j(s^b_{\bar{p}})| < C_1 \epsilon$ for some finite constant $C_1$.  

(30)

Similar proof can be done for $s^b_{\bar{p}}$ and that completes the proof.

Proof for Theorem 2

Proof: Suppose consensus is not reached. This means that there exists at least one pair of agents $(l_0, l_m) \in V \times V$ such that $x_{l_0}^\infty \neq x_{l_m}^\infty$ or $\lim_{l \to l_m} h^\alpha_{l_0, l_m}(k) \neq 0$. By (A5) and Lemma 3, there must be a persistent path from $l_0$ to $l_m$ in the form of $(l_0, l_1, l_1, l_2, \cdots, l_{m-1}, l_m)$. Since $x_{l_0}^\infty \neq x_{l_m}^\infty$, there must be a persistent edge $(l_k, l_{k+1})$ for some $0 \leq k \leq m-1$ such that $x_{l_k}^\infty \neq x_{l_{k+1}}^\infty$. In addition, at least one of $(l_k, l_{k+1})$ is in $\mathcal{I}$ or $\overline{\mathcal{I}}$ following (ii) of Lemma 4, since $x_{l_k}^\infty = x_{l_{k+1}}^\infty$ if both are in $\mathcal{I}$. Hence, $\mathcal{I} \cap \overline{\mathcal{I}} \neq \emptyset$. Only three cases needed to be considered.

Case I: $\mathcal{I} = \emptyset$ and $\overline{\mathcal{I}} \neq \emptyset$. This case corresponds to (i) of Lemma 5 which states that

$$\min_{i \in \mathcal{I}} x_i^\infty = x_{\min}^\infty := \min_{i \in \mathcal{I}} x_i^\infty.$$  

(31)

Since consensus is not reach, there must exist at least one $j \in \mathbb{Z}_n$ such that

$$x_j^\infty > \min_{i \in \mathcal{I}} x_i^\infty.$$  

(32)

By definition of $x_{\min}^\infty$, $x_j^\infty \geq x_{\min}^\infty$ for all $i \in \mathbb{Z}_n$. Adding these $n$ inequalities yields

$$x_1^\infty + \cdots + x_j^\infty + \cdots + x_n^\infty \geq x_j^\infty + (n-1)x_{\min}^\infty > n \min_{i \in \mathcal{I}} x_i^\infty = n \min_{i \in \mathcal{I}} \bar{x}_i,$$

where the second inequality follows from (31) and (32). The above also implies that $\frac{1}{n} \sum_{i=1}^{k} x_i^\infty > \min_{i \in \mathcal{I}} \bar{x}_i$. Since $\frac{1}{n} \sum_{i=1}^{k} x_i^\infty = \frac{1}{n} \sum_{i=1}^{k} x_i(0)$ from property (i) and (iii) of Theorem 1, this implies $\frac{1}{n} \sum_{i \in \mathcal{I}} x_i^\infty = \mathcal{X} \neq X$ which contradicts (A4).

Case II: $\mathcal{I} \neq \emptyset$ and $\overline{\mathcal{I}} = \emptyset$. Contradiction to (A4) can be achieved using a similar reasoning as Case I.

Case III: $\mathcal{I} \neq \emptyset$ and $\overline{\mathcal{I}} \neq \emptyset$. $X = \{ i \in \mathbb{Z}_n \}$ also means that $\max\{x_i| x \in X\} = \min_{i \in \mathcal{I}} \bar{x}_i$, $\min\{x_i| x \in X\} = \max_{\mathcal{I}} \bar{x}_i$. Hence, $\text{int} X \neq \emptyset$ of (A3) is equivalent to

$$\min\{x_i| i \in \mathbb{Z}_n\} > \max\{x_i| i \in \mathcal{I}\}.$$  

(33)

The case of $\mathcal{I} \neq \emptyset$ and $\overline{\mathcal{I}} \neq \emptyset$ corresponds to case (ii) of Lemma 5 which states that $\min_{i \in \mathcal{I}} \bar{x}_i \leq \max_{\mathcal{I}} \bar{x}_i \leq \max_{\mathcal{I}} \bar{x}_i \leq \max_{\mathcal{I}} \bar{x}_i = \max_{\mathcal{I}} \bar{x}_i$. It then follows that $\min_{i \in \mathcal{I}} \bar{x}_i \leq \max_{\mathcal{I}} \bar{x}_i \leq \max_{\mathcal{I}} \bar{x}_i \leq \max_{\mathcal{I}} \bar{x}_i$, which contradicts (33) or assumption (A3).