Near Optimality of Greedy Strategies for String Submodular Functions with Forward and Backward Curvature Constraints

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Abstract—The problem of choosing a string of actions to optimize an objective function that is string submodular has been considered in [1]. There it is shown that the greedy strategy, consisting of a string of actions that only locally maximizes the step-wise gain in the objective function, achieves at least a \((1 - e^{-\epsilon})\)-approximation to the optimal strategy. This paper improves this approximation by introducing additional constraints on curvature, namely, total backward curvature, total forward curvature, and elemental forward curvature. We show that if the objective function has total backward curvature \(\sigma\), then the greedy strategy achieves at least a \((1 - e^{-\epsilon})\)-approximation to the optimal strategy. If the objective function has total forward curvature \(\epsilon\), then the greedy strategy achieves at least a \((1 - \epsilon)\)-approximation of the optimal strategy. Moreover, we consider a generalization of the diminishing-return property by defining the elemental forward curvature. We also introduce the notion of string-matroid and consider the problem of maximizing the objective function subject to a string-matroid constraint.

I. INTRODUCTION

A. Background

We consider the problem of optimally choosing a string of actions over a finite horizon to maximize an objective function. Let \(A\) be a set of all possible actions. At each stage \(i\), we choose an action \(a_i\) from \(A\). We use \(A = (a_1, a_2, \ldots, a_k)\) to denote a string of actions taken over \(k\) consecutive stages, where \(a_i \in A\) for \(i = 1, 2, \ldots, k\). We use \(A^*\) to denote the set of all possible strings of actions (of arbitrary length, including the empty string). Let \(f : A^* \rightarrow \mathbb{R}\) be an objective function, where \(\mathbb{R}\) denotes the real numbers. Our goal is to find a string \(M \in A^*\), with a length \(|M|\) not larger than \(K\), to maximize the objective function:

\[
\text{maximize } f(M) \\
\text{subject to } M \in A^*, |M| \leq K.
\]

The solution to (1), which we call the optimal strategy, can be found using dynamic programing (see, e.g., [2]). More specifically, this solution can be expressed with Bellman’s equations. However, the computational complexity of finding an optimal strategy grows exponentially with respect to the size of \(A\) and the length constraint \(K\). On the other hand, the greedy strategy, though suboptimal in general, is easy to compute because at each stage, we only have to find an action to maximize the step-wise gain in the objective function. The question we are interested in is: How good is the greedy strategy compared to the optimal strategy in terms of the objective function? This question has attracted widespread interest, which we will review in the next section.

In this paper, we extend the concept of set submodularity in combinatorial optimization to bound the performance of the greedy strategy with respect to that of the optimal strategy. Moreover, we will introduce additional constraints on curvatures, namely, total backward curvature, total forward curvature, and elemental forward curvature, such that the greedy strategy achieves at least a factor of the performance of the optimal strategy. Therefore, the greedy strategy serves as a good approximation to the optimal strategy. We will investigate the relationship between the approximation bounds for the greedy strategy and the values of the curvature constraints. These results have many potential applications in closed-loop control problems such as portfolio management (see, e.g., [3]), sensor management (see, e.g., [4]), and influence in social networks (see, e.g., [5]).

B. Related Work

Submodular set functions play an important role in combinatorial optimization. Let \(X\) be a ground set and \(g : 2^X \rightarrow \mathbb{R}\) be an objective function defined on the power set \(2^X\) of \(X\). Let \(I\) be a non-empty collection of subsets of \(X\). Suppose that \(I\) has the hereditary and augmentation properties (see [6] for the definitions). Then, we call \((X, I)\) a matroid. The goal is to find a set in \(I\) to maximize the objective function:

\[
\text{maximize } g(N) \\
\text{subject to } N \in I.
\]

Suppose that \(I = \{S \subset X : \text{card}(S) \leq k\}\) for a given \(k\), where \(\text{card}(S)\) denotes the cardinality of \(S\). Then, we call \((X, I)\) a uniform matroid.

The main difference between (1) and (2) is that the objective function in (1) depends on the order of elements in the string \(M\), while the objective function in (2) is independent of the order of elements in the set \(N\). To further explain the difference, we use \(P(M)\) to denote a permutation of a string \(M\). Note that for \(M\) with length \(k\), there exist \(k!\) permutations. In (1), suppose that for any \(M \in A^*\) we have \(f(M) = f(P(M))\) for any \(P\). Then, under these special circumstances, problem (1) is equivalent to problem (2). In other words, we can view the second problem as a special case of the first problem. Moreover, there can be repeated
identical elements in a string, while a set does not contain identical elements (but we note that this difference can be bridged by allowing the notion of multisets in the formulation of submodular set functions).

Finding the solution to (2) is NP-hard—a tractable alternative is to use a greedy algorithm. The greedy algorithm starts with the empty set, and incrementally adds an element to the current solution giving the largest gain in the objective function. Theories for maximizing submodular set functions and their applications have been intensively studied in recent years [7]–[31]. The main idea is to compare the performance of the greedy algorithm with that of the optimal solution. Suppose that the set objective function \( g \) is non-decreasing: \( g(A) \leq g(B) \) for all \( A \subseteq B \); and \( g(\emptyset) = 0 \) where \( \emptyset \) denotes the empty set. Moreover, suppose that the function has the diminishing-return property: For all \( A \subseteq B \subseteq X \) and \( j \in X \setminus B \), we have \( g(A \cup \{j\}) - g(A) \geq g(B \cup \{j\}) - g(B) \).

Then, we say that \( g \) is a submodular set function. Nemhauser et al. [7] showed that the greedy algorithm achieves at least a \((1 - e^{-1})\)-approximation for the optimal solution given that \( (X, \mathcal{I}) \) is a uniform matroid and the objective function is submodular. (By this we mean that the ratio of the objective function value of the greedy solution to that of the optimal solution is at least \((1 - e^{-1})\).) Fisher et al. [8] proved that the greedy algorithm provides at least a \(1/2\)-approximation of the optimal solution for a general matroid. Conforti and Cornuéjols [9] showed that if the function \( g \) has a total curvature \( c \), then the greedy algorithm achieves at least \( \frac{1}{c}(1 - e^{-c}) \) and \( \frac{1}{1+c} \)-approximations of the optimal solution given that \( (X, \mathcal{I}) \) is a uniform matroid and a general matroid, respectively. Note that \( c \in [0, 1] \) for a submodular set function, and if \( c = 0 \), then the greedy algorithm is optimal; if \( c = 1 \), then the result is the same as that in [7]. Vondrák [10] showed that the continuous greedy algorithm achieves at least a \( \frac{1}{2}(1 - e^{-c}) \)-approximation for any matroid.

Wang et al. [11] provided approximation bounds in the case where the function has an elemental curvature \( \alpha \), defined as

\[
\alpha = \max_{S \subseteq X, i,j \in X, i \neq j} \left\{ \frac{g(S \cup \{i, j\}) - g(S \cup \{i\})}{g(S \cup \{j\}) - g(S)} \right\}.
\]

The notion of elemental curvature generalizes the notion of diminishing return.

Some recent papers [1], [12]–[14] have extended the notion of set submodularity to problem (1). Streeter and Golovin [12] showed that if the function \( f \) is forward and backward monotone:

\[
f(M \oplus N) \geq f(M) \quad \text{and} \quad f(M \oplus N) \geq f(N)
\]

for all \( M, N \in \mathbb{A}^* \), where \( \oplus \) means string concatenation, and \( f \) has the diminishing-return property:

\[
f(M \oplus (a)) - f(M) \geq f(N \oplus (a)) - f(N)
\]

for all \( a \in \mathbb{A}, M, N \in \mathbb{A}^* \) such that \( M \) is a prefix of \( N \), then the greedy strategy achieves at least a \((1 - e^{-1})\)-approximation of the optimal strategy. The notion of string submodularity and weaker sufficient conditions are established in [1] under which the greedy strategy still achieves at least a \((1 - e^{-1})\)-approximation of the optimal strategy. Golovin and Krause [14] introduced adaptive submodularity for solving stochastic optimization problems under partial observability.

C. Contributions

In this paper, we study the problem of maximizing submodular functions defined on strings. We impose additional constraints on curvatures, namely, total backward curvature, total forward curvatures, and elemental forward curvature, which will be rigorously defined in Section II. The notion of total forward and backward curvatures is inspired by the work of Conforti and Cornuéjols [9]. However, the forward and backward algebraic structures are not exposed in the setting of set functions because the objective function defined on set is independent of the order of elements in a set. The notion of elemental forward curvature is inspired by the work of Wang et al. [11]. We have exposed the forward algebraic structure of this elemental curvature in the setting of string functions. Moreover, the result and technical approach in [11] are different from those in this paper. More specifically, the result in [11] uses the fact that the value of a set function evaluated at a given set does not change with respect to any permutation of this set. However, the value of a string function evaluated at a given string might change with respect to a permutation of this string.

In Section III, we consider the maximization problem in the case where the strings are chosen from a uniform structure. For this case, our results are summarized as follows. Suppose that the string submodular function \( f \) has total backward curvature \( \sigma(O) \) with respect to the optimal strategy. Then, the greedy strategy achieves at least a \( \sigma(O)/(1 - e^{-\sigma(O)}) \)-approximation of the optimal strategy. Suppose that the string submodular function \( f \) has total forward curvature \( \epsilon \). Then, the greedy strategy achieves at least a \((1 - \epsilon)\)-approximation of the optimal strategy. We also generalize the notion of diminishing return by defining elemental forward curvature \( \eta \). The greedy strategy achieves at least a \( 1 - (1 - \frac{1}{K_\eta})^K \)-approximation, where \( K_\eta = (1 - \eta^K)/(1 - \eta) \) if \( \eta \neq 1 \) and \( K_\eta = K \) if \( \eta = 1 \).

In Section IV, we consider the maximization problem in the case where the strings are chosen from a non-uniform structure by introducing the notion of string-matroid. Our results for this case are as follows. Suppose that the string submodular function \( f \) has total backward curvature \( \sigma(O) \) with respect to the optimal strategy. Then, the greedy strategy achieves at least a \( 1/(1 + \sigma(O)) \)-approximation. We also provide approximation bounds for the greedy strategy when the function has total forward curvature and elemental forward curvature.
II. STRING SUBMODULARITY, CURVATURE, AND STRATEGIES

A. String Submodularity

We now introduce notation (same as those in [1]) to define string submodularity. Consider a set $\mathcal{A}$ of all possible actions. At each stage $i$, we choose an action $a_i$ from $\mathcal{A}$. Let $A = (a_1,a_2,\ldots,a_k)$ be a string of actions taken over $k$ stages, where $a_i \in \mathcal{A},$ $i = 1,2,\ldots,k$. Let the set of all possible strings of actions be

$$\mathcal{A}^* = \{(a_1,a_2,\ldots,a_k) | k = 0,1,\ldots \text{ and } a_i \in \mathcal{A}, \ i = 1,2,\ldots,k\}.$$ 

Note that $k = 0$ corresponds to the empty string (no action taken), denoted by $\emptyset$. For a given string $A = (a_1,a_2,\ldots,a_k)$, we define its string length as $k$, denoted $|A| = k$. If $M = (a_1^m,a_2^m,\ldots,a_k^m)$ and $N = (a_1^n,a_2^n,\ldots,a_k^n)$ are two strings in $\mathcal{A}^*$, we say $M \preceq N$ if $|M| = |N|$ and $a_i^m = a_i^n$ for each $i = 1,2,\ldots,|M|$. Moreover, we define string concatenation as follows:

$$M \oplus N = (a_1^m,a_2^m,\ldots,a_i^m,a_{i+1}^m,\ldots,a_k^m).$$

If $M$ and $N$ are two strings in $\mathcal{A}^*$, we write $M \preceq N$ if we have $N = M \oplus L$, for some $L \in \mathcal{A}^*$. In other words, $M$ is a prefix of $N$.

A function from strings to real numbers, $f : \mathcal{A}^* \rightarrow \mathbb{R}$, is string submodular if

i. $f$ has the forward-monotone property, i.e.,

$$\forall M,N \in \mathcal{A}^*, \quad f(M \oplus N) \geq f(M).$$

ii. $f$ has the diminishing-return property, i.e.,

$$\forall M \preceq N \in \mathcal{A}^*, \forall a \in \mathcal{A},$$

$$f(M \oplus (a)) - f(M) \geq f(N \oplus (a)) - f(N).$$

In the rest of the paper, we assume that $f(\emptyset) = 0$. Otherwise, we can replace $f$ with the marginalized function $f - f(\emptyset)$. From the forward-monotone property, we know that $f(M) \geq 0$ for all $M \in \mathcal{A}^*$.

We first state an immediate result from the definition of string submodularity. We note that all the proofs in this paper are omitted for the lack of space.

Lemma 1: Suppose that $f$ is string submodular. Then, for any string $N = (n_1,n_2,\ldots,n_{|N|})$, we have $f(N) \leq \sum_{i=1}^{|N|} f((n_i))$.

B. Curvature

We define the total backward curvature of $f$ by

$$\sigma = \max_{a \in \mathcal{A},M \in \mathcal{A}^*} \left\{1 - \frac{f((a) \oplus M) - f(M)}{f((a)) - f(\emptyset)} \right\}. \quad (3)$$

We define the total backward curvature of $f$ with respect to string $M \in \mathcal{A}^*$ by

$$\sigma(M) = \max_{N \in \mathcal{A}^*,0 \leq |N| \leq |K|} \left\{1 - \frac{f(N \oplus M) - f(M)}{f(N) - f(\emptyset)} \right\}, \quad (4)$$

where $K$ is the length constraint in (1). Suppose that $f$ is backward-monotone; i.e., $\forall M,N \in \mathcal{A}^*, f(M \oplus N) \geq f(N)$.

Then, we have $\sigma \leq 1$ and $f$ has total curvature at most $\sigma$ with respect to any $M \in \mathcal{A}^*$; i.e., $\sigma(M) \leq \sigma \ \forall M \in \mathcal{A}^*$. This fact can be shown using a simple derivation: For any $N \in \mathcal{A}^*$, we have

$$f(N \oplus M) - f(M) = \sum_{i=1}^{|N|} f((n_i) \oplus M) - f((n_i+1) \oplus M),$$

where $n_i$ represents the $i$th element of $N$. From the definition of total backward curvature and Lemma 1, we obtain

$$f(N \oplus M) - f(M) \geq \sum_{i=1}^{|N|} (1 - \sigma)f((n_i)) \geq (1 - \sigma)f(N),$$

which implies that $\sigma(M) \leq \sigma \leq 1$. We will give an lower bound for $\sigma(M)$ in the next section.

Symmetrically, we define the total forward curvature of $f$ by

$$\epsilon = \max_{a \in \mathcal{A},M \in \mathcal{A}^*} \left\{1 - \frac{f(M \oplus (a)) - f(M)}{f((a)) - f(\emptyset)} \right\}. \quad (5)$$

Moreover, we define the total forward curvature with respect to $M$ by

$$\epsilon(M) = \max_{N \in \mathcal{A}^*,0 \leq |N| \leq K} \left\{1 - \frac{f(M \oplus N) - f(M)}{f(N) - f(\emptyset)} \right\}. \quad (6)$$

If $f$ is string submodular and has total forward curvature $\epsilon$, then it has total forward curvature at most $\epsilon$ with respect to any $M \in \mathcal{A}^*$; i.e., $\epsilon(M) \leq \epsilon \ \forall M \in \mathcal{A}^*$. Moreover, for a string submodular function $f$, it is easy to see that for any $M$, we have $\epsilon(M) \leq \epsilon \leq 1$ because of the forward-monotone property and $\epsilon(M) \geq 0$ because of the diminishing-return property.

We define the elemental forward curvature of the string submodular function by

$$\eta = \max_{a_i,a_j \in \mathcal{A},M \in \mathcal{A}^*} \frac{f(M \oplus (a_i) \oplus (a_j)) - f(M \oplus (a_i))}{f(M \oplus (a_j)) - f(M)}.$$ \quad (7)

Moreover, we define the $K$-elemental forward curvature as follows:

$$\hat{\eta} = \max_{a_i,a_j \in \mathcal{A},M \in \mathcal{A}^*,|M| \leq 2K-2} \frac{f(M \oplus (a_i) \oplus (a_j)) - f(M \oplus (a_i))}{f(M \oplus (a_j)) - f(M)}.$$ \quad (8)

For a forward-monotone function, we have $\eta \geq 0$. Moreover, note that the diminishing-return property is equivalent to the condition $\eta \leq 1$. By the definitions, we know that $\hat{\eta} \leq \eta$ for all $K$.

We note that the definitions of $\sigma(M), \epsilon(M)$, and $\hat{\eta}$ depend on the length constraint $K$ of the optimal control problem (1), whereas $\sigma, \epsilon, \text{ and } \eta$ are independent of $K$. In other words, $\sigma, \epsilon, \text{ and } \eta$ can be treated as the universal upper bounds for $\sigma(M), \epsilon(M)$, and $\hat{\eta}$, respectively.
C. Strategies

We will consider the following two strategies.

1) Optimal strategy: Consider the problem (1) of finding a string that maximizes \( f \) under the constraint that the string length is not larger than \( K \). We call a solution of this problem an optimal strategy (a term we already have used repeatedly before). Note that because the function \( f \) is forward monotone, it suffices to just find the optimal strategy subject to the stronger constraint that the string length is equal to \( K \). In other words, if there exists an optimal strategy, then there exists one with length \( K \).

2) Greedy strategy: A string \( G_k = (a_1^*, a_2^*, \ldots, a_k^*) \) is called greedy if \( \forall i = 1, 2, \ldots, k \),

\[
a_i^* = \arg \max_{a_i \in k} f((a_1^*, a_2^*, \ldots, a_{i-1}^*, a_i)) - f((a_1^*, a_2^*, \ldots, a_{i-1}^*)).
\]

Notice that the greedy strategy only maximizes the step-wise gain in the objective function. In general, the greedy strategy (also called the greedy string) is not an optimal solution to (1). In this paper, we establish theorems which state that the greedy strategy achieves at least a factor of the performance of the optimal strategy, and therefore serves in some sense to approximate an optimal strategy.

III. Uniform Structure

Let \( I \) be the subset of \( \mathcal{A}^* \) with maximal string length \( K \): \( I = \{ A \in \mathcal{A}^* : |A| \leq K \} \). We call \( I \) a uniform structure. Note that the way we define uniform structure is similar to the way we define independent sets associated with uniform matroids. We will investigate the case of non-uniform structure in the next section. Now (1) can be rewritten as

maximize \( f(M) \)

subject to \( M \in I \).

We first consider the relationship between the total curvatures and the approximation bounds for the greedy strategy.

**Theorem 1:** Consider a string submodular function \( f \). Let \( O \) be a solution to (1). Then, any greedy string \( G_K \) satisfies

\[
f(G_K) \geq \frac{1}{\sigma(O)} \left(1 - \left(1 - \frac{\sigma(O)}{K}\right)^K\right) f(O)
\]

and

\[
f(G_K) \geq (1 - \max_{i=1,\ldots,K-1} \epsilon(G_i)) f(O).
\]

Under the framework of maximizing submodular set functions, similar results are reported in [9]. However, the forward and backward algebraic structures are not exposed in [9] because the total curvature there does not depend on the order of the elements in a set. In the setting of maximizing string submodular functions, the above theorem exposes the roles of forward and backward algebraic structures in bounding the greedy strategy. To explain further, let us state the results in a symmetric fashion. Suppose that the diminishing-return property is stated in a backward way: \( f((a) \oplus M) - f(M) \geq f((a) \oplus N) - f(N) \) for all \( a \in \mathcal{A} \) and \( M, N \in \mathcal{A}^* \) such that \( N = (a_1, \ldots, a_k) \oplus M \). Moreover, a string \( G_k = (a_1^*, a_2^*, \ldots, a_k^*) \) is called backward-greedy if

\[
a_i^* = \arg \max_{a_i \in k} f((a_i, a_{i+1}^*, \ldots, a_k^*)) - f((a_i^*, \ldots, a_k^*)) \quad \forall i = 1, 2, \ldots, k.
\]

Then, we can derive bounds in the same way as Theorem 1, and the results are symmetric.

The results in Theorem 1 implies that for a string submodular function, we have \( \sigma(O) \geq 0 \). Otherwise, part (i) of Theorem 1 would imply that \( f(G_K) \geq f(O) \), which is absurd. Moreover, recall that if the function is backward monotone, then \( \sigma(O) \leq \sigma \leq 1 \). From these facts and part (i) of Theorem 1, we get the following result, which is also derived in [12].

**Corollary 1:** Suppose that \( f \) is string submodular and backward monotone. Then,

\[
f(G_K) \geq (1 - (1 - \frac{1}{K})) f(O) > (1 - e^{-1}) f(O).
\]

Another immediate result follows from the facts that \( \sigma(O) \leq \sigma \) and \( \epsilon(G_i) \leq \epsilon \) for all \( i \).

**Corollary 2:** Suppose that \( f \) is string submodular and backward monotone. Then,

\[
f(G_K) \geq \frac{1}{\sigma} \left(1 - \left(1 - \frac{\sigma}{K}\right)^K\right) f(O)
\]

and

\[
f(G_K) \geq (1 - \epsilon) f(O),
\]

We note that the bounds \( \frac{1}{\sigma}(1 - e^{-\sigma}) \) and \( (1 - \epsilon) \) are independent of the length constraint \( K \). Therefore, the above bounds can be treated as universal lower bounds of the greedy strategy for all possible length constraints.

Next, we use the notion of elemental forward curvature to generalize the diminishing-return property and we investigate the approximation bound using the elemental forward curvature.

**Theorem 2:** Consider a forward-monotone function \( f \) with \( K \)-elemental forward curvature \( \eta \) and elemental forward curvature \( \eta \). Let \( O \) be an optimal solution to (1). Suppose that \( f(G_i \oplus O) \geq f(O) \) for \( i = 1, 2, \ldots, K-1 \). Then, any greedy string \( G_K \) satisfies

\[
f(G_K) \geq f(O) \left(1 - \left(1 - \frac{1}{K_\eta}\right)^K\right)
\]

where \( K_\eta = (1 - \eta)K/(1 - \eta) \) if \( \eta \neq 1 \) and \( K_\eta = K \) if \( \eta = 1 \); \( K_{\eta} = (1 - \eta)K/(1 - \eta) \) if \( \eta \neq 1 \) and \( K_\eta = K \) if \( \eta = 1 \).

Recall that \( \eta \) depends on the length constraint \( K \), whereas \( \eta \) does not. Therefore, the lower bound using \( K_\eta \) can be treated as a universal lower bound of the greedy strategy.

Suppose that \( f \) is string submodular. Then, we have \( \eta \leq 1 \). Because \( 1 - (1 - \frac{1}{K_\eta})^K \) is decreasing as a function of \( \eta \), we obtain the following result, which is reported in [1].

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Corollary 3: Consider a string submodular function \( f \). Let \( O \) be a solution to (1). Suppose that \( f(G_i \oplus O) \geq f(O) \) for \( i = 1, 2, \ldots, K - 1 \). Then, any greedy string \( G_K \) satisfies
\[
f(G_K) \geq (1 - (1 - \frac{1}{K})^K) f(O) > (1 - e^{-1}) f(O).
\]

The second inequality in the above corollary is given by the fact that \( 1 - (1 - \frac{1}{K})^K \to 1 - e^{-1} \) from above, as \( K \) goes to infinity. Next we combine the results in Theorems 1 and 2 to get the following result.

Proposition 1: Consider a forward-monotone function \( f \) with elemental forward curvature \( \eta \) and \( K \)-elemental forward curvature \( \hat{\eta} \). Let \( O \) be a solution to (1). Then, any greedy string \( G_K \) satisfies
\[
f(G_K) \geq \frac{1}{\sigma(O)} \left( 1 - \left( 1 - \frac{\sigma(O)}{K \eta} \right)^K \right) f(O),
\]
and
\[
f(G_K) \geq (1 - \max_{i=1,\ldots,K-1} \epsilon(G_i)) \frac{K^{\hat{\eta}}}{K^{\eta}} f(O).
\]

We note that the condition in Theorem 2, \( f(G_i \oplus O) \geq f(O) \) for \( i = 1, \ldots, K - 1 \), is essentially captured by \( \sigma(O) \).

In other words, even if the condition \( f(G_i \oplus O) \geq f(O) \) is violated, we can still provide approximation bound using \( \sigma(O) \), which is larger than 1 in this case.

IV. NON-UNIFORM STRUCTURE

In the last section, we considered the case where \( I \) is a uniform structure. In this section, we consider the case of non-uniform structures.

We first need the following definition. Let \( M = (m_1, \ldots, m_{|M|}) \) and \( N = (n_1, \ldots, n_{|N|}) \) be two strings in \( \mathbb{K}^* \). We write \( M \prec N \) if there exists a sequence of strings \( L_i \in \mathbb{K}^* \) such that
\[
N = L_1 \oplus (m_1, \ldots, m_i) \oplus L_2 \oplus (m_i+1, \ldots, m_{i+2}) \oplus \ldots \oplus (m_{i+k-1}, \ldots, m_{|M|}) \oplus L_{k+1}.
\]

In other words, we can remove some elements in \( N \) to get \( M \). Note that \( \prec \) is a weaker notion of dominance than \( \preceq \) defined earlier in Section II. In other words, \( M \preceq N \) implies that \( M \prec N \) but the converse is not necessarily true.

Now we state the definition of a non-uniform structure, analogous to the definition of independent sets in matroid theory. A subset \( I \) of \( \mathbb{K}^* \) is called a non-uniform structure if it satisfies the following conditions:

1. \( I \) is non-empty;
2. Hereditary: \( \forall M \in I, N \preceq M \) implies that \( N \in I \);
3. Augmentation: \( \forall M, N \in I \) and \( |M| < |N| \), there exists an element \( x \in \mathbb{K} \) in the string \( N \) such that \( M \oplus (x) \in I \).

By analogy with the definition of a matroid, we call the pair \( (\mathbb{K}, I) \) a string-matroid. We assume that there exists \( K \) such that for all \( M \in I \) we have \( |M| \leq K \) and there exists a \( N \in I \) such that \( |N| = K \). We call such a string \( N \) a maximal string. We are interested in the following optimization problem:

\[
\text{maximize } f(N) \text{ subject to } N \in I.
\]

Note that if the function is forward monotone, then the maximum of the function subject to a string-matroid constraint is achieved at a maximal string in the matroid. The greedy strategy \( G_k = (a_1^*, \ldots, a_k^*) \) in this case is given by \( a_i^* = \arg \max_{a_i \in \mathbb{A}} \) and \( (a_1^*, \ldots, a_{i-1}^*, a_i) \in f((a_1^*, a_2^*, \ldots, a_{i-1}^*, a_i)) - f((a_1^*, a_2^*, \ldots, a_{i-1}^*)) \forall i = 1, 2, \ldots, k \). Compared with (1), at each stage \( i \), instead of choosing \( a_i \) arbitrarily in \( \mathbb{A} \) to maximize the step-wise gain in the objective function, we also have to choose the action \( a_i \) such that the concatenated string \( (a_1^*, \ldots, a_{i-1}^*, a_i) \) is an element of the non-uniform structure \( I \). We first establish the following theorem.

Theorem 3: For any \( N \in I \), there exists a permutation of \( N \), denoted by \( P(N) = (\tilde{n}_1, \tilde{n}_2, \ldots, \tilde{n}_{|N|}) \), such that for \( i = 1, 2, \ldots, |N| \) we have \( f(G_i \oplus (\tilde{n}_i)) - f(G_{i-1}) \leq f(G_i) - f(G_{i-1}) \).

Next we investigate the approximation bounds for the greedy strategy using the total curvatures.

Theorem 4: Let \( O \) be an optimal strategy for (9). Suppose that \( f \) is a string submodular function. Then, a greedy strategy \( G_K \) satisfies

(i) \( f(G_K) \geq \frac{1}{1 + \sigma(O)} f(O), \)

(ii) \( f(G_K) \geq (1 - \epsilon(G_K)) f(O). \)

The inequality in (i) above is a generalization of a result on maximizing submodular set functions with a general matroid constraint [8]. The submodular set counterpart involves total curvature, whereas the string version involves total backward curvature. Note that if \( f \) is backward monotone, then \( \sigma(O) \leq \sigma \leq 1 \). We now state an immediate corollary of Theorem 4.

Corollary 4: Suppose that \( f \) is string submodular and backward monotone. Then, the greedy strategy achieves at least a 1/2-approximation of the optimal strategy.

Another immediate result follows from the facts that \( \sigma(O) \leq \sigma \) and \( \epsilon(G_K) \leq \epsilon \).

Corollary 5: Suppose that \( f \) is string submodular and backward monotone. Then, we have

(i) \( f(G_K) \geq \frac{1}{1 + \sigma} f(O), \)

(ii) \( f(G_K) \geq (1 - \epsilon) f(O). \)

Next we generalize the diminishing-return property using the elemental forward curvature.

Theorem 5: Suppose that \( f \) is a forward-monotone function with elemental forward curvature \( \eta \) and \( K \)-elemental forward curvature \( \hat{\eta} \). Suppose that \( f(G_K \oplus O) \geq f(O) \). If \( \hat{\eta} \leq 1 \), then \( f(G_K) \geq \frac{1}{1 + \hat{\eta}} f(O) \geq \frac{1}{1 + \eta} f(O) \). If \( \hat{\eta} > 1 \), then \( f(G_K) \geq \frac{1}{1 + \hat{\eta} K} f(O) \geq \frac{1}{1 + \eta} f(O) \).

This result is similar to that in [11]. However, the second bound in Theorem 5 is different from that in [11]. This is because the proof in [11] uses the fact that the value of a set function evaluated at a given set does not change with respect to any permutation of this set. However, the value of a string function evaluated at a given string might change.
with respect to a permutation of this string. Recall that the elemental forward curvature for a string submodular function is not larger than 1. We obtain the following result.

**Corollary 6:** Suppose that \( f \) is a string submodular function and \( f(G_K + O) \geq f(O) \). Then, the greedy strategy achieves at least a \( 1/2 \)-approximation of the optimal strategy.

Now we combine the results for total and elemental curvatures to get the following.

**Proposition 2:** Suppose that \( f \) is a forward-monotone function with \( K \)-elemental forward curvature \( \tilde{\eta} \) and elemental forward curvature \( \eta \). Then, a greedy strategy \( G_K \) satisfies

(i) \[ f(G_K) \geq \frac{1}{\sigma(O)+h(\tilde{\eta})} f(O) \geq \frac{1}{\pi(O)+h(\eta)} f(O), \]

(ii) \[ f(G_K) \geq \frac{1-\epsilon(G_K)}{h(\eta)} f(O) \geq \frac{1-\epsilon(G_K)}{h(\eta)} f(O), \]

where \( h(\tilde{\eta}) = \tilde{\eta} \) and \( h(\eta) = \eta \) if \( \tilde{\eta} \leq 1 \); \( h(\tilde{\eta}) = \tilde{\eta}^{2K-1} \) and \( h(\eta) = \eta^{2K-1} \) if \( \tilde{\eta} > 1 \).

From these results, we know that when \( f \) is string submodular, \( \tilde{\eta} \in [0,1] \) and we must have \( \sigma(O) + \tilde{\eta} \geq 1 \) and \( \epsilon(G_K) + \eta \geq 1 \). From Theorems 1, 2, 4, and 5, we know that the approximations of greedy relative to optimal get better as the total forward/backward curvature or the elemental forward curvature decreases to 0. However, the above inequalities indicate that the approximations with total curvature constraints and elemental forward curvature constraint cannot get arbitrarily good simultaneously. When equality in either case holds, the greedy strategy is optimal. A special case for this scenario is when the objective function is string-linear: \( f(M \oplus N) = f(M) + f(N) \) for all \( M, N \in \mathbb{R}^* \), i.e., \( \eta = 1 \) and \( \sigma = \epsilon = 0 \). Recall that \( 0 \leq \sigma(O) \leq \sigma, 0 \leq \epsilon(G_K) \leq \epsilon, \) and \( 0 \leq \tilde{\eta} \leq \eta \). Therefore, we have \( \sigma(O) = \epsilon(G_K) = 0 \) and \( \tilde{\eta} = 1 \).

V. CONCLUSION

In this paper, we have introduced the notion of total forward/backward and elemental forward curvature for functions defined on strings. We have derived several variants of lower bounds, in terms of these curvature values, for the greedy strategy with respect to the optimal strategy. Our results contribute significantly to our understanding of the underlying algebraic structure of string submodular functions.

REFERENCES


