Polytope joint Lyapunov functions for positive LSS∗

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Abstract

We consider switched linear systems of odes. \( \dot{x}(t) = A(u(t))x(t) \) where \( A(u(t)) \in \mathcal{A} \), a compact set of matrices. In this paper we propose a new method for the approximation of the upper Lyapunov exponent and lower Lyapunov exponent of the LSS when the matrices in \( \mathcal{A} \) are Metzler matrices (or the generalization of them for arbitrary cone), arising in many interesting applications (see e.g. [9]). The method is based on the iterative construction of invariant positive polytopes for a sequence of discretized systems obtained by forcing the switching instants to be multiple of \( \Delta(k) \), \( t \rightarrow 0 \) as \( k \rightarrow \infty \). These polytopes are then used to generate a monotone piecewise-linear joint Lyapunov function on the positive orthant, which gives tight upper and lower bounds for the Lyapunov exponents. As a byproduct we detect whether the considered system is stabilizable or uniformly stable. The efficiency of this approach is demonstrated in numerical examples, including some of relatively large dimensions.

1. BACKGROUND

We consider the following LSS:

\[ \dot{x}(t) = A(u(t))x(t) \quad (1) \]

where \( A(u(t)) \in \mathcal{A}, \ t \geq 0 \). Here \( u(\cdot) \) is a control function, \( A(u(t)) \) is assumed to be a summable function (the space of such function will be denoted as \( L_1 \)) that takes values on a given compact set of matrices. Since the range of the function \( A(\cdot) \) is compact, the summability of \( A(\cdot) \) is equivalent to its measurability. The set of control functions on an interval \([a,b]\) will be denoted by \( \mathcal{U}[a,b] \). We use the short notation \( \mathcal{U}[0,\infty) = \mathcal{U} \).

The upper Lyapunov exponent \( \hat{\sigma}(\mathcal{A}) \) is the infimum of numbers \( \alpha \) such that \( \|x(t)\| \leq Ce^{\alpha t} \) for every trajectory of (1). The system is (uniformly) stable if \( \|x(t)\| \rightarrow 0 \) as \( t \rightarrow \infty \) for every trajectory of (1). Obviously, if \( \hat{\sigma}(\mathcal{A}) < 0 \), then the system is stable, and, conversely, the stability implies that \( \hat{\sigma}(\mathcal{A}) \leq 0 \). This small gap between the necessary and sufficient condition can be handled: actually the system is stable if and only if \( \hat{\sigma}(\mathcal{A}) < 0 \) (see, for instance, [18, 2]). The lower Lyapunov exponent \( \check{\sigma}(\mathcal{A}) \) is the infimum of numbers \( \alpha \), for which there exists a control function \( u(t) \) with \( A(u(t)) \in L_1 \) such that every corresponding trajectory of (1) satisfies \( \|x(t)\| \leq Ce^{\alpha t} \). The system is stabilizable if there is a control function \( u(t) \in \mathcal{U} \) such that \( \|x(t)\| \rightarrow 0 \) as \( t \rightarrow \infty \) for every corresponding trajectory. The stabilizability is equivalent to the condition \( \check{\sigma}(\mathcal{A}) < 0 \) (see e.g. [11]). The following inequalities are checked directly:

\[ \hat{\sigma}(\mathcal{A} + \alpha I) = \hat{\sigma}(\mathcal{A}) + \alpha, \check{\sigma}(\mathcal{A} + \alpha I) = \check{\sigma}(\mathcal{A}) + \alpha. \quad (2) \]

We are interested to compute \( \hat{\sigma}(\mathcal{A}) \) and \( \check{\sigma}(\mathcal{A}) \). There is a wide bibliography on this subject (mainly for \( \hat{\sigma}(\mathcal{A}) \)); in this paper we take into consideration Metzler systems, appearing in many applications (for example [6]), and make use of a polytope based approach. There are several interesting relevant papers in the literature, among which we quote [3, 5, 17, 10, 11, 21] (this list far from being exhaustive) where both analytical and numerical procedures are proposed. The main contribute of this paper is the setting of a numerical methodology allowing to approximately compute the Lyapunov exponents by discretizing the system and exactly computing the Lyapunov exponents of the associated discrete system by means of the recent methods proposed in [12]. This allows us to compute a sequence of lower and upper bounds which converge to the exact values as the discretization stepsize tends to zero.

1.1. Extremal and invariant norms

In the sequel we shall make use of the following definition (where \( \text{co}(\mathcal{A}) \) is the convex hull of \( \mathcal{A} \)).
Definition 1 A generalized trajectory associated to (1) is obtained replacing $\mathcal{A}$ by $\text{co}(\mathcal{A})$ and extending the range of $u(t)$ so that $A(u(t)) \in \text{co}(\mathcal{A})$.

We start with the upper Lyapunov exponents; main tools here are extremal and invariant norms.

Definition 2 A norm $\| \cdot \|$ is called extremal for $\mathcal{A}$ if for any trajectory of (1), $\|x(t)\| \leq e^{\hat{\sigma}(\mathcal{A})^t} \|x(0)\|$, $t \geq 0$. An extremal norm is called invariant if for every $x_0 \in \mathbb{R}^d$ there exists a generalized trajectory $\tilde{x}(t)$ with $\tilde{x}(0) = x_0$ such that $\|x(t)\| = e^{\hat{\sigma}(\mathcal{A})^t} \|x_0\|$, $t \geq 0$.

Since every point $x(\Delta t)$ can be considered as a starting point of a new trajectory (after the shift of the argument $t' = t - \Delta t$), for an extremal norm the function $e^{-\hat{\sigma}(\mathcal{A})^t} \|x(t)\|$ is non-increasing in $t$ on every trajectory. For an invariant norm this function is identically constant on some trajectory, and for every point $x_0 \in \mathbb{R}^d$ there is such a trajectory starting in it. In particular, for $\hat{\sigma}(\mathcal{A}) = 0$ we have:

Corollary 1 In case $\hat{\sigma}(\mathcal{A}) = 0$ a norm is extremal for $\mathcal{A}$ if and only if it is non-increasing in $t$ on every trajectory of (1). An extremal norm is invariant if and only if for every $x_0 \in \mathbb{R}^d$ there exists a generalized trajectory $\tilde{x}(t)$ with $\tilde{x}(0) = x_0$, on which this norm is constant.

If we take a unit ball $B$ of that norm, we see that a norm is extremal if and only if every trajectory starting on the unit sphere $\partial B$ never leaves the ball $B$. This norm is invariant if for each point of the sphere there exists a generalized trajectory starting at this point that remains on the sphere. A set of operators $\mathcal{A}$ is called irreducible if these operators do not share a nontrivial common invariant subspace. The following theorem originated with N.Barabanov in [2].

Theorem A. An irreducible set of operators possesses an invariant norm.

1.2. Invariant cones and $K$-Metzler operators

Let $K \subset \mathbb{R}^d$ be a cone. In the sequel every cone will be assumed convex, closed, solid, pointed, and with an apex at the origin.

Definition 3 A linear operator $A$ in $\mathbb{R}^d$ is called Metzler with respect to a cone $K \subset \mathbb{R}^d$ (or, in short notation, $K$-Metzler) if there exists $h > 0$ such that $(I + hA)K \subset K$.

A vector $x$ is $K$-nonnegative ($x \geq_K 0$) if it belongs to this cone, and an operator $A$ is $K$-nonnegative ($A \geq_K 0$) if it leaves the cone $K$ invariant. If $I + hA \geq_K 0$, then $I + tA \geq_K 0$ for all $t \in [0, h]$. Indeed, $I + tA = \frac{h-t}{h} I + \frac{t}{h} (I + hA) \geq_K 0$, since both terms are $K$-nonnegative.

In the sequel of this section we assume a cone $K$ to be fixed, and write “nonnegative” and “Metzler” instead of “$K$-nonnegative” and “$K$-Metzler” respectively. We start with a well-known in the case $K = \mathbb{R}^d_+$.

Lemma 1 Let $K$ be an arbitrary cone. (1). If an operator $A$ is $K$-Metzler, then the operator $e^{tA}$ is nonnegative $\forall t > 0$. (2). If a compact set $\mathcal{A}$ consists of Metzler operators, then for every trajectory of (1) such that $x_0 \in K$ we have $x(t) \in K$, $t \geq 0$.

Corollary 2 If a compact set $\mathcal{A}$ consists of Metzler operators, then for every control function $A(\cdot)$ inequality $y_0 \geq_K x_0$ implies $y(t) \geq_K x(t)$ for every $t \geq 0$.

A face of a cone $K$ is the intersection of $K$ with a hyperplane passing through the apex. The apex is a face of dimension 0, this is a trivial face, all others are nontrivial. All generatrices are faces of dimension 1. A face plane is a linear span of a face.

Definition 4 A $K$-Metzler operator is called irreducible with respect $K$ (in short, $K$-irreducible) if it has no invariant subspace among the nontrivial face planes of $K$. A family of $K$-Metzler operators $\mathcal{A}$ is $K$-irreducible if there is no nontrivial face plane of $K$ invariant for all operators from $\mathcal{A}$.

Proposition 1 [22] A Metzler operator is irreducible with respect to a given cone $K$ if and only if it does not have eigenvectors on the boundary of $K$.

1.3. Monotone invariant norms

A norm on a cone $K$ is called monotone if for every $x, y \in K$ the inequality $x \geq_K y$ implies $\|x\| \geq \|y\|$. The aim of this section is to sharpen Barabanov’s theorem for Metzler operators with a cone $K$. We have that in this case there exists an invariant norm that is monotone with respect to $K$. Moreover, the irreducibility assumption can be now weakened to $K$-irreducibility. Thus, even if the operators share common invariant subspaces, they have an invariant norm, unless one of those subspaces is a face plane for $K$. The extremal norm on a cone $K$ for $K$-Metzler operators is defined in the same way as in Definition 2. The only difference is that now we consider only those trajectories starting in the cone $K$ (and hence, entirely lying in $K$). The definition of invariant norm on a cone $K$ also remains the same, we only write $x_0 \in K$ instead of $x_0 \in \mathbb{R}^d$.

Theorem 1 Every $K$-irreducible set of Metzler operators possesses an invariant monotone norm on $K$.

The proof is given in the upcoming paper [13] (for the positive orthant it is also proved in [1]). In [1] the authors proved the existence of an extremal monotone norm...
norm, but not necessarily invariant. In fact they write
"Indeed, in the situations we consider we cannot guar-
antee that a Barabanov norm exists". So, Theorem 1
provides a solution of the problem posed in [1].

1.4. Monotone invariant antinorms

It is not difficult to formulate analogues to the no-
tions of extremal and invariant norms for the lower Lyap-
unov exponent and stabilizable systems. For \( \sigma(\mathcal{A}) = 0 \) one could say as follows: a norm is extremal if it
is non-decreasing in \( t \) on every trajectory; an extremal
norm is invariant if for every starting point there exists
a generalized trajectory, on which the norm is identi-
cally constant. However, simple examples show that ex-
tremal norms may not exist already in dimension \( d = 2 \),
even for irreducible sets \( \mathcal{A} \) of totally positive ma-
trices. The reason is that in the proof of Theorems 1
an extremal norm is constructed as supremum of some
convex functionals. This is natural, because taking the
supremum respects convexity: suprema of convex func-
tionals are also convex functionals. For the lower Lyap-
unov exponent the supremum has to be replaced by the
infimum, but this does not preserve convexity. That is
why a functional with the "extreme" property (to be
non-decreasing on all trajectories) in most cases cannot
be convex. One might suggest a way out by replacing
convexity by concavity. However, there are no posi-
tive homogeneous concave functionals on \( \mathbb{R}^d \).
Nevertheless, such functions exist on cones. That is why, the
lower Lyapunov exponent of Metzler operators can be
analyzed using concave functionals on cones.

Definition 5 An antinorm \( f \) on a cone \( K \) is a nontrivial nonnegative concave homogeneous functional on \( K \). An antinorm is called positive if \( f(x) > 0 \ \forall x \in K \setminus \{0\} \).

The concept of antinorm originated in [19] (for Lyap-
unov exponents) and applied for the lower spectral ra-
dius in [12].

Definition 6 Let all operators of \( \mathcal{A} \) be \( K \) Metz-
zer. An antinorm \( f(\cdot) \) on \( K \) is said extremal
if \( f(x(t)) \geq e^{\sigma(\mathcal{A})t} f(x(0)), \ t \geq 0 \) for every trajectory
of \( f \) starting in \( K \) at \( t = 0 \). An extremal antinorm is said invari-
ant if \( \forall x_0 \in K \) there exists a generalized trajectory \( \tilde{x}(t) \)
with \( \tilde{x}(0) = x_0 \) such that \( f(x(t)) = e^{\sigma(\mathcal{A})t} f(x(0)), t \geq 0 \).

Thus, for an extremal antinorm the function \( e^{-\sigma(\mathcal{A})t} f(x(t)) \) is non-decreasing in \( t \) on every
trajectory. For an invariant antinorm this function is
identically constant on some generalized trajectory, and
for every point \( x_0 \in K \) there is such a trajectory starting
in it. A special case is the following.

Corollary 3 In case \( \sigma(\mathcal{A}) = 0 \) an antinorm is ex-
tremal for \( \mathcal{A} \) if and only if it is non-decreasing in \( t \) on
every trajectory of (1) in the cone. An extremal anti-
norm is invariant if and only if for every \( x_0 \in K \) there
exists a generalized trajectory \( \bar{x}(t) \) with \( \bar{x}(0) = x_0 \), on
which this antinorm is identically constant.

Consider the unit level set \( D = \{ x \in K | f(x) \geq 1 \} \) of
this antinorm. This is a convex unbounded subset of \( K \).
The antinorm is extremal if and only if every trajectory
starting on the boundary \( \partial D \) never leaves the set \( D \). The
antinorm is invariant if for each point of the boundary
there exists a trajectory starting at this point that re-
mains on the boundary. We formulate the existence re-
sult for extremal and invariant antinorms here. Its proof
is given in the upcoming paper [13].

Theorem 2 Let \( K \) be a cone. Every compact set \( \mathcal{A} \)
of \( K \)-Metzler operators possesses an extremal mono-
tone antinorm on \( K \). If, in addition, every \( A \in \mathcal{A} \) is
\( K \)-irreducible, then there exists a positive monotone in-
variant antinorm on \( K \).

1.5. Geometric conditions for stability and sta-
bilizability

Let \( K \) be a cone and \( \mathcal{A} \) be a set of \( K \)-Metzler oper-
ators. For a given set \( M \subset K \) we denote
\[
\text{co}_-(M) = (\text{co}(M) - K) \cap K = \{ x \in K | x = y - z, \ y \in \text{co}(M), z \in K \}
\]
\[
\text{co}_+(M) = (\text{co}(M) + K) = \{ y + z | y \in \text{co}(M), z \in K \}
\]

A monotone body is a convex body such that \( G = \text{co}_-(G) \). A monotone infinite body is a set such that
\( G = \text{co}_+(G) \). Clearly, if an arbitrary compact set \( M \subset K \)
contains at least one interior point of \( K \), then \( \text{co}_-(M) \) is
a monotone body. For an arbitrary closed set \( M \subset K \)
the set \( \text{co}_+(M) \) is a monotone infinite body. For a monotone
body \( G \), we denote by \( \partial(G) \) the boundary of \( G \) with re-
spect to the cone \( K \), i.e., \( \partial(G) \) consists of points from \( G \)
that are limits of sequences from \( K \setminus G \). The boundary
of an infinite body is defined in the same way. A mono-
tone body is the unit ball of a monotone norm, and \( \partial(G) \)
is the corresponding sphere; an infinite monotone body is
the unit ball of a monotone antinorm, and \( \partial(G) \) is
the corresponding sphere. If \( M \) is a finite set of points,
then the sets \( \text{co}_-(M) \) and \( \text{co}_+(M) \) are called monotone
polytope and infinite monotone polytope.

Theorem 3 (1) If there exists a monotone body \( P \) such
that for every point \( x \in \partial(P) \), all the vectors \( Ax, A \in \mathcal{A} \)
are oriented inside \( P \), then \( \mathcal{A} \) is stable. (2) For every
stable family \( \mathcal{A} \) there exists a monotone body \( P \) such that for every point \( x \in \partial(P) \), all the vectors \( Ax \), \( A \in \mathcal{A} \) are oriented inside \( P \).

In the simplest case, when \( K = \mathbb{R}^d_+ \), an operator \( A \) is Metzler if it is represented by a Metzler matrix, that is a matrix having all nonnegative off-diagonal elements. Thus, all the results of Sections IV and V hold for \( K = \mathbb{R}^d_+ \) and for a compact set \( \mathcal{A} \) of Metzler matrices. We need only to clarify two notions for the case \( K = \mathbb{R}^d_+ \).

Irreducibility. In this case the \( K \)-irreducibility coincides with the usual positive irreducibility of nonnegative matrices. A set of matrices is positively irreducible if none of the coordinate planes (i.e., linear spans of several basis vectors) is a common invariant subspace for those matrices. For positively reducible sets of matrices there always exists a permutation of basis vectors, transforming them into block upper triangular form.

2. Discretization and algorithms

We consider here a finite family \( \mathcal{A} = \{A_1, A_2, \ldots, A_n\} \) of Metzler matrices. Let \( u \in \mathcal{U}_\mathcal{M} \) be the restriction to the space of piecewise constant functions on the discrete grid \( \{t_j = j\Delta t\}_{j \geq 0} \). This transforms the problem of computing the upper Lyapunov exponent into that of computing the joint spectral radius (see e.g. the monograph [16]) of the family

\[
\mathcal{B}_\mathcal{M} = \{B_1, \ldots, B_m\} := \{e^{A_1 \Delta t}, \ldots, e^{A_n \Delta t}\}.
\]  

2.1. Joint spectral radius

Let \( \mathcal{B} = \{B_1, \ldots, B_m\} \) and \( \|\cdot\| \) a norm on \( \mathbb{R}^d \) and \( \|B\| = \max_{\|x\|=1} \|Bx\| \) the induced matrix norm. Let \( \mathcal{I} = \{1, \ldots, m\} \). Then, for \( k = 1, 2, \ldots \), consider the set

\[ \Sigma_k(\mathcal{B}) = \{B_{i_1} \cdots B_{i_k} \ | \ i_1, \ldots, i_k \in \mathcal{I}\} \]

of all products of degree \( k \) and the number \( \hat{\rho}_k(\mathcal{B}) = \max_{P \in \Sigma_k(\mathcal{B})} \|P\|^{1/k} \).

**Definition 7 (joint spectral radius (j.s.r.) [20])** The number \( \hat{\rho}(\mathcal{B}) \) defined as

\[ \hat{\rho}(\mathcal{B}) = \limsup_{k \to \infty} \hat{\rho}_k(\mathcal{B}) \]

is said the joint spectral radius of the family \( \mathcal{B} \).

Analogously, let \( \hat{\rho}(\cdot) \) denote the spectral radius of a \( d \times d \)-matrix and then, for each \( k = 1, 2, \ldots \), consider \( \tilde{\rho}_k(\mathcal{B}) = \sup_{P \in \Sigma_k(\mathcal{B})} \rho(P)^{1/k} \).

**Definition 8 (generalized spectral radius (g.s.r.) [7])** The number \( \tilde{\rho}(\mathcal{B}) = \limsup_{k \to \infty} \tilde{\rho}_k(\mathcal{B}) \) is said generalized spectral radius of \( \mathcal{B} \).

In their paper [7], Daubechies and Lagarias proved that

\[ \tilde{\rho}_k(\mathcal{B}) \leq \hat{\rho}(\mathcal{B}) \leq \rho(\mathcal{B}) \leq \hat{\rho}_k(\mathcal{B}) \]

for all \( k \geq 1 \). (4)

The fundamental equality \( \hat{\rho}(\mathcal{B}) = \rho(\mathcal{B}) \) has been proved later by Berger and Wang. Consequently we can simply denote as \( \rho(\mathcal{B}) \) the spectral radius of \( \mathcal{B} \). An important characterization of the spectral radius \( \rho(\mathcal{B}) \) of a matrix family is the generalization of Gelfand’s formula. In order to state this characterization, we define

\[ \|\mathcal{B}\| = \hat{\rho}_1(\mathcal{B}) = \max_{i \in \mathcal{I}} \|B_i\|. \]

**Proposition 2** (see [20, 8]) The j.s.r. of a bounded family \( \mathcal{B} \) of \( d \times d \)-matrices is characterized by \( \rho(\mathcal{B}) = \inf_{\|\cdot\| \in \mathcal{O}_\mathcal{M}} \|\mathcal{B}\| \), where \( \mathcal{O}_\mathcal{M} \) is the set of operator norms.

In order to establish whether the infimum in Proposition 2 is a minimum, we give the following definition.

**Definition 9 (Extremal norm)** We say that a norm \( \|\cdot\| \), satisfying \( \|\mathcal{B}\| = \rho(\mathcal{B}) \) is extremal for \( \mathcal{B} \).

For families with a common invariant cone \( K \) it suffices to construct an extremal monotone norm \( g \) on \( K \).

2.2. Bounds for the upper Lyapunov exponent

In [12] an algorithm is proposed to exactly compute \( \rho(\mathcal{B}_\mathcal{M}) \) by means of a monotone polytope extremal norm. We name \( \mathcal{P}_\mathcal{M} \) the computed extremal polytope and \( \mathcal{V}_\mathcal{M} \) its vertices. An obvious lower bound for \( \hat{\sigma}(\mathcal{A}) \) is

\[ \hat{\sigma}_\mathcal{M} = \frac{1}{\Delta t} \log(\rho(\mathcal{B}_\mathcal{M})) \]

that is the upper Lyapunov exponent restricted to \( \mathcal{V}_\mathcal{M} \). It is useful to define the shifted family

\[ \hat{\mathcal{A}}_\mathcal{M} := \{\hat{A}_i\}_{i=1}^n \text{ with } \hat{A}_i = A_i - \hat{\sigma}_\mathcal{M} \]

having upper Lyapunov exponent \( \hat{\sigma}(\hat{\mathcal{A}}_\mathcal{M}) \geq 0 \). If the vectorfield \( \hat{A}_i \nu \) applied to \( v \in \partial(\mathcal{P}_\mathcal{M}) \) is oriented inside \( \mathcal{V}_\mathcal{M} \) for all \( i \) and \( \mathcal{P}_\mathcal{M} \) is positively invariant and every trajectory is bounded so that \( \hat{\sigma}(\hat{\mathcal{A}}_\mathcal{M}) = 0 \) (see Proposition 3) and the extremal control function for (1) belongs to \( \mathcal{V}_\mathcal{M} \). Moreover \( \mathcal{P}_\mathcal{M} \) is the unit ball of an extremal monotone polytope norm for the continuous system. Of course this is extremely unlikely. However, by the simple equality (2), it is possible to get an upper bound for the upper Lyapunov exponent by shifting the matrices \( A_i \) by \( \alpha I \), \( \alpha > 0 \) so that the vectorfield \( (A_i - \alpha I) \nu \) applied to \( v \in \partial(\mathcal{P}_\mathcal{M}) \) is oriented inside \( \mathcal{V}_\mathcal{M} \) for all \( v \) and \( i \). This determines, by a finite minimization w.r.t. \( \alpha \), the upper bound

\[ \hat{\sigma}(\mathcal{A}) \leq \frac{1}{\Delta t} \log(\rho(\mathcal{B}_\mathcal{M})) + \alpha_M := \gamma_M. \]

2.3. Estimating the upper Lyapunov exponent

Let \( \mathcal{V}_\mathcal{M} = \{v_i\} \) be an essential system of vertices of \( \mathcal{P}_\mathcal{M} \). In order to compute an upper bound we solve the following optimization problem (with a small \( \delta > 0 \)).
for $i = 1, \ldots, m$ do
  Solve the LP problems (w.r.t. $\{t_i\}, \alpha_i$)
  \[
  \min \alpha_i \\
  \text{s.t. } w + \delta(\tilde{A}_i - \alpha_i I)w \leq \sum_{v \in V_\Delta} t_v v \quad \forall w \in V_\Delta \\
  \text{and } \sum_{v \in V_\Delta} t_v \geq 1, \quad t_v \geq 0 \quad \forall v \in V_\Delta 
  \]
end

Algorithm 1: Computing an optimal upper bound

### 2.4. Lower spectral radius

Consider the family $\mathcal{B}$ given in (3) and let $\hat{\rho}_k(\mathcal{B}) = \min_{P \in \Sigma_k(\mathcal{B})} \|P\|^{1/k}$.

**Definition 10 (lower spectral radius [15])** The number

\[
\hat{\rho}(\mathcal{B}) = \liminf_{k \to \infty} \hat{\rho}_k(\mathcal{B})
\]

is said the lower spectral radius (l.s.r.) of the family $\mathcal{B}$.

Thus the l.s.r. is the exponent of asymptotic growth of the minimal product of operators from the family $\mathcal{B}$. The limit in (6) always exists and does not depend on the norm. A simple observation is that the l.s.r. can be estimated by the usual spectral radii as follows:

\[
\hat{\rho}(\mathcal{B}) \leq \min_{P \in \Sigma_1(\mathcal{B})} \rho(P)^{1/k} \leq \min_{P \in \Sigma_1(\mathcal{B})} \|P\|^{1/k}. \tag{7}
\]

In contrast to inequality (4) for the j.s.r., estimation (7) gives only upper bounds. The role of norms is replaced by antinorms. The following results are proved in [12].

**Proposition 3** If for an antinorm $f$ and a constant $\lambda$, $f(B, x) \geq \lambda f(x), \forall x \in K$ and $\forall B_i \in \mathcal{B}$, then $\hat{\rho}(\mathcal{B}) \geq \lambda$.

**Definition 11** An antinorm is called extremal for $\mathcal{B}$ if $f(B, x) \geq \hat{\rho}(\mathcal{B}) f(x), \forall x \in K$ and $\forall B_i \in \mathcal{B}$.

**Theorem 4** A family of matrices with a common invariant cone $K$ admits a monotone extremal antinorm on $K$.

In [12] sufficient conditions for the existence of an extremal monotone polytope antinorm are given and an algorithm is presented to compute the infinite polytope determining the set $f(x) \geq 1$.

### 2.5. Bounds for the lower Lyapunov exponent

For the family (3) we denote as $\mathcal{P}_\Delta$ the infinite polytope obtained by the algorithm computing $\hat{\rho}(\mathcal{B}_\Delta)$ and $V_\Delta$ its vertices. A natural upper bound for $\hat{\sigma}(\mathcal{A})$ is

\[
\hat{\sigma}_\Delta = \frac{1}{\Delta t} \log(\hat{\rho}(\mathcal{B}_\Delta))
\]

that is the lower Lyapunov exponent restricted to $\mathcal{P}_\Delta$. Again, it is useful to define the shifted family

\[
\hat{\mathcal{P}}_{\Delta} := \{\tilde{A}_i\}_{i=1}^m \quad \text{with} \quad \tilde{A}_i = A_i - \hat{\sigma}_\Delta
\]

having lower Lyapunov exponent $\hat{\sigma}(\hat{\mathcal{P}}_{\Delta}) \leq 0$. Still making use of (2), we can get a lower bound for the lower Lyapunov exponent by positively shifting the matrices $A_i$ by $\alpha I, \alpha > 0$ so that the vectorfield $(A_i + \alpha I)v$ applied to $v \in \partial(\mathcal{P}_\Delta)$ is oriented inside $\mathcal{P}_\Delta$ for all $v$ and $i$. A minimization w.r.t. $\alpha$ provides the lower bound

\[
\hat{\sigma}_\Delta \geq \frac{1}{\Delta t} \log(\hat{\rho}(\mathcal{B}_\Delta)) + \hat{\sigma}_\Delta := \hat{\beta}_\Delta.
\]

### 2.6. Estimating the lower Lyapunov exponent

Let $V_\Delta = \{v_i\}$ be an essential system of vertices of $\mathcal{P}_\Delta$. Then we solve the problem (with a small $\delta > 0$):

for $i = 1, \ldots, m$ do
  Solve the LP problems (w.r.t. $\{t_i\}, \alpha_i$)
  \[
  \min \alpha_i \\
  \text{s.t. } w + \delta(\tilde{A}_i + \alpha_i I)w \geq \sum_{v \in V_\Delta} t_v v \quad \forall w \in V_\Delta \\
  \text{and } \sum_{v \in V_\Delta} t_v \geq 1, \quad t_v \geq 0 \quad \forall v \in V_\Delta 
  \]
end

Algorithm 2: Computing an optimal lower bound

### 3. RATE OF CONVERGENCE OF THE ALGORITHMS AND EXAMPLES

Main convergence results (proved in [13]) follow.

**Theorem 5** For every compact irreducible family $\mathcal{A}$ of Metzler matrices, there exist constants $C_1, C_2$ such that

\[
\hat{\gamma}_\Delta - \hat{\sigma}_\Delta \leq C_1 \Delta t, \quad \Delta t > 0 \\
\hat{\sigma}_\Delta - \hat{\beta}_\Delta \leq C_2 \Delta t, \quad \Delta t > 0,
\]

where $\hat{\gamma}_\Delta, \hat{\sigma}_\Delta$ are the upper and lower bounds produced by the polytope algorithm in [12] coupled with Algorithm 1 and similarly $\hat{\sigma}_\Delta, \hat{\beta}_\Delta$ are the upper and lower bounds produced by the polytope algorithm in [12] and Algorithm 2, with the parameter $\Delta t$.

### 3.1. Illustrative examples

Example 1. Let $\mathcal{A} = \{A_1, A_2\}$ with $A_1 = \log(B_1), A_2 = \log(B_2)$,

\[
B_1 = \begin{pmatrix} 7 & 0 \\ 2 & 3 \end{pmatrix}, \quad B_2 = \begin{pmatrix} 2 & 4 \\ 0 & 8 \end{pmatrix}.
\]
For $\Delta t = 1$ we get that $B\Delta t = \{B_1, B_2\}$. By means of the mentioned algorithm for computing the l.s.r. we are able to prove that the product of degree equal to 8, $P = B_1 B_2 (B_1^2 B_2^2)^2$ is spectrum minimizing, which means that $\tilde{\rho}(B\Delta t) = \rho(P)^{1/8} = 6.009313489 \ldots$, giving the upper bound $\tilde{\sigma}_\Delta = 1.793310513 \ldots$. Applying Algorithm 2 we obtain $\alpha_\Delta = 0.1323026 \ldots$ yielding the estimate $1.661007914 \leq \tilde{\sigma}(\mathcal{A}) \leq 1.793310513 \ldots$

Example 3. Consider the family $\mathcal{A} = \{A_1, A_2\}$,

$$A_1 = \begin{pmatrix}
-15 & 1 & 0 & 3 & 2 & 0 & 0 \\
2 & 9 & 3 & 2 & 1 & 2 & 1 \\
1 & 3 & -13 & 2 & 1 & 1 & 0 \\
2 & 0 & 1 & -7 & 1 & 0 & 0 \\
1 & 0 & 1 & 1 & -8 & 0 & 1 \\
1 & 3 & 1 & 2 & 3 & -11 & 2 \\
1 & 3 & 1 & 3 & 1 & 1 & -10 \\
2 & 1 & 3 & 2 & 3 & 2 & -11
\end{pmatrix}$$

$$A_2 = \begin{pmatrix}
-10 & 2 & 2 & 0 & 1 & 3 & 2 \\
0 & -16 & 2 & 1 & 2 & 3 & 1 \\
2 & 2 & -14 & 3 & 1 & 2 & 3 \\
0 & 3 & 3 & -13 & 3 & 2 & 0 \\
3 & 2 & 1 & 2 & 3 & -9 & 0 \\
1 & 3 & 0 & 0 & 1 & 7 & 0 \\
0 & 2 & 3 & 2 & 2 & 3 & -17 \\
2 & 2 & 2 & 2 & 2 & 3 & 2 & -17
\end{pmatrix}$$

with $\sigma(A_1) = -0.89470735 \ldots$, $\sigma(A_2) = -1.22136422 \ldots$. Our aim is determining whether the switched system (1) is stable. Table 1 reports the obtained computational results.

### Table 1. Approximation of the upper exponent

<table>
<thead>
<tr>
<th>$\Delta t$</th>
<th>$\tilde{\sigma}_\Delta$</th>
<th>$\tilde{\gamma}_\Delta$</th>
<th>#V</th>
</tr>
</thead>
<tbody>
<tr>
<td>1/4</td>
<td>-0.81422339</td>
<td>1.607661866</td>
<td>4</td>
</tr>
<tr>
<td>1/8</td>
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<td>0.581116076</td>
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<td>-0.76212368</td>
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<td>43</td>
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<tr>
<td>1/32</td>
<td>-0.76212368</td>
<td>-0.33813367</td>
<td>700</td>
</tr>
</tbody>
</table>

### Acknowledgment

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### References


