PD Control of a Second-order System with Hysteretic Actuator
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Abstract—In this paper, we design a proportional and derivative (PD) controller for a mass-damper-spring system interconnected with a hysteretic actuator, such as, the piezo-actuated stage. The hysteretic actuator is assumed to have a counterclockwise (CCW) input-output (I/O) dynamics (e.g. piezo-actuator) and is modeled by a Duhem operator. Based on the CCW of the hysteretic actuator, we provide sufficient conditions on the controller gains that depend on known intervals where the plant and hysteresis parameters lie such that the velocity of the linear plant converges to the origin. Robustness analysis and experimental results for tracking a constant reference velocity are also presented.

I. INTRODUCTION

Nowadays, many actuators are made of “smart material”, such as piezoceramic material, magnetostrictive material and magnetic shape memory alloy. While they can offer a novel type of actuation, these smart materials exhibit hysteresis phenomena which need to be taken into account in the control design. Without a proper compensation, it has been known that hysteresis can cause problems in applications, such as, limiting the accuracy and performance degradation [?].

There are many different approaches proposed in recent literatures for compensating the hysteresis in the feedback loop, see, among many others, [7], [19], [20], [6]. In [7], an inverse Preisach operator is used to compensate a hysteretic system. In [19], an adaptive control design for compensating an unknown backlash-type hysteresis for systems with known plant is proposed. In [20], the stability of a linear system with an backlash operator is analyzed by using generalized sector conditions of the backlash-type hysteresis and the stability conditions are given based on linear matrix inequalities (LMIs). In [6], the dissipativity of the Preisach operator is derived and a controller which is strictly passive is designed for the smart actuators.

In this paper we are focusing on the Duhem hysteresis operator (the definition of the Duhem hysteresis operator will be given in Section II). Our main results are based on our previous results in [9],[16] and [17], where we study the dissipativity property of the Duhem hysteresis operator with counterclockwise (CCW) and clockwise (CW) input-output (I/O) behavior, and the absolute stability of a feedback interconnection between a linear system and a Duhem hysteresis operator based on the CCW or CW property of each subsystem. In particular, we design a PD controller for a second-order mass-damper-spring system interconnected with a CCW Duhem hysteretic actuator. We provide the sufficient conditions on the controller based on the parameters of the linear plant such that the velocity of the mass-damper-spring system converges to the origin. Based on this analysis, we can provide a method to determine the control gains based on the parameters of the linear plant. The robustness of the closed-loop system is also discussed with respect to an input disturbance using the integral input-to-state stability concept. Our main results are different from our previous result in [14] in the following aspects: (i). The actuator in [14] is considered to be clockwise which is motivated by friction-based actuator and has a limited number of applications since this class of actuators is rather small. In this paper, we consider CCW actuators which include many well-known actuators, such as, piezoactuator which we will use in the experimental results; (ii). In [14], one of the main tools in the control design is to make the cascaded linear system counterclockwise which is rather straightforward. Adopting this to our present context, designing a linear controller, such that the cascaded linear system is clockwise, is not trivial. In particular, a PD controller with a non-zero integrator gain cannot make the cascaded system clockwise.

One motivating application for our control design is the raster scanning in an atomic force microscopy [11]. For such scanning, the sample is placed on the stage which is driven by a piezo-actuator and a fixed cantilever is placed on the top of the sample. The stage follows a wave signal (move back and forth at a constant speed) and the roughness of the sample will be measured by the deflection of the cantilever using laser.

II. PRELIMINARIES

Denote $C^1(\mathbb{R}_+)$ the space of continuously differentiable functions $f : \mathbb{R}_+ \to \mathbb{R}$. Denote $AC(\mathbb{R}_+)$ the space of absolutely continuous functions $f : \mathbb{R}_+ \to \mathbb{R}$. Define

$$\frac{dz(t)}{dt} := \lim_{h \to 0^+} \frac{z(t+h)-z(t)}{h}.$$  

A function $\alpha : \mathbb{R}_+ \to \mathbb{R}_+$ belongs to class $\mathcal{K}$ if it is continuous, strictly increasing and $\alpha(0) = 0$. A function $\alpha$ belongs to class $\mathcal{K}_\infty$ if $\alpha \in \mathcal{K}$ and $\lim_{s \to \infty} \alpha(s) = \infty$. A function $\beta : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ belongs to class $\mathcal{KL}$ if for each fixed $s \in \mathbb{R}_+$, $\beta(\cdot, s) \in \mathcal{K}$ and for each fixed $\xi \in \mathbb{R}_+$, $\beta(s, \cdot) \in \mathcal{K}_\infty$. $\beta(\cdot, s)$ is decreasing and $\lim_{s \to \infty} \beta(s, \cdot) = 0$.

A. integral input-to-state stability

The concept of integral input-to-state stability is introduced for the stability analysis of nonautonomous nonlinear systems given a bounded-energy input signal in [18], [2]. It is an integral variant of the input-to-state stability (ISS) property and is closely related to the $L^2$ stability of the
nonlinear dynamical systems. It is shown in [2] that a system is integral input-to-state stable (iISS) if it is (a) 0-GAS and (b) dissipative with respect to supply rate $\sigma(u)$, where $\sigma$ is class $K$ function. The relation between the iISS gain $\gamma$ and the supply rate $\sigma(u)$ is then discussed in [8] and it is shown that for a class of dissipative systems, the supply rate $\sigma(u)$ is essentially the iISS gain function.

In this paper, a modified iISS, so-called $A$-iISS, is used which will be defined below. Compare to the standard iISS definition, the solution $x$ of the system converges to an invariant set $A$ instead of the origin in the notion of $A$-iISS when a bounded-energy input signal is used.

Definition 2.1: Consider a system $\dot{\zeta} = f(\zeta, u)$, $\zeta(0) = \zeta_0$, where $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ is locally Lipschitz. Let $A \subset \mathbb{R}^n$ be a nonempty and closed set. It is said to be integral input-to-state stable with respect to $A$ ($A$-iISS) if there exist functions $\alpha \in K_{\infty}$, $\beta \in K$ and $\gamma \in K$ such that, for every $\zeta_0 \in \mathbb{R}^n$ and $u \in L^T(\mathbb{R}^m)$, the unique maximal solution $\zeta$ is global and

$$\alpha(\|\zeta(t)\|_A) \leq \beta(\|\zeta_0\|_A, t) + \int_0^t \gamma(\|u(s)\|)ds \quad \forall t \in \mathbb{R}_+,$$

where $\gamma$ is referred to as $A$-iISS gain.

B. Counterclockwise and Clockwise dynamics

In the following, we provide the definitions of the CCW and CW dynamics which are based on the work by Angeli in [1] and by Padthe, Oh and Bernstein in [13].

Definition 2.2: [1] A (nonlinear) operator $G : AC(\mathbb{R}_+) \to AC(\mathbb{R}_+)$ is counterclockwise (CCW) if for every $u \in AC(\mathbb{R}_+)$ with the corresponding output map $y := G(u)$, the following inequality holds

$$\lim \inf_{T \to \infty} \int_0^T \dot{y}(t)u(t)dt > -\infty.$$  

(2)

For a nonlinear operator $G$, inequality (2) holds if there exists a function $H_G : \mathbb{R}^2 \to \mathbb{R}_+$ such that for every input signal $u \in AC(\mathbb{R}_+)$, the inequality

$$\frac{dH_G(y(t), u(t))}{dt} \leq \dot{y}(t)u(t),$$

holds for almost every $t$ where the output signal $y := G(u)$.

Definition 2.3: [13] A (nonlinear) operator $G : AC(\mathbb{R}_+) \to AC(\mathbb{R}_+)$ is clockwise (CW) if for every $u \in AC(\mathbb{R}_+)$ with the corresponding output map $y := G(u)$, the following inequality holds

$$\lim \inf_{T \to \infty} \int_0^T y(t)\dot{u}(t)dt > -\infty.$$  

(4)

Consequently, for a nonlinear operator $G$, inequality (4) holds if there exists a function $H_G : \mathbb{R}^2 \to \mathbb{R}_+$ such that for every input signal $u \in AC(\mathbb{R}_+)$, the inequality

$$\frac{dH_G(y(t), u(t))}{dt} \leq y(t)\dot{u}(t),$$

holds for a.e. $t$ where the output signal $y := G(u)$.

C. Duhamel hysteresis operator

The Duhamel operator $\Phi : AC(\mathbb{R}_+) \times \mathbb{R} \to AC(\mathbb{R}_+), u_\phi \mapsto \Phi(u_\phi, y_{u_\phi}) := y_\phi$ is described by the following differential equation

$$\dot{y}_\phi(t) = f_1(y_\phi(t), u_\phi(t)) + f_2(y_\phi(t), u_\phi(t))\dot{y}_\phi(t),$$

$$y_\phi(0) = y_{u_\phi},$$

(6)

where $f_1, f_2 : \mathbb{R}^2 \to \mathbb{R}_+$ are assumed to be $C^1$ functions, $\dot{u}_\phi(t) := \max\{0, \dot{u}_\phi(t)\}$, $\dot{u}_\phi(t) := \min\{0, \dot{u}_\phi(t)\}$ (see also, [10], [12], [23]). Assume that there exist a unique solution to (6).

Fig. 1. The input-output dynamics of the Bouc-Wen model with $k_x = 1.2117 \times 10^{-7}$, $k_w = -5.08 \times 10^{-6}$, $n = 1.27$, $\alpha = 8.95 \times 10^{-3}$, $\beta = 6.6082 \times 10^{-3}$ and $\lambda = 2.321810^{-3}$.

An example of the CCW hysteresis model is the Bouc-Wen model [4], which is commonly used to model the hysteresis phenomena in piezo-actuators and will be used in our numerical and experimental results. The general representation of the Bouc-Wen model is given by

$$y_\phi(t) = k_x \dot{u}_\phi(t) + k_w h(t),$$

$$\dot{h}(t) = \rho \dot{u}_\phi(t) - \rho\sigma\dot{u}_\phi(t)\|h(t)\|^{n-1}h - \rho(1 - \sigma)\dot{u}_\phi(t)|h(t)|^n,$$

where $u_\phi$ denotes the voltage, $y_\phi$ denotes the displacement, $n \geq 1$ and $k_x, k_w, \rho, \sigma$ are the parameters determining the shape of the hysteresis curve.

The Bouc-Wen model can be described by the Duhamel hysteretic operator (6) with

$$f_1(\gamma, \nu) = k_x + k_w \rho - k_w \rho\sigma|\gamma - k_w \nu|^{n-1}k_w \nu| - k_w \rho(1 - \sigma)|\gamma - k_w \nu|^{n-1}k_w \nu,$$

$$f_2(\gamma, \nu) = k_x + k_w \rho + k_w \rho\sigma|\gamma - k_w \nu|^{n-1}k_w \nu| - k_w \rho(1 - \sigma)|\gamma - k_w \nu|^{n-1}k_w \nu.$$  

(7)
In Figure 1, we illustrate the behavior of the Bouc-Wen model with $k_x = 1.2117 \times 10^{-7}$, $k_w = -5.08 \times 10^{-6}$, $n = 1.27$, $\rho = 8.93 \times 10^{-3}$ and $\sigma = 0.74$.

III. MAIN RESULTS

Consider a feedback interconnection of a mass-damper-spring system with a PD controller and a hysteretic actuator as shown in Figure 2, where $P$ represents the linear plant, $C$ is the PD controller and $Φ$ represents the Duhamel hysteresis operator. In Figure 2, the exogeneous input disturbance signal is represented by $d$. The closed-loop system as shown in Figure 2, is the PD controller $C$, $\Phi$ and $P$.

The closed-loop system as shown in Figure 2, is the PD controller $C$, $\Phi$ and $P$.

Using (10), it can be checked that $V_G$ is proper in $x$ and $y_Φ$. Based on the LaSalle’s invariance principle and the compactness of $(x, y_Φ)$, the trajectories converge to the largest invariant set $O$ contained in $M := \{(x, y_Φ) \in \mathbb{R}^3 | Lx - wy_Φ = 0\}$ as $t \to \infty$.

In the invariant set $O$ in $M$, $y_Φ = -\frac{L}{w} x$, i.e., $u = \frac{L}{w} x$.

Substituting $u = \frac{L}{w} x$ in $P$, we have that in the invariant set $O$

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ -\frac{1}{m} & -\frac{1}{m} \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u,$$ (8)

where $x = [x_1 x_2]^T$ with $x_1$ be the position and $x_2$ be the velocity, and $c_1, c_2$ are the measurement gains of the position and velocity, respectively. For the PD controller $C$, $k_p > 0$ and $k_d > 0$ are the controller gains.

First, we consider the case where $d = 0$, i.e. there is no input disturbance in the closed-loop system. Let $G$ denote the cascaded system of $P$ and $C$, then we have

$$\begin{align*}
G : \begin{cases}
\dot{x} &= Ax + Bu, \\
y_c &= Cx + Du,
\end{cases}
\end{align*}$$ (9)

where $A = \begin{bmatrix} 0 & 1 \\ -\frac{1}{m} & -\frac{1}{m} \end{bmatrix}$, $B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, $C = \begin{bmatrix} k_p c_1 - \frac{k_p c_2 + k_d c_1}{m} \\ k_p c_2 + k_d c_1 - \frac{k_p c_2}{m} \end{bmatrix}$ and $D = \frac{k_d c_1}{m}$.

**Theorem 3.1:** Consider the closed-loop system in (8) with $d = 0$. Assume that for the Duhamel hysteresis operator $Φ$ there exists $H_{OC} : \mathbb{R}^2 \to \mathbb{R}_+$ such that for every $u_Φ \in AC(\mathbb{R}_+)$ and for every admissible $y_Φ$, (3) holds, where $y_Φ = Φ(u_Φ, y_Φ)$. Suppose that either (i) the function $H_{OC}(\cdot, \xi)$ is proper (or radially unbounded) for every $ξ$; or (ii) the solution $y_Φ = Φ(u_Φ, y_Φ)$ is bounded for every $u_Φ \in AC(\mathbb{R}_+)$ and every admissible $y_Φ$. If there exists $Q = Q^T > 0$, $L = [l_1 l_2]$, $w$ $k_p > 0$ and $k_d > 0$ such that the following inequalities hold.

$$\begin{align*}
Q &> 0, \\
A^T Q + QA + L^T L &\leq 0, \\
QB + A^T C^T &= L^T w, \\
2CB &= -w^2,
\end{align*}$$ (10) (11) (12) (13)

and $-\frac{k_p}{m} + \frac{k_d}{w} < 0$. Then for every initial conditions $x_0 \in \mathbb{R}^2$ and every admissible $y_Φ \in \mathbb{R}$, the state trajectories of the closed-loop system (8) are bounded and converge to $A := \{(x_1, x_2, y_Φ) \in \mathbb{R}^3 | x_2 = 0\}$, i.e. the closed-loop system is globally asymptotic stable with respect to $A (A-GAS)$.

**Proof:**

By the assumption of the theorem, the Duhamel operator $Φ$ is CCW and there exists a function $H_{OC} : \mathbb{R}^2 \to \mathbb{R}_+$ such that

$$\begin{align*}
H_{OC}(y_Φ, t, u_Φ) &\leq \hat{y}_Φ(t)u_Φ(t),
\end{align*}$$ (14)

Using $V_G = \frac{1}{2} x^T Q x + (y_Φ - Du_Φ)u + \frac{1}{2} u^2$ as the Lyapunov function for the linear systems $G$, and (11)-(13), we have

$$\begin{align*}
\dot{V}_G &= \frac{1}{2} x^T (A^T Q + QA)x + x^T Q Bu \\
&\quad + x^T A^T C^T u + u CBu + y_Φ u, \\
&\leq y_Φ u - \frac{1}{2} (Lx - wu)^2.
\end{align*}$$

Based on Definition 2.3, it can be easily checked that $G$ is CW.

Now take $V_G(x, y_Φ) = V_G + H_{OC}(y_Φ, y_Φ)$ as the Lyapunov function of the closed-loop system (8) and substituting the interconnection conditions $u_Φ = -y_Φ$ and $u = y_Φ$, we obtain

$$\begin{align*}
\dot{V}_G &= \dot{V}_G + \dot{H}_{OC}, \\
&\leq y_Φ u - \frac{1}{2} (Lx - w u)^2 + \dot{y}_Φ u_Φ, \\
&= -\frac{1}{2} (Lx - w u)^2.
\end{align*}$$ (15)

Using (10), it can be checked that $V_G$ is proper in $x$ and $y_Φ$. Based on the LaSalle’s invariance principle and the compactness of $(x, y_Φ)$, the trajectories converge to the largest invariant set $O$ contained in $M := \{(x, y_Φ) \in \mathbb{R}^3 | Lx - wy_Φ = 0\}$ as $t \to \infty$.

In the invariant set $O$ in $M$, $y_Φ = -\frac{L}{w} x$, i.e., $u = \frac{L}{w} x$.

Substituting $u = \frac{L}{w} x$ in $P$, we have that in the invariant set $O$

$$\begin{align*}
\dot{x} &= \begin{bmatrix} 0 & 1 \\ -\frac{1}{m} & -\frac{1}{m} \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u
\end{align*}$$ (16)

From (13), we have that $w = \frac{\sqrt{2k_p c_2 - 2m(k_p c_2 + k_d c_1)}}{m}$. Using (11) and (12), it can be computed that $l_1$ has a unique solution $l_1 = \frac{k_d}{2k_p c_2 - 2m(k_p c_2 + k_d c_1)}$ if $w = \frac{\sqrt{2k_p c_2 - 2m(k_p c_2 + k_d c_1)}}{m}$ or $l_1 = -\frac{k_d}{2k_p c_2 - 2m(k_p c_2 + k_d c_1)}$ if $w = -\frac{\sqrt{2k_p c_2 - 2m(k_p c_2 + k_d c_1)}}{m}$. Substituting these $w$ and $l_1$ into (15), we get that in the invariant set $O$
holds which implies that \( x_2 = 0 \) in the invariant set \( O \). By La-Salle’s principle, this implies that \( x_2(t) \to 0 \) as \( t \to \infty \) if \( \frac{1}{\sigma} + \frac{1}{\omega_m} < 0 \). Hence, the closed-loop system is GAS with respect to \( A := \{(x_1, x_2, y_\Phi) \in \mathbb{R}^3 | x_2 = 0 \} \).

**Remark 3.2:** If all the hypotheses in Theorem 3.1 are satisfied, then the LMI conditions given in (10)-(13) indicates the interval of the control parameters such that the closed-loop system is \( \mathcal{A} \)-GAS. For instance, if \( k_p \leq \frac{\sqrt{2}}{c_2 m} \), \( k_p \leq \frac{b_d C_d}{c_2 m} \) and \( k_p > k_p C_d m - k_p c_1 m - k_d c_2 \), then the closed-loop system (8) satisfies (10)-(13) with \( L = \left[ \frac{\sqrt{2}}{m} \quad 0 \right] \), \( w = \sqrt{\frac{2}{m}} \) and

\[
Q = \begin{bmatrix}
\frac{k_d c_2^2}{m} - k_p c_1 k & \frac{1}{m} \\
\frac{1}{m} & k_d c_2 - k_p c_1 m - \frac{1}{m}
\end{bmatrix}
\]

where \( q = (k_d c_2^2 - k_p c_2 m - k_p c_1 m) \).

In the rest of this section, we proceed our analysis with \( d \neq 0 \), i.e., we study the robustness of the closed-loop system (8) by adding disturbances \( d \) to the input of the plant. The concept of \( \mathcal{A} \)-iISS is applied here to show the robustness of the closed-loop system. Let us first study the dissipativity property of (8).

**Lemma 3.3:** Consider the closed-loop system in (8) with \( d \in C^1(\mathbb{R}_+) \) s.t. \( d \in AC(\mathbb{R}_+) \). Assume that the Duben hysteresis operator satisfies the hypotheses given in Theorem 3.1. If there exists \( Q = Q^T > 0 \), \( L = \left[ \begin{array}{c} I \end{array} \right] \), \( w, k_p > 0 \) and \( k_d > 0 \) such that the following inequalities hold:

\[
A^T Q + QA + L^T L + \epsilon_1 I \leq 0
\]

\[
A^T Q + QA + L^T L + \epsilon_1 \leq 0
\]

\[
Q + C^T D = 0
\]

\[
\epsilon_2 \leq \epsilon_2 - \epsilon_2 - \epsilon_2,
\]

\[
2CB = -w^2 - \epsilon_2,
\]

for some positive constants \( \epsilon_1 \) and \( \epsilon_2 \). Suppose \( -\frac{1}{\sigma} + \frac{1}{\omega_m} < 0 \), then there exists \( \mu > 0 \) such that the closed-loop system is dissipative with respect to the supply function \( \sigma(d) = \mu \parallel d \parallel^2 \).

**Proof:** Let \( V_d = V_Q + H_\varphi \) be the Lyapunov function for the closed-loop system, where \( V_Q \) and \( H_\varphi \) have the same descriptions as in the proof of Theorem 3.1. Then using (17)-(20) we have

\[
\dot{V}_c = \dot{V}_Q + H_\varphi,
\]

\[
\leq y_\varphi \dot{u} - \frac{1}{2}(Lx - wu)^T (Lx - wu) + y_\Phi u_x
\]

\[
- \frac{1}{2} \epsilon_1 x^T x - \frac{1}{2} \epsilon_2 u^2,
\]

\[
= -\frac{1}{2}(Lx - wu)^T (Lx - wu) + y_\Phi \dot{d} - \frac{1}{2} \epsilon_1 x^T x - \frac{1}{2} \epsilon_2 u^2
\]

where the last equality is obtained since \( u = d - y_\Phi \). Using Young’s inequality, \( u = y_\Phi \) and \( \epsilon_3 > 0 \), we have

\[
\dot{V}_c \leq \frac{\eta}{2} d^2 + \frac{1}{2} \epsilon_1 y_\Phi^2 - \frac{1}{2} \epsilon_1 x^T x - \frac{1}{2} \epsilon_2 u^2
\]

\[
\leq \frac{\eta}{2} d^2 + \frac{1}{2} \epsilon_1 x^T C^T C x - \frac{1}{2} \epsilon_1 x^T x - \frac{1}{2} \epsilon_2 u^2
\]

\[
\leq \frac{\eta}{2} d^2 + \frac{1}{2} d^2
\]

where the last inequality is obtained since \( \eta \) is an arbitrary positive constant. This implies that system (8) is dissipative with respect to \( \sigma \) where \( \sigma(d, d) = \mu \parallel d \parallel^2 \) and \( \mu := \eta/2 \).

**Theorem 3.4:** Consider the closed-loop system in (8) with \( d \in C^1(\mathbb{R}_+) \) s.t. \( d \in AC(\mathbb{R}_+) \). Assume that the hypotheses in Lemma 3.3 hold. Then (8) is iISS with respect to \( \mathcal{A} \) (A-iISS), with iISS gain \( \gamma(\parallel d \parallel) = \mu \parallel d \parallel^2 \), where \( \mu > 0 \).

**Proof:** The proof of the Theorem 3.4 is similar to the proof of Theorem 3.1 in [8]. First of all, consider the system (8) with \( d = 0 \) which has the following form

\[
\begin{bmatrix}
\dot{x} \\
\dot{y}_\varphi
\end{bmatrix}
=\begin{bmatrix}
A_{x,y_\varphi} - B_{x,y_\varphi} \\
\frac{f_1(y_\varphi, u_\Phi)}{1 + D_{f_1}(y_\varphi, u_\Phi)} (CA_{x} + CB_{y_\varphi}) + \\
\frac{f_2(y_\varphi, u_\Phi)}{1 + D_{f_2}(y_\varphi, u_\Phi)} (CA_{x} + CB_{y_\varphi})
\end{bmatrix}, \tag{21}
\]

where \( u = -y_\Phi \). It can be checked that the RHS of equation (21) is locally Lipschitz. Let us write (21) by \( \dot{\zeta} = f(\zeta) \) where \( \zeta = [x^T \ y_\Phi] \). Based on the converse Lyapunov theorem [21, Corollary 2], Theorem 3.1 implies that there exists a smooth Lyapunov function \( V : \mathbb{R}^3 \to \mathbb{R}_+ \) such that

- there exist \( K_\infty \) functions \( \alpha_1 \) and \( \alpha_2 \) such that

\[
\alpha_1(\parallel \zeta \parallel_A) \leq V(\zeta) \leq \alpha_2(\parallel \zeta \parallel_A) \quad \forall \zeta \in \mathbb{R}^3.
\]

- there exists a continuous, positive definite function \( \alpha_3 \) such that

\[
\frac{dV(\zeta)}{d\zeta} f(\zeta) \leq -\alpha_3(\parallel \zeta \parallel_A) \quad \forall \zeta \in \mathbb{R}^3.
\]

In the case when the input disturbance \( d \neq 0 \), the system (8) can be written as \( \dot{\zeta} = f(\zeta, d) \) where

\[
f(\zeta, d, d) = \begin{bmatrix}
\frac{A_x + B_y d - y_\Phi}{1 + D_{f_1}(y_\varphi, u_\Phi)} (CA_{x} + CB_{y_\varphi}) + D_{d_+} + D_{d_-} \\
\frac{f_1(y_\varphi, u_\Phi)}{1 + D_{f_1}(y_\varphi, u_\Phi)} (CA_{x} + CB_{y_\varphi}) + D_{d_+} + D_{d_-}
\end{bmatrix},
\]

\[
u = d - y_\Phi.
\]

Since \( f_1(y_\varphi, u_\Phi) \geq 0 \) and \( f_2(y_\varphi, u_\Phi) \geq 0 \) according to the definition of the Duben operator given in Section II, then \( f_1(y_\varphi, u_\Phi) \in [0, 1] \) and \( f_2(y_\varphi, u_\Phi) \in [0, 1] \).

Hence, we have that for every \( \mathbb{B}_1^4 \), \( \parallel f(c, d, d) \parallel \leq c \left( 1 + \mu \parallel d \parallel^2 \right) \)

\[
\forall \zeta, d, d \in \mathbb{B}_1^4 \times \mathbb{R}^2.
\]

The rest of the proof can use the same arguments as in [8] where the compact set \( K \subset \mathbb{R}^n \) in [8] is replaced by \( \mathbb{B}_1^4 \). It can be checked that the same lemmas as given in
[8, Lemma 3.2, Lemma 3.3 and Lemma 3.4] can also be obtained in this case. Applying the Lemmas and the converse Lyapunov function for $A$-GAS system, we can obtain the $A$-ISS Lyapunov function similar to the proof of [8, Theorem 3.1].

**IV. SIMULATION AND EXPERIMENTAL RESULTS**

In the previous sections, stability for a second-order hysteretic system was shown by using a PD controller. In particular, we give the sufficient conditions on the control parameters such that the closed-loop system is $A$-GAS. In case the linear plant is the mass-damper-spring system and the position is measured output, then applying our approach, the velocity of the linear plant converges to the origin. The robustness property has also been discussed with respect to the input disturbance. In this section, we apply these results to develop a PD controller for a piezo-actuated stage.

To evaluate the controller experimentally, the piezo-actuated stage $P611.2S$ from Physik Instrumente (PI) is used combined with the piezo amplifier $E610.S0$ from Physik Instrumente (PI). The displacement of the stage is measured by a strain gauge sensor. The input voltage is controlled by a PC computer where the MATLAB Real-Time Workshop is used. The PD controller is implemented within Simulink.

The piezo-actuated stage can be considered as a hysteretic actuator connected to a mass-damper-spring plant as shown in Figure 3.

![Fig. 3. Mass-damper-spring system connected with a hysteretic actuator.](image)

Then the piezo-actuated system can be given as follows

$$
\mathbf{P} : \dot{x} = \begin{bmatrix} 0 & - \frac{k}{m} & 0 \\ - \frac{k}{m} & 1 & - \frac{b}{m} \\ 0 & - \frac{b}{m} & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} u,
$$

$$
\Phi : \dot{y}_\Phi = f_1(y_\Phi, u_\Phi) \hat{u}_{\Phi +} + f_2(y_\Phi, u_\Phi) \hat{u}_{\Phi -},
$$

where $x = [x_1, x_2]^T$, $x_1$ denotes the displacement and $x_2$ denotes the velocity. The hysteresis system $\Phi$ is represented by the Bouc-Wen model as introduced in (7) in Section II. The identification process is separated into two parts: the identification of the Bouc-Wen model and the identification of the mass-damper-spring system.

First of all, we identify the hysteresis system by applying low frequent input signals, in which case the linear plant can be considered as a DC gain. To identify the Bouc-Wen model, we first determine the initial parameters of the Bouc-Wen model based on the limit cycle approach as proposed in [5]. Then we use the initial parameters and apply the least square optimization toolbox (lscurvefit) in MATLAB to estimate the real parameters of the Bouc-Wen model.

Based on these two steps, the identified parameters for the Bouc-Wen model (7) are: $k_x = 0.2379$, $k_{wx} = -0.0723$, $\rho = 2.57$, $\sigma = 3.53$ and $n = 1.46$.

Based on the parameters of the Bouc-Wen model, the parameters of the mass-damper-spring system can also be identified by using lscurvefit, since boundaries of the parameters can be given. The parameters we obtained are $m = 0.0352$, $b = 80.0543$ and $k = 2.5541 \times 10^4$. The identified parameters are validated by a mixed frequency signal as shown in Figure 4.

![Fig. 4. Identification results for the piezo-actuated stage, where the red line is the output of the model and the blue line is the measured output from the setup.](image)

The control parameters for the PD controller can then be obtained based on the parameters of the linear plant. It can be checked that with $k_p = 2000$, $k_d = 1$, $L = \left[3.1981 \times 10^6\ 0\right]$, and $w = 125.215$, we can obtain the matrix

$$
Q = \begin{bmatrix} 1.85 \times 10^{10} & 7.04 \times 10^6 \\ 7.04 \times 10^6 & 3.4 \times 10^3 \end{bmatrix},
$$

which satisfies the conditions given in (10)-(13). The simulation result is given in Figure 5 (a) with initial condition $x_0 = \left[0 \ 10^{-5}\right]^T$.

To show the robustness of the system, we add a disturbance signal $d(t) = \frac{10}{t^2}$, which satisfies $d, d \in L^2$, to the input of the linear plant. The simulation result is shown in Figure 5 (b).

We will now apply our results to the velocity tracking problem where the position of piezoeactuator must track a ramp reference signal. Consider again the closed-loop system (8) with $d = 0$. Let $x_2 = \bar{v}$ denote the desired constant velocity and the corresponding steady-state displacement is
then given by \( \bar{x}_1 = \bar{v}t \) (up to a constant) with \( t \in \mathbb{R}_+ \). It follows from the state equations of the closed-loop system in (8) that in the steady-state, \( \bar{y}_q = -k\bar{v}t - \bar{b}\bar{v} \) (up to a constant). Denote \( \bar{\zeta} = [x_1 \ x_2 \ y_\Phi]^T, \ e = [e_1 \ e_2 \ e_\Phi]^T \), where \( e_1 = x_1 - \bar{x}_1, e_2 = x_2 - \bar{x}_2 \) and \( e_\Phi = y_\Phi - \bar{y}_\Phi \). Let \( \bar{u}_\Phi = u_\Phi - \bar{u}_\Phi \) where \( \bar{u}_\Phi \) satisfies \(-k_w + k_d \bar{u}_\Phi = \bar{y}_\Phi \). It can be computed that, in the neighborhood of \( (e_\Phi, \bar{u}_\Phi) = (0, 0) \), the operator \( e_\Phi = \Phi(\bar{u}_\Phi) \) can be approximated by a Duham operator. Hence the error dynamics can be described by

\[
f(e) = \left[ -\frac{k}{\bar{m}} e_1 - \frac{e_2}{\bar{m}} - \frac{e_\Phi}{\bar{c}_d}, f_1(e_\Phi, \bar{u}_\Phi) \right]_+ + f_2(e_\Phi, \bar{u}_\Phi)(\bar{u}_\Phi)_-.
\] (24)

Applying Theorem 3.1, we can obtain that the error system (24) is locally asymptotically stable with respect to \( A := \{ (e_1, e_2, e_\Phi) \in \mathbb{R}^3 | e_2 = 0 \} \). We remark that in the original coordinate, the applied control input is given by \( u_\Phi = \bar{u}_\Phi + \bar{u}_\Phi \) where \( \bar{u}_\Phi := k_p(y - \bar{v}t - \bar{v}) + k_d(y - \bar{v}) \) is the output of the PD controller. The simulation result of tracking a constant velocity \( \bar{v} = 5\mu\text{m/s} \) is shown in Figure 6(a). The experimental result is shown in Figure 6(b) where we use constant reference velocity of \( \bar{v} = 5\mu\text{m/s} \).

![Simulation results of the piezo-actuated stage system with initial condition \( x_0 = [0 \ 10^{-5}]^T \).](image)

![Simulation results of tracking a constant velocity](image)

**V. CONCLUSION**

In this work we investigate the stability for a second-order mass-damper-spring system driven by a hysteretic actuator which can be represented by a CCW Duham hysteresis operator. The hysteretic actuator is controlled by a PD controller, where we give the sufficient conditions on the control parameters such that the closed-loop is \( A \)-GAS. The sufficient conditions on the control parameters are obtained based on the CW property of the cascaded system of the linear plant and the PD controller. Furthermore, the robustness of the closed-loop system with respect to the input disturbance is also studied by applying the concept of \( A \)-iISS. To evaluate our approach, we conduct an experiment on the piezo-actuated stage where a constant velocity is tracked.

**REFERENCES**


