Proportional Integral Retarded Control of Second Order Linear Systems

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Abstract—This study presents a derivative-free control algorithm called Proportional Integral Retarded (PIR) controller, and its application to second order linear systems. Using a frequency domain analysis, the stability map of the closed loop system is constructed; moreover, the exploration of the regions where stability switches occur leads to a tuning strategy assigning a triple real dominant root that corresponds to the maximum achievable exponential decay. The paper also presents an analysis of the main PIR controller properties. Finally, the proposed control law is compared to a proportional integral derivative controller using an experimental platform.

I. INTRODUCTION

Proportional Integral Derivative (PID) controllers are widely employed in industry. Regardless of the simplicity of its structure, the functionality of its terms is sufficient to provide an efficient solution in most real-world control problems, eliminating steady-state errors through the integral term and improving transient response by means of the derivative action [1], [2]. Although the tuning of a PID control law appears to be intuitive, it is a difficult task in practice. One of the main problems is tuning the derivative term, which may amplify high-frequency measurement noise; in fact, most control loops avoid the use of this term [3].

Control loops regularly operate even in the presence of delays; in this case, traditional tuning rules for PID controllers provide poor closed loop system performance [3]. However, the presence of delays not always induces instability or bad performance; in fact, its deliberate introduction can be used to obtain simple and easy to implement control laws [4]. As mentioned in [5], the idea of introducing delays for control purposes is not new, and it has been used for stabilizing chaotic and oscillatory systems, chains of integrators, robotic arms, among other systems, see [4]-[8].

This paper addresses the Proportional Integral Retarded (PIR) control of second order linear systems and its properties. The proposed algorithm introduces a delayed proportional action in the closed loop dynamics. The main features of this approach are the following. First, the closed loop desired exponential decay is achieved as a result of a frequency domain study. Second, measurement noise amplification problems are removed since the PIR controller does not rely on derivative terms, thus allowing a transparent implementation that obviates the need of additional considerations such as measurements filtering. Third, robustness against constant perturbations is ensured due to the integral action term. Fourth, the method provides simple tuning formulae.

The ideas in this paper were inspired by previous contributions [9] and [10], where the Proportional Retarded (PR) and the Integral Retarded (IR) controllers are introduced, thus the desirable properties of both, namely, the noise attenuation and the disturbance rejection are retained. It is worthy of mention that, due to the complexity of the characteristic equation, the analysis is not a straightforward extension of the above mentioned contributions, but it is the outcome of a careful analysis based on both, discriminants of polynomials and the Descartes’ rule of signs, see [11] and [12].

II. DERIVATIVE-FREE CONTROL

A. System dynamics

Consider the second order linear system described by

\[ \dot{y}(t) + ay(t) + by(t) = cu(t), \] (1)

where \( a, b, c \in \mathbb{R}^+ \). Defining the error \( e(t) := r - y(t) \), equation (1) can be written as

\[ \dot{e}(t) + ae(t) + be(t) = c [d - u(t)], \] (2)

here, \( r \) is a constant reference, and \( d = br/c \) is considered as a constant disturbance.

B. PIR controller

The PIR control law given by

\[ u(t) = k_p e(t) + k_i \int_0^t e(\tau)d\tau - k_r e(t - h) \] (3)

avoids the use of derivative terms by including a proportional retarded action, thus measurement noise amplification is removed while the disturbance rejection property and steady state-errors elimination of a standard PID controller are preserved. Here \( k_p, k_i \) and \( k_r \) are respectively the proportional, integral and retarded gains.

C. Closed loop dynamics

The characteristic equation describing the closed loop dynamics is given by the quasipolynomial

\[ p(s) = s^3 + as^2 + (b + ck_p)s + ck_i - ck_r se^{-sh} = 0 \] (4)

The change of variable \( s \to (s - \sigma) \) allows the \( \sigma \)-stability analysis of \( p(s) \), [9]. Thus, the quasipolynomial (4) becomes

\[ \tilde{p}(s) = s^3 - (3\sigma - a)s^2 + (3\sigma^2 - 2a\sigma + b + ck_p)s - \sigma^3 + a\sigma^2 - (b + ck_p)\sigma + ck_i - ck_r e^{-h(s-\sigma)}(s - \sigma) = 0, \]

where \( \sigma \) is the desired closed loop exponential decay.

Remark 1: Observe that the attenuation of (1) is given by \( a/2 \), [13]. Thus, the analysis is restricted to the case \( \sigma > a/2 \).
III. STABILITY REGIONS

A. Stability crossing boundaries

This subsection presents the construction of the stability map in the \((h, k_r)\) parametric space using the \(D\)-subdivision method. Based on the continuity properties of the closed loop system spectrum, the boundaries where stability switches may occur are detected by analyzing the behavior of the characteristic roots along the imaginary axis, \([14]\).

The crossing boundaries at the origin associated to \(\tilde{p}(s)\) are given by \(\tilde{p}(0) = 0\). Crossings occur when

\[
k_r(h) = \frac{\sigma^3 - a\sigma^2 + (b + ck_p)\sigma - ck_i}{ce^{\sigma h} \sigma}.
\]

(6)

The crossing boundaries at the imaginary axis associated to \(\tilde{p}(s)\) are described by \(\tilde{p}(j\omega) = 0\). This is true when \(\Re [\tilde{p}(j\omega)] = 3|\tilde{p}(j\omega)| = 0\). It follows from the above that

\[
\text{cot} \left[ h \omega - \arctan \left( \frac{\sigma}{\omega} \right) \pm n\pi \right] = f(\omega)
\]

for \(n = 0, 1, 2, \ldots\), where

\[
f(\omega) := \frac{-\omega^3 + (3\sigma^2 - 2a\sigma + b + ck_p)\omega}{(3\sigma - a)\omega^2 - \sigma^2 + a\sigma^2 - (b + ck_p)\sigma + ck_i}.
\]

The parametric equations

\[
\begin{align*}
h(\omega) &= \frac{1}{\omega} \text{arccot} \left[ f(\omega, \sigma) + \text{arctan} \left( \frac{\sigma}{\omega} \right) + \frac{n\pi}{\omega} \right] \\
nk_r(\omega) &= \frac{\sigma^3 - a\sigma^2 + (b + ck_p)\sigma - (3\sigma - a)\omega^2 - ck_i}{ce^{\sigma h(\omega)} \left| \sigma \cos(h(\omega)\omega) - \omega \sin(h(\omega)\omega) \right|}
\end{align*}
\]

are obtained by algebraic manipulation of \(\Re[\tilde{p}(j\omega)] = 0\) and \(7\). Therefore, the exact domain of stability is described by the hypersurfaces generated by equations \(6\), \(8\) and \(9\).

Fig. 1 shows the level contours of the stability crossing boundaries for several values of \(\sigma\) with \(a = 5\), \(b = 0\), \(c = 45\), \(k_p = 34.38\), \(k_i = 30.27\). The stability region related to a particular \(\sigma\) is detected by applying the necessary conditions described in \([15]\). Here, it can be observed that as \(\sigma\) increases, the corresponding \(\sigma\)-stability region is trimmed by the hypersurface generated from \(6\), this hypersurface separates the parametric space into two regions where \(\sigma\)-stability cannot be achieved below this boundary. This behavior reduces the stability map to a single zone named the Main Stability Region (MSR) bounded from below by \(6\) and from above by \(8\) and \(9\). By analyzing this region, it can be seen that the MSR is reduced as \(\sigma\) increases due to the fact that the lower and upper boundaries approach each other. On the one hand, the lower boundary obtained from \(6\) is formed when a root of \(\tilde{p}(s)\) is at \(-\sigma\). On the other hand, the upper boundary obtained from \(8\) and \(9\) is formed if there is a complex root at \(-\sigma \pm j\omega\). Since the upper and lower boundaries approach each other, it implies that at the collapse point \(\omega \to 0\) and the upper boundary is generated by a double root at \(-\sigma_m\) while the lower boundary is generated by a single root at \(-\sigma_m\); it follows that the collapse point is characterized by a triple root at \(-\sigma_m\). This point corresponds to the maximum achievable closed loop exponential decay.

B. Locus of the collapse point

In the following, the locus of the collapse point is characterized by using mathematical tools from Elimination Theory, namely, discriminants of polynomials, and the Descartes’ rule of signs. Our goal is to find an extremum for \(\sigma\) such that the stability crossing boundaries collapse for a closed loop exponential decay greater than \(a/2\). This characterization is used to prove the existence of a triple root of \(p(s)\) at \(-\sigma_m\) and to present a tuning procedure for the PIR controller.

**Theorem 1:** Let \(\bar{\sigma}\) be the value of \(\sigma\) where the stability regions collapse. Assume that the parameters \(a\), \(b\) and \(c\) of \(1\) are known, then \(\bar{\sigma}\) is contained in the discriminant

\[
\Delta(\sigma) = 2a^6 - 2a\sigma^5 + [a^2 - 2(b + ck_p)]\sigma^4
\]

\[
+ 8ck_k^3 - 4a\sigma^2 - 4ck_k^2 + 2ck_k(b + ck_p)\sigma - c^2 k_i^2,
\]

if \(k_p, k_i \in \mathbb{R}^+\) satisfy

\[
k_p + \frac{b}{c} - \frac{k_i}{a} > 0,
\]

with the gains \(k_p\) and \(k_i\) given by

\[
k_p = \frac{(a - \sigma)^2 + 2(\sigma^2 - b)}{2c} + \lambda,
\]

\[
k_i = \frac{\sigma}{c} \left[ 2\sigma(2\sigma - a) + (b + ck_p) - \sqrt{g(\sigma, k_p)} \right]
\]

where \(\lambda > 0\) is a free parameter and

\[
g(\sigma, k_p) = [(\sigma(3\sigma - 2a) + b + ck_p)^2 + \sigma^2 (3\sigma - a)^2].
\]

**Outline of the proof:** Evaluating \(s = j\omega\) in \(5\) gives \(\tilde{p}(j\omega) = 0\). Without any loss of generality the analysis is restricted to \(\omega \geq 0\). Since \(a\), \(b\) and \(c\) are known and the gains \(k_p\) and \(k_i\) are chosen a priori; thus, \(\tilde{p}(\sigma, \omega, h, k_r)\) implicitly depends on four variables. If \(\sigma\) exhibits an extremum in the \((\omega, h, k_r)\) domain, then the conditions \(d\sigma/d\omega = 0\), \(d\sigma/dk_r = 0\) and \(d\sigma/dh = 0\) must hold. Notice that the condition \(d\sigma/dh = 0\) can be obtained from \(\sigma\) using the implicit function theorem \([16]\).

Here, \(d\sigma/d\omega = 0\) holds if \(d\tilde{p}/d\omega = 0\); i.e., \(d\tilde{p}/d\omega = 0\) implies that \(d\sigma/d\omega = 0\). Hence, the conditions

\[
\tilde{p} = 0 \quad \text{and} \quad \frac{\partial \tilde{p}}{\partial \omega} = 0,
\]

(15)
must be satisfied in order to find an extremum for \( \sigma \). Using (15), the variables \( h \) and \( k_r \) can be eliminated, and the bivariate polynomial \( \tilde{q}(\sigma, \omega) = 0 \) is obtained by preserving all the factors that may lead to a real value for \( \sigma \). Using the polynomial \( \tilde{q} = \tilde{q}(\sigma, \omega) \) the value for \( \tilde{\sigma} \) cannot be obtained yet because the solution of \( \tilde{q} \) for \( \sigma \) still depends on the unknown \( \omega \); therefore, if \( \sigma \) has an extremum in the domain of \( \omega \), the condition \( \partial \tilde{q}/\partial \omega = 0 \) must hold; then, using once more the implicit function theorem, the term \( \partial \sigma/\partial \omega = 0 \) can be studied through \( \partial \tilde{q}/\partial \omega = 0 \). Hence, the conditions

\[
\tilde{q} = 0 \quad \text{and} \quad \frac{\partial \tilde{q}}{\partial \omega} = 0, \tag{16}
\]

must be satisfied for \( \sigma \) to be an extremum. At this point \( \omega \) may be eliminated using the discriminant of \( \tilde{q} \), namely \( D_\omega(\tilde{q}) = 0 \). Then, neglecting all the factors that do not contain a value that qualifies as an extremum for \( \sigma \), the discriminant \( D_\omega(\tilde{q}) = 0 \) is reduced to the analysis of the condition \( \Delta(\sigma) = 0 \) which produces the extremum \( \tilde{\sigma} \).

**Lemma 2:** The stability region for collapses of the extremum

\[
\tilde{\sigma} := \min_{n=1,2,\ldots} \left\{ r_n \in \mathbb{R}^+ : \Delta(r_n) = 0, \quad r_n > a/2 \right\}. \tag{17}
\]

**C. Maximum exponential decay**

The above analysis provides an insight about the relation between the roots of \( \tilde{p}(s) \) and \( \tilde{\sigma} \). On the one hand, finding \( \tilde{\sigma} \) requires that \( \tilde{p}(s) \) and its derivatives be concurrently zero as stated in (15) and (16). On the other hand, by inspecting Fig. 1 it is found that the collapse point of the MSR is characterized by a triple root at \( -\tilde{\sigma}_m \). In order to relate these statements, consider the following theorem.

**Theorem 3:** Let the parameters \( a, \ b, \ c \in \mathbb{R}^+ \) and the gains \( k_p, k_i \in \mathbb{R}^+ \) be fixed such that (11) is satisfied, then the maximum exponential decay for the closed loop system is

\[
\sigma_m \equiv \tilde{\sigma}, \tag{18}
\]

which is achieved in the \((h, k_r)\) parametric space with

\[
h = \frac{2\sigma_m^3 - a\sigma_m^2 + c k_i}{\sigma_m^3 - a\sigma_m^2 + (b + c k_p)\sigma_m - c k_i \sigma_m}, \tag{19}
\]

\[
k_r = \frac{2\sigma_m^3 - c k_i}{c h^2 \sigma_m^3 e^{h \sigma_m}}, \tag{20}
\]

Moreover, at \( \sigma_m \), the stability crossing boundaries collapse into a point characterized by a triple real root at \( -\sigma_m \) on the real axis of the complex plane.

**Proof:** At this locus, if \( \tilde{p}(s) \) has three roots at \( -\sigma_m \), it implies that \( \tilde{p}(s) \) has three roots at \( s = 0 \); hence, (5) must satisfy \( \tilde{p}(s)|_{s=0} = \tilde{p}'(s)|_{s=0} = \tilde{p}''(s)|_{s=0} = 0 \), where \( \tilde{p}'(s) = d(\tilde{p}(s))/ds \) and \( \tilde{p}''(s) = d(\tilde{p}'(s))/ds \).

It follows from \( \tilde{p}(s)|_{s=0} = 0 \) and \( \tilde{p}'(s)|_{s=0} = 0 \) that

\[
\sigma h = \frac{2\sigma^3 - a\sigma^2 + c k_i}{\sigma^3 - a\sigma^2 + (b + c k_p)\sigma - c k_i}; \tag{21}
\]

Equations \( \tilde{p}(s)|_{s=0} = 0 \) and \( \tilde{p}''(s)|_{s=0} = 0 \) imply that

\[
(\sigma h)^2 = \frac{2(\sigma^3 - c k_i)}{\sigma^3 - a\sigma^2 + (b + c k_p)\sigma - c k_i}, \tag{22}
\]

Eliminating \( \sigma h \) from (21) and (22) and some algebraic manipulations lead to the polynomial

\[
\Lambda(\sigma) = 2\sigma^6 - 2a\sigma^5 + [a^2 - 2(b + c k_p)]\sigma^4 \tag{23}
\]

\[+8ck_i\sigma^3 - 4a ck_i\sigma^2 + 2ck_i(b + c k_p)\sigma - c^2 k_i^2,\]

which coincides with the discriminant \( \Delta(\sigma) \) in equation (10), where \( \tilde{\sigma} \) is the value of \( \sigma \) where the stability crossing boundaries collapse; thus, defining \( \sigma_m := \tilde{\sigma} \) leads to equation (18). Equations (19) and (20) follow from solving \( \tilde{p}''(s)|_{s=0} = 0 \) and (21) for \( k_r \) and \( h \), respectively.

**IV. ANALYSIS OF THE PIR CONTROLLER**

**A. Tuning**

The assignment of a reduced number of dominant roots defines the delay system dynamics [17]. Particularly, the characteristic of the collapse point allows a triple dominant root assignment at \( -\sigma_m \) as stated in the proposition below.

**Proposition 4:** Given a desired exponential decay \( \sigma_d \) \( a/2 \), a triple dominant root at \( -\sigma_d \) in the complex plane is assigned by the following tuning of the PIR controller

\[
k_p = \frac{1}{2c^2} \left( (\sigma_d - a)^2 + 2(\sigma_d^2 - b) + \xi (\varphi - \xi) \right), \tag{24}
\]

\[
k_i = \frac{\sigma_d^2}{2c^2} \left[ 2\sigma_d^2 - 2\xi (\sigma_d + \xi) + \xi (\varphi - \xi) \right], \tag{25}
\]

\[
h = \frac{1}{3c^2 \sigma_d} [\varphi - 3\xi], \tag{26}
\]

\[
k_r = \frac{\xi}{ch^2 \sigma_d^2 e^{h \sigma_d}} \left[ 2(\sigma_d + \xi) - (\varphi - \xi) \right], \tag{27}
\]

where \( \xi = 3\sigma_d - a \) and \( \varphi = \sqrt{9\xi^2 + 12\xi \sigma_d} \).

**Proof:** Consider the gains \( k_p \) and \( k_i \) in (14) yields

\[
g(\sigma, \lambda) = 4c^2 [\alpha^2 + \beta^2] \tag{29}
\]

where \( \alpha = -(3\sigma - a)^2/(2c) \) and \( \beta = -\sigma (3\sigma - a)/c \).

Then, consider the following polynomial

\[
g_d(\sigma, \lambda) = 4c^2 (3\alpha + \beta)^2. \tag{30}
\]

Matching (29) with (30) and solving for \( \sigma \) give

\[
\lambda = \xi (\varphi - \xi)/(2c), \tag{31}
\]

where \( \xi = 3\sigma - a \) and \( \varphi = \sqrt{9\xi^2 + 12\xi \sigma} \).

Since \( \sigma > a/2 \), it implies that \( \varphi > \xi > 0 \); then, it follows that \( \lambda > 0 \) satisfies the positivity restriction of the free parameter. Thus, substituting \( \lambda \) into (12), and then (12) into (13), equations (24) and (25) follow. Substituting these gains into inequality (11) leads to \( r(\sigma) = 3\sigma^2 \xi + 4\sigma (\sigma - a)^2 - \varphi \xi (\sigma - a) > 0 \) in the interval \( \zeta = [a/2 < \sigma < 17a] \), hence (11) holds since \(-r(\sigma)\) is convex in \( \zeta \), and \( r(a/2) > 0 \) and \( r(17a) > 0 \). Finally, substituting (24) and (25) into (19) and (20) and defining \( \sigma_d := \sigma_m \), (26) and (27) follow.
B. Disturbance Rejection

The PIR control law is inspired by the PID controller, where the integral term corrects steady-state offsets while providing robustness against constant disturbances. Next, it is shown that the PIR controller preserves this property.

Consider the perturbed system in the error coordinates (2) and the PIR control law written in the frequency domain.

\[
\hat{u}(s) = \hat{e}(s) \left[ \frac{k_p s + k_i - k_r s e^{-h s}}{s} \right].
\]

The corresponding input-output transfer function is

\[
\hat{y}(s) = \frac{c \left( k_p s + k_i - k_r s e^{-h s} \right)}{s^3 + a s^2 + (b + c k_p) s + c (k_i - k_r s e^{-h s})}.
\]

and the perturbation-output transfer function is

\[
\frac{\hat{y}(s)}{d(s)} = \frac{c s}{s^3 + a s^2 + (b + c k_p) s + c (k_i - k_r s e^{-h s})}.
\]

According to the principle of superposition, the output \( \hat{y}(s) \) is given by

\[
\hat{y}(s) = \frac{1}{s^3 + a s^2 + (b + c k_p) s + c (k_i - k_r s e^{-h s})} \left[ r (k_p s + k_i - k_r s e^{-h s}) + ds \right],
\]

where the reference signal and disturbance are assumed to be constants, i.e. \( \hat{r}(s) = \tau / s \) and \( \hat{d}(s) = d / s \). Using the final value theorem, it is proved that the steady-state error is zero, i.e. \( e_s = r - y_s = 0 \) where \( y_s \) is the steady-state output.

C. Noise attenuation

In practice, a pure derivative control term cannot be implemented due to measurement noise amplification [18]. However, the PIR controller does not rely on the time derivative of the error thus removing undesired noise amplification. Consider now the transfer function of the PIR controller

\[
\frac{\hat{u}(s)}{-\hat{y}(s)} = \frac{k_p s + k_i - k_r s e^{-h s}}{s}.
\]

The bode plot of (36) is depicted in Fig. 2 for the parameters of Table I. The noise attenuation property of the proposed control law, which is observed as the magnitude of the PIR controller, is not large at high frequencies. Clearly, there is no need for additional filtering or controller structure modifications, thus allowing a simple implementation.

D. \( \sigma \)-Stability

According to the argument principle, \( \phi(s) \) is stable if and only if the inequality

\[
\pi \leq \phi_{[e, j R]} \leq 2\pi
\]

holds, where \( \phi_{[e, j R]} \) denotes the net change of the argument of \( \phi(s) \) along the segment \( [e, j R] \) for \( R \gg 1 \). [19]. Here, the Nyquist contour is modified using a semicircle of infinitesimal radius \( e \) skipping the singularity at the origin generated by the triple real root assignment. Thus, substituting the parameters obtained in Proposition 4 permits drawing the Mikhailov plot of the quasipolynomial (5) which is depicted in Fig. 3 using weighted polar coordinates, so that, the magnitude of the Mikhailov curve is scaled while its angle is preserved with the objective of properly observing the change of the argument. In this figure, one can see that \( \Phi_{[e,jR]} \) lies within the interval \((\pi, 2\pi)\); thus, inequality (37) is satisfied; hence, there are no roots in the open right half complex plane for \( \phi(s) \) and it follows that (4) is \( \sigma \)-stable.

V. EXPERIMENTAL RESULTS

Conventional PID controllers are widely employed in industry due to their simplicity and functionality. For this reason the proposed PIR controller is compared to a PID controller, both tested on an experimental platform. The platform is composed by a DC brushed servomotor, a power amplifier and a position sensor, its dynamics is described by

\[
J \ddot{\theta}(t) + f \dot{\theta}(t) = ku(t)
\]

where \( \theta(t), \dot{\theta}(t) \) and \( \ddot{\theta}(t) \) are the angular position, velocity and acceleration respectively, \( u(t) \) the control input voltage, \( J \) the motor and load inertia, \( f \) the viscous friction and \( k \) is the gain associated to the amplifier and the motor.

Defining the error \( e(t) := r - \theta(t) \), (38) is rewritten as

\[
J \ddot{e}(t) + f \dot{e}(t) = -ku(t).
\]

Notice that dividing (39) by \( J \) and defining \( a := f / J \) and \( c := k / J \) the above reduces to the form (2) with \( b = d = 0 \). The PID controller is given by

\[
u(t) = k_1 e(t) + k_2 \int_0^t e(\tau) d\tau + k_3 x(t)
\]

\[
x(t) = \nu(t) + f_0 e(t)
\]
with \( \dot{v}(t) = -f_x x(t) \) and \( f_e > 0 \). Parameters \( k_1, k_2 \) and \( k_3 \) are respectively, the proportional, integral and derivative gains. Notice that from (41) yields \( \dot{x}(t) = -f_x [x(t) - \dot{e}(t)] \). Here, the state \( x(t) \) is an estimate of the error derivative \( \dot{e}(t) \) obtained by filtering the error position. The characteristic equation of (2) in closed loop with (40) and (41) is

\[
\sigma(s) = s^4 + (a + f_c)s^3 + (b + af_e + ck_1 + ck_3 f_c)s^2 + (bf_e + ck_1 f_e + ck_2)s + ck_2 f_c.
\]

The parameters of the PIR controller are chosen as

\[
f_c = 13\sigma_d - a \\
k_2 = 10\sigma_d^2 / (cf_e) \\
k_1 = (31\sigma_d^3 - bf_e - ck_2) / (cf_e) \\
k_3 = (33\sigma_d^2 - b - af_e - ck_1) / (cf_e)
\]

which assign three poles of the closed loop system at \(-\sigma_d\) and one pole at \(-10\sigma_d\), thus providing essentially the same dynamic response obtained with the PIR controller.

The system parameters \( a = 5 \) and \( c = 45 \) are estimated using the identification algorithm in [20]. The gains and the delay of the PIR control law given in Table I are tuned based on the triple real root assignment presented in Section 4 for \( \sigma_d = 12.30 \). The parameters of the PIR controller given in Table II are obtained from (42) using the same value of \( \sigma_d \) which is chosen to obtain low values of the controller gains, thus avoiding control saturation.

**Table I. PIR controller parameters**

<table>
<thead>
<tr>
<th>( k_p )</th>
<th>( k_i )</th>
<th>( k_d )</th>
<th>( h )</th>
</tr>
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<tbody>
<tr>
<td>34.38</td>
<td>30.27</td>
<td>26.93</td>
<td>18.73 ms</td>
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**Table II. PIR controller parameters**

<table>
<thead>
<tr>
<th>( k_1 )</th>
<th>( k_2 )</th>
<th>( k_3 )</th>
<th>( f_c )</th>
</tr>
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<tbody>
<tr>
<td>8.06</td>
<td>32.83</td>
<td>0.55</td>
<td>154.90</td>
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</table>

### A. Experiments

The experimental platform consists of a DC brushed motor from Clifton Precision, model JDTH-2250-BQ-IC, driven by an analog power amplifier from Copley Controls, model 413, configured in current mode. A BEI optical encoder, model L15 with 2500 pulses per revolution measures the servomotor position. Data acquisition is performed by a MultiQ-3 card from Quanser Consulting which multiplies the encoder resolution by 4. The controllers are coded in Simulink under the Linux software from Quanser Consulting. The software runs on a personal computer with an Intel(R) Core(TM)2 Duo CPU @3.06GHz processor. The sampling period in Simulink is fixed to 1 ms using the ODES5 solver.

Fig. 4 and 5 show the dynamic response of the servo system under both control schemes, where the position reference is a sequence of steps used to verify regulation at different angular positions. The reference is filtered by a low pass filter with cut-off frequency of 10 Hz to obtain a smoother reference signal. In Fig. 4(a) and 5(a) the output \( \theta(t) \) is stabilized despite the simulated constant torque disturbances; here, when the disturbances are introduced, \( \dot{\theta}(t) \) returns to its reference and the position error tends to zero in 900 ms in both control schemes as can be seen in Fig. 4(c) and 5(c).

The Integral of Square of Error (ISE)

\[
ISE = \int_0^T e(t)^2 dt,
\]

and the Integral of the Scaled Absolute Error (ISAE) serves as a performance metric of the tracking quality. Since a motor shaft turn corresponds to 10000 pulses, the ISAE can be expressed in encoder pulses as follows

\[
ISAE = \int_0^T |10000 e(t)| dt.
\]

The ISE and the ISAE are obtained using a quadrature method with \( T = 15 s \) which corresponds to 15000 samples of the error \( e(t) \). Table III summarizes the ISE and the ISAE for both control schemes. Since the shaft rotates 1.1 turns, the tracking errors in encoder pulses of the PIR and the PID controllers correspond to 3.89% and 4.06% of the total shaft turns, respectively. One observes on Fig. 4(a) and (c), Fig. 5(a) and (c), and Table III an adequate performance obtained using both control algorithms and negligible differences of the two dynamic responses.

**Table III. Performance measures**

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<thead>
<tr>
<th></th>
<th>ISE (motor shaft turns)</th>
<th>ISAE (encoder pulses)</th>
</tr>
</thead>
<tbody>
<tr>
<td>PIR</td>
<td>( 6.35 \times 10^{-4} )</td>
<td>428</td>
</tr>
<tr>
<td>PID</td>
<td>( 6.70 \times 10^{-4} )</td>
<td>447</td>
</tr>
</tbody>
</table>

A further advantage of the proposed controller is shown in Fig. 4(b) where a smooth control signal is produced by the PIR controller. In contrast, Fig. 5(b) shows a noisy control signal produced by the PID controller in spite of the fact that the error derivative \( \dot{e}(t) \) is estimated through a high pass filter applied to the error position.

**VI. CONCLUSIONS**

The use of elimination theory permits obtaining explicit tuning rules for the parameters of a PIR controller employed in the stabilization of a second order linear system.

The delay-based PIR controller, applied to a second order linear system, is able to produce satisfactory tracking of step responses as demonstrated by means of experimental tests conducted through a servo system. Results validate the proposed tuning method based on a three real dominant poles assignment corresponding to a desired maximum exponential decay rate. The experiments also show that the closed loop response does not exhibit overshoots, and the PIR controller produces a smooth control signal thus reducing servo motor stress.

The simplicity of the PIR controller provides an alternative to using the standard PID controller applied to second order linear systems with two important differences. First, only position measurements are needed, thus avoiding expensive velocity acquisition hardware. Second, regarding to real-time implementation, the PIR controller only uses a few kilobytes of memory allocation, since there is no need of velocity estimations or reconstructions which requires the on-line solution of a dynamic system.
Fig. 4. Dynamic response of the servomechanism in closed loop with the PIR controller (3). a) Position $\theta(t)$. b) Control signal $u(t)$. c) Error $e(t)$.

Fig. 5. Dynamic response of the servomechanism in closed loop with the PID controller (40). a) Position $\theta(t)$. b) Control signal $u(t)$. c) Error $e(t)$.

REFERENCES


