On Optimal Thermal Control of an Idealized Room including Hard Limits on Zone-Temperature and a Max-control Cost Term

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Abstract—We study an open-loop optimal control problem for energy efficient cooling of a simple room. The goal is to provide insights for the development of an implementable control scheme based on a Model Predictive Control strategy. The cost function includes a non-standard Mayer-term \( F(\max, u(t)) \), that is, a term depending on the peak value of the control. Comfort requirements are imposed as state inequality constraints.

I. ENERGY AND ECONOMIC COSTS

Buildings account for about 40% of the US energy budget, so that even modest savings are significant. Beyond energy savings, in the presence of time-dependent rates and peak-demand power charges, pre-cooling (load shifting) becomes an important cost saving strategy [1], [2], [3]. Note that pre-cooling can also reduce energy consumption through enhanced efficiency achieved by operating mechanical refrigeration equipment at more favorable reservoir conditions.

Practical feedback in building energy control is commonly implemented via Model Predictive Control (MPC) [4]. MPC is predicated on the solution of a sequence of open-loop optimal control problems from the current state over a forward planning horizon. To facilitate the reliable solution of these open-loop problems the dynamics are treated in discrete-time form, controls are commonly subject to move limits, and solutions realized by iterative NonLinear Programming (NLP) procedures. Among the existing software tools for MPC, the Berkeley Library for Optimization Modeling [5] provides interfaces to SIMULINK-based modeling and to the open-source Ipopt software [6]. At present however, BLOM does not treat the periodic boundary conditions in our problem.

II. ROOM MODEL

In the present discussion we consider a simple thermal model for a room and concentrate on indirect-theory for the open-loop optimal control [7], [8]. Our primary objective is to characterize a continuous-time optimal open-loop control for comparison to NLP results.

To focus ideas we consider a (summer) cooling scenario with

- **exterior wall** -
- **thermal storage** - high thermal capacitance features of the building interior
- **room air** - temperature, occupied zones

A notional view of the room is shown in Figure 1. Thermal energy storage is modeled in the solid circles which depict:

- wall interior temperature \( (T_i) \)
- storage temperature \( (T_s) \)
- room-air temperature \( (T_a) \)

Four energy exchange mechanisms are modeled:

1) \( q_{\text{cond}} \) - conduction through the exterior wall from \( T_i \) to \( T_e \)
2) \( q_{\text{rad}} \) - radiation from \( T_i \) to \( T_s \)
3) \( q_{\text{conv, i}} \) - convection from \( T_i \) to \( T_a \)
4) \( q_{\text{conv, s}} \) - convection from \( T_s \) to \( T_a \)

With these energy-exchange mechanisms, the lumped dynamic model is

\[
\begin{align*}
C_i \frac{dT_i}{dt} &= - (q_{\text{cond}} + q_{\text{rad}} + q_{\text{conv,i}}) \\
C_s \frac{dT_s}{dt} &= (q_{\text{rad}} - q_{\text{conv,s}}) \\
C_a \frac{dT_a}{dt} &= (q_{\text{conv,i}} + q_{\text{conv,s}} - C(u))
\end{align*}
\]

where the \( C_k \) values (\( k \in \{i, s, a\} \)) are thermal capacitances, and \( C(u) \) is the cooling delivered to the room-air using electrical input power \( u \).

The conduction term is given by a simple one-dimensional model as

\[
q_{\text{cond}} = \frac{k A}{\Delta} (T_i - T_e) .
\]

Radiant exchange between the two interior surfaces is modeled as

\[
q_{\text{rad}} = \sigma A \left( \frac{\epsilon_i \epsilon_s}{\epsilon_i + \epsilon_s - \epsilon_i \epsilon_s} \right) (T_i^4 - T_s^4) .
\]

If the surface temperatures are reasonably close, the radiant

\[ 
\]

Fig. 1. Notional room configuration

- wall interior temperature \( (T_i) \)
- storage temperature \( (T_s) \)
- room-air temperature \( (T_a) \)
exchange can be approximated by the linear model
\[ q_{\text{rad}} = \sigma A \hat{T}^3 \left( \frac{\epsilon_i \epsilon_s}{\epsilon_i + \epsilon_s - \epsilon_i \epsilon_s} \right) (T_i - T_s) \text{,} \]
where \( \hat{T} \) is a suitable mean temperature. Finally, the convection terms are given by
\[ q_{\text{conv}, k} = (hA)_{k} (T_k - T_a) \text{,} \quad k \in \{i, s\} \text{.} \]

In summary, with the linearized model for radiant exchange the dynamics of our system can be written as
\[
\dot{z}(t) = f(t, z(t), u(t)) = A z(t) + \begin{bmatrix}
\frac{kA}{C_i \Delta T_e(t)} \\
0
\end{bmatrix},
\]
\[ u(t) \in \Omega \equiv \{0 \cup \left[ U_r, U \right] \}, \tag{1} \]
where
\[ z(t) = \begin{bmatrix} T_i(t) \\ T_s(t) \\ T_a(t) \end{bmatrix}. \tag{2} \]

The specific structure of the \( A \)-matrix may be deduced from the previous discussion.

A. Quadratic \( \text{cop}(u) \)

The cooling function \( C(u) \) is represented as the product of the applied power \( (u) \), and a function that characterizes the system’s coefficient of performance \( (\text{cop}(u)) \). We consider a case wherein
\[ \text{cop}(u) = \alpha + \beta u + \gamma u^2 \]
with data specifications:
- \( \text{cop}(0) = \alpha > 0 \)
- \( \arg \max \text{cop}(u) = \ell, \quad U < \ell < U \)
- \( \max \text{cop}(u) = v > \alpha \)

From these specifications we find that:
\[ \beta = \frac{2(v - \alpha)}{\ell} > 0 \quad \text{and} \quad \gamma = \frac{-(v - \alpha)}{\ell^2} < 0 \]

It’s useful to introduce a scaled control variable
\[ \hat{u} \equiv \frac{u}{\ell} \text{,} \]
so that
\[ \text{cop}(\hat{u}) = \alpha + (v - \alpha) \hat{u} [2 - \hat{u}] \text{,} \]
and
\[ C(\hat{u}) = \ell \hat{u} \text{cop}(\hat{u}) \text{.} \tag{3} \]

In the next section we formulate an optimal control problem for these dynamics. In the following we drop the \( \hat{u} \) notation so that \( u \) is the scaled applied power.

III. Optimal Control Problem

Whereas the primary focus of our optimization study is minimal energy use, to avoid trivialities it is necessary to place some restrictions on the temperature histories. To this end we formulate bounds on the zone-air temperature \( (T_a) \)
\[ T_{\min} \leq T_a(t) \leq T_{\max} \quad \text{for} \quad t \in [t_b, t_e] \text{,} \tag{4} \]
where the \( [t_b, t_e] \) is the occupied interval, the period when human comfort must be considered.

In formulating the power cost we admit dependence on peak-demand and time-varying power rates to define our cost-functional as
\[ J[u] = F(\overline{U}) + \ell \int_0^{t_f} r(t) u(t) \, dt \text{,} \tag{5} \]
where \( r > 0 \) is a given function specifying the time-varying cost of power. The function \( F \) admits dependence on the maximum value; we assume that \( F \) is strictly convex. To study this feature we include \( \overline{U} \) as a 4th state and incorporate a state-dependent control constraint
\[ \overline{U} = 0 \text{,} \quad u(t) - \overline{U}/\ell \leq 0 \text{.} \]

We anticipate that the exterior wall temperature \( T_e \), and the power-rate \( r \), will be given over a 24 hour period. We take \( t_f = 24 \text{h} \) and seek periodic boundary conditions \( z(0) = z(t_f) \) and a control function \( u^* \) to minimize the cost (5) subject to the dynamics (1) and the state-inequality constraints (4).

IV. Necessary Conditions

We begin by considering the problem without the Mayer term \( F(\overline{U}) \), and apply the Minimum Principle [7], [8] to our problem. The variational-Hamiltonian is given by
\[
H(t, \lambda_z, z, u) = \ell r(t) u + \left\langle \lambda_z, \left[ A z + \begin{bmatrix}
\frac{kA}{C_i \Delta T_e(t)} \\
0
\end{bmatrix} \right] \right \rangle \text{.} \tag{6}
\]

The treatment by Gamkrelidze [8, see Chap 6] decomposes the problem into a sequence of arcs; constrained arcs where the constraint (4) is active, and unconstrained arcs. At the junctions of such arcs certain jump conditions are required. Necessary conditions for these problems are also found in [9], [10].

A. Unconstrained Arc

Evolution of the co-state \( (\lambda_z) \) is governed by the homogeneous linear system:
\[ \lambda'_z(t) = -A^T \lambda_z(t) \text{.} \tag{7} \]

We consider that part of the variational Hamiltonian that depends (explicitly) on the control \( (u) \). Noting that the
electrical power cost function \( r \) is positive, we factor it out
\[
\hat{H}(u) = \left( \frac{1}{r(t)} \right) H_{\text{cont}}(u) = \lambda_0 u - \left( \frac{\lambda_a}{C_a r(t)} \right) C(u),
\]
where \( C(u) \) is the cooling rate from (3).

The Minimum Principle requires that we characterize the value(s) of the control that minimize \( \hat{H} \) over the set \( \Omega(U) \).
Since \( \hat{H} \) is continuous and the domain \( (\Omega) \) is compact, minimizers exist.

For current purposes it’s useful to exploit the general structure of the variational Hamiltonian and to interpret the \( \arg \min H \) -operation geometrically. The control \( (u) \) dependent part of the variational Hamiltonian (8) is
\[
\hat{H}(u) = \left\langle \left( \frac{\lambda_0}{\lambda_a} \right), \left( \frac{u}{-C(u)} \right) \right\rangle.
\]
The vector on the right in this inner-product is the (augmented) state-rate and captures the time-rates of the control-dependent parts of the cost function and the state (normalized as in (8)). The locus of admissible points is shown in Figure 2 for a case with \( U/\ell = 1.4 \). This locus of admissible state-rates is called the Velocity Set [12, see Chap 4] or Hodo- graph [13], [14]. In the present case it consists of the point at the origin, and an operating line of points corresponding to control values \( u \in [U/\ell, \bar{U}/\ell] \). Note that the other terms in the variational-Hamiltonian are independent of the control and do not affect the \( \min - H \) operation.

Next, consider a fixed vector \( \vec{\lambda}_a \equiv \left( \frac{1}{\lambda_a} \right) \), where \( \vec{\lambda}_a \) is given in (8). The orthogonal complement \( \{ \vec{\lambda}_a \}^\perp \) is the subspace of vectors orthogonal to (the span of) \( \lambda_a \), and for points in this subspace we have \( H_c = 0 \) (see Figure 3).
Any translation of this subspace along the \( \vec{\lambda}_a \) direction has \( H_c = \chi \), a positive real, whereas any translation along the opposite direction is a set of points with negative real values for the function \( H_c \).

The ideas underlying Figures 2 and 3 are combined in Figure 4. In the case shown the minimizing control is \( u^* = 1.2 \) which is between the control for maximum \( \text{cop} \) and the upper bound \( U/\ell = 1.4 \). Figure 5 shows the case \( \tilde{\lambda}_a = \tilde{\lambda}_a^I \) wherein the separating plane is tangent to the Velocity Set at \( u = 1.4 = \bar{U}/\ell \). For larger values of \( \tilde{\lambda}_a \) the separating plane still goes through the operating point at \( u = 1.4 \), although the plane is no longer tangent to the operating line. As \( \lambda_a \) decreases from \( \lambda_a^I \), the vector \( \lambda_a \) rotates clockwise; the separating plane remains tangent to the Velocity Set and the extremal control decreases from \( U/\ell \). This continues until we reach the case \( \lambda_a = \lambda_a^I \) shown in Figure 6. Note that in this case the separating plane also passes through the operating point at the origin. Further decreasing \( \lambda_a \) (i.e. further clockwise rotation of \( \tilde{\lambda}_a \)) requires operating at \( u^* = 0. \)

From this analysis we find there are two critical values of \( \lambda_a \)
\[
\tilde{\lambda}_a^I = \left. \frac{1}{\alpha + 4(v - \alpha) 1/(U/\ell) - 3(v - \alpha)} \right|_{U/\ell} \frac{1}{U/\ell}^2
\]
and
\[
\tilde{\lambda}_a^{II} = \left. \frac{1}{\alpha + 4(v - \alpha) 1/(U/\ell) - 3(v - \alpha)} \right|_{U/\ell} \frac{1}{U/\ell}^2
\]
We assume that \( U/\ell > 1 \) so that \( \tilde{\lambda}_a^{II} > \tilde{\lambda}_a^I \). If the upper bound is less than the value for maximum coefficient of performance \( (U < \ell) \) then extremal controls take values at either \( U/\ell \) or at 0.
In summary the results of the min $H$-operation are

$$u^* (\lambda_a) = \begin{cases} 0 & \text{if } \hat{\lambda}_a < \lambda^I_a \\ \frac{U}{\ell} u_{\text{int}} & \text{if } \lambda^I_a < \hat{\lambda}_a < \lambda^H_a \\ \lambda_a & \text{if } \hat{\lambda}_a \leq \lambda_a \end{cases} \quad (9)$$

where

$$u_{\text{int}} = \frac{2 + \sqrt{4 - 3p}}{3} \quad \text{and} \quad p = \frac{\lambda_a - \alpha}{(v - \alpha)}.$$

The choice of sign (+) guarantees that the Hessian ($\partial^2 H$) is positive-definite, so that $u_{\text{int}}$ is (at least) a local minimizer. Our geometric analysis confirms that we have a global minimizer over $\Omega(U)$.

The interior control (scaled) takes values in the interval $[1, U/\ell]$. Thus, it is never optimal to take control values in the interval $[U/\ell, 1]$. This statement is equivalent to the classical Weierstrass Necessary Condition [15, see Appendix C.4].

In constructing Figures 2 - 6 we have used $\alpha = 4$, $v = 8$. Nonetheless, the geometric features, such as the origin lying on the tangent line in Figure 6, and the characterization of extremal controls in Equation (9) are generic. Existence theorems require (e.g. [12]) require convexity of the Velocity Set. In the present case it’s sufficient to replace the part of the cooling curve below $u = 1$ with the line segment between the operating point at the origin, and the point at $u = 1$.

B. Constrained Arc

Since the constraint (4) depends on the zone-air temperature ($T_a$) and time only we display the $(t, T_a)$ projection of the admissible region in Figure 7. Motion along the vertical boundaries is not feasible; we focus on the upper-horizontal boundary: $T_a(t) \leq T_{\text{max}}$, so that on constrained arcs the zone-air temperature is constant. From the dynamics (1) this implies that the state/control pair $(z, u)$ satisfy:

$$\dot{T}_a(t) = 0 \rightarrow C_a A(3, :) z(t) - C(z) = 0.$$  

Since the control $(u)$ appears explicitly, the state constraint is of first-order. Under our hypotheses the cooling-function $C$ is invertible on its range. Along the constrained arc the adjoint equations are modified [7, see §13.10] [8, Chap. 6].

The augmented Hamiltonian is

$$H^a(t, \lambda_z, z, u, \nu) = H(t, \lambda_z, z, u) + \nu [C_a A(3, :) z - C(z)] .$$

The Valentine multiplier $(\nu, [16])$ follows from $\frac{\partial H^a}{\partial u} = 0$, viz

$$\nu = \frac{\partial H}{\partial u} / C'(u) = \left[ r(t) - \frac{\lambda_a C'(u)}{C_a} \right] .$$

The adjoint equation (7) is modified to

$$\lambda_z(t) = - \frac{\partial H^a}{\partial z} = - A^T \lambda_z$$

$$+ \left[ r(t) - \frac{\lambda_a C'(u)}{C_a} \right] C_a A(3, :)^T . \quad (10)$$

C. Transversality

The periodic boundary conditions $(z(0) = z(t_f))$ imply that the adjoints are also periodic, viz:

$$\lambda_z(0) = \lambda_z(t_f) . \quad (11)$$

At the junction of unconstrained and constrained arcs

$$H(t_en^-) = H(t_en^+)$$

$$\lambda_z(t_en^-) = \lambda_z(t_en^+) + \pi \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} .$$
Since the entry time is free, the Hamiltonian is continuous there. The jump in $\lambda_a$ can occur at either the entry or the exit point. The Hamiltonian at the exit point need not be continuous.

D. $F(\max_t u(t))$

We return now to the consideration of the optimal choice for the control bound $U$. In terms of the $z$ variable the variational-Hamiltonian is

$$H(t, \lambda_0, \lambda_z, \lambda_T, z, U, u) = \langle \lambda_z, Az \rangle - \lambda_a \frac{C(u)}{C_a} + \lambda_0 (r(t) \ell u) + \nu (u - U/\ell)$$

where $z$ is given by (2) and the components of the adjoint variable $\lambda_z$ are defined similarly. The last term in (12) accounts for the state-dependent control constraint; $\nu$ is the Valentine multiplier [16].

Evolution of the co-state ($\lambda_z$) is governed by the linear system:

$$\dot{\lambda}_z(t) = -A^T \lambda_z(t).$$

For the augmented state ($U$) the corresponding adjoint equation is

$$\dot{\lambda}_T = \nu/\ell,$$

where $\nu$ follows from $\frac{\partial H}{\partial u} = 0$; viz

$$\nu = \frac{\lambda_a(t) C'(U)}{C_a} - \lambda_0 r(t).$$

Since the 4-th state ($U$) is unspecified at $t = 0$, transversality implies that

$$\lambda_T(0) = 0,$$

whereas at $t = t_f$ we have

$$\lambda_T(t_f) = \lambda_0 \overset{=}{=} F'(U).$$

Combining these conditions with (13, 14) leads to

$$\lambda_0 F'(U) = \int_0^{t_f} \dot{\lambda}_T(t) \, dt = \int_{t | u(t) \neq 0} \left[ \frac{\lambda_a(t) C'(U)}{C_a} - r(t) \right] \, dt. \quad (15)$$

Equation (15) is an optimality condition for the unknown parameter $U$.

V. NUMERICAL RESULTS

A. Indirect Method: Optimality System

The dynamic model (1), the adjoint system (7, 10), the optimality result (9), along with the periodic boundary condition and associated transversality condition (15) form a multi-point boundary value problem (MPBVP) in the composite pair $z \triangleq (z, \lambda_z)$. For numerical solution we transform to a two-point BVP $s \in [0, 1]$

$$w(s) = \begin{bmatrix} w_1(s) \\ w_2(s) \\ w_3(s) \\ w_4(s) \end{bmatrix} = \begin{bmatrix} z(st_{en}) \\ z((1-s)t_{en} + st_e) \\ z((1-s)t_e + st_f) \end{bmatrix}$$

We have an ODE system of 18 dependent variables and one parameter: $w'(s) = \frac{dw}{ds} = F(s, w(s), u(s), t_{en})$.

It is well-known that numerical solution of this MPBVP can be difficult to obtain. One reason for this behavior is the Hamiltonian structure of the combined state/adjoint system. Specifically, if $\sigma \in C$ is an eigenvalue of the (linearized) system, then so is $-\sigma$. Indeed, for typical system parameters we find the eigenvalues of the state/adjoint system are: $\sigma = \pm 0.015/h, \pm 0.153/h, \pm 19.579/h$. For the fastest growing of these over the time interval [0,24h] the growth is $\exp(19.579/h \times 24h) > 10^{204}$! Our numerical studies make use of the MATLAB procedure $bvp4c$ which implements the three-stage Lobatto IIIa formula; it is a collocation method and provides a $C^1$-continuous solution that is uniformly fourth-order accurate. Mesh selection and error control are based on the residual of the continuous solution. Unknown parameters are treated explicitly. The underlying Newton problem has dimension 19 + 18 * $N$, where $N$ is the number of internal collocation points. Typically, $N \sim O(100)$. Good initial estimates are crucial for convergence of the Newton problem.

Fortunately, for this problem it’s relatively easy to construct an initial estimate for a nearby problem. Specifically, for a fixed forcing temperature ($T_e(t)$) it’s clear that with the $T_{\max}$ parameter sufficiently large the optimal control is $u \equiv 0$. With this solution in hand we can systematically lower $T_{\max}$ to a desired value. Shown in Figure 8 are the temperature histories of the case $T_{\max} = 27^\circC$. Note that the control (not shown) is non-zero only along the constrained arc ($\approx [16\,00, 18\,30]$). Figures 9, 10 display the temperature and control profiles (respectively), with $T_{\max} = 25^\circC$. In this case, the control comes on $\approx [04\,00, 07\,00]$ to pre-cool the air, and again to follow the state constraint. The power-cost function $r$ (not shown) begins at unity, increases almost linearly to 3 on 7:00 to 13:00, and decreases back to unity from 18:00 to 20:00. In these calculations we have used: $\ell = 40w, \alpha = 2, \nu = 4$.

Figure 11 displays the control history when the power-cost function peaks at 4. Note that the pre-cooling segment...
VI. CONCLUSIONS

In our model with quadratic $\text{cop}(u)$ admissible controls can take values in $\Omega = \{0 \cup [U, \ell]\}$, where $0 < U < \ell < \overline{U}$ and $\ell = \arg\max_{\Omega} \text{cop}(u)$. Analysis shows that optimal controls never take values in $[U, \ell)$; that is it is never optimal to use positive power less than that required for maximum $\text{cop}$ on unconstrained arcs.

The model admits two mechanisms to decrease the power-cost. The first exploits the variation in efficiency ($\text{cop}$) by judicious choice of control power. The second, more significant mechanism, is achieved by shifting the cooling to off-peak hours. A more complete model would include increased efficiency by cooling when the outside air temperature is lower, and allow for night-ventilation of cool outside air.

Finally, we note equation (15), our new optimality condition for the unknown parameter $\overline{U}$ in the case with a max-power Mayer cost term.

VII. ACKNOWLEDGEMENTS

Support by DOE under Award Number DE-EE0004261 and by AFOSR under Grant FA9550-10-1-0201 is gratefully acknowledged.

REFERENCES

[6] https://projects.coin-or.org/ipopt

Fig. 9. Temperature histories $T_{\text{max}} = 25^\circ\text{C}$

Fig. 10. Control history $T_{\text{max}} = 25^\circ\text{C}$, $\text{max}_r = 3$

Fig. 11. Control history $T_{\text{max}} = 25^\circ\text{C}$, $\text{max}_r = 4$ is longer and the control during the peak-cost period has been diminished from the earlier case.