Some Considerations about Discrete-Time AFC Controllers

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Abstract—Resonant Control or Adaptive Feedforward Cancellation is a linear control technique dealing with periodic signals. It is based on the Internal Model Principle and let asymptotically track/reject periodic references/disturbances. Many papers studying resonators in the continuous-time domain have been published, nevertheless, the increasing use of microcontrollers motivates its discrete-time design; sampled-data control systems are a rich subject of study and discretizing resonators from continuous-time is not enough to take into account all the sampling effects. In this paper, a review of discrete-time AFC controllers is presented and some results of the classical continuous-time design techniques for resonators are translated into discrete-time.

I. INTRODUCTION

For the asymptotically output tracking and disturbances rejection, the Internal Model Principle (IMP) establishes that the open-loop transfer function of a system must contain a model of the references to be tracked and the disturbances to be rejected [1]. Referring to periodic signals, the corollary of the IMP states that this is equivalent to have an open-loop transfer function which presents an infinity gain at the main frequency of the tracking/perturbation signal and its harmonics. This is the reason why classical controllers as PI or PID are not good enough to face periodic disturbances since they give infinite gain only at zero frequency. On the contrary, linear techniques as Repetitive Control (RC), Proportional Resonant Control (PRC) and Adaptive Feedforward Cancellation (AFC) fulfill the IMP requirements and have been proved to work excellently in many domains as mechanical [2][3] or electrical [4][5].

The interest of the AFC with respect to the other techniques is its flexibility. A repetitive controller cannot directly follow frequency changes in signals, which may cause big control action losses. Whereas, a resonator is constantly tuned to the appropriate frequency by means of the carriers, supplied by, for example, a phase-locked loop (PLL). Another AFC technique advantage with respect to RC is that high frequencies are not naturally amplified, since one can choose the number of resonators in the control loop. Moreover, each resonator let place a zero allowing to increase the robustness of the system, which is not directly possible with PRC.

On the other hand, there are more and more applications involving discrete-time controllers, due to the rapid technological advances on Digital Signal Processors (DSP). Digital controllers have many advantages that analogical ones do not have (memory, mathematical calculations, adaptability, etc.). Nevertheless, the use of a DSP to implement the controllers transforms the plant into a sampled-data control system, which has also some disadvantages (computational delays, time discretization, etc.). Therefore, in order to take all these effects into account, it seems logical to carry out the whole controller design directly in discrete time, and not to translate it from continuous time.

It is obvious that discrete-time resonators must have the same properties than the continuous-time ones, i.e., infinite gain at the exact frequency. However, an approximate discretization process will not fulfill this condition, which is a problem for the control system [6]. Note that resonators’ poles position is crucial: they must be marginally stable to give infinite gain and they must be at the right frequency. Therefore, the mapping must be perfect or the system will lose its properties.

The main problem handling resonators is its design; each resonator adds two poles and one zero to the control-loop, so it becomes rapidly a high order system. In literature, many guidelines to carry out a continuous-time AFC controllers design can be found [7][8], but this is not the case in discrete-time. Some works have been published concerning discrete-time resonators [9][5], but the subject needs a more wide discussion.

The layout of the article is as follows. In Section II, the discrete-time resonator and its properties are presented. In section III, the control structure is presented and the design of resonators for low and large gains is addressed. Then, an application example is shown in Section IV using both presented design techniques. Finally, a conclusion sums up all the contributions.

II. DISCRETE-TIME RESONATOR

Fig. 1 shows the block diagram of a continuous-time resonator. In spite of the origin of AFC control as an adaptive technique, it has been shown [10] that, assuming that $\omega_k$ is constant, a resonator is equivalent to the LTI system

$$R_k(s) = \frac{g_k s \cos (\phi_k) + \omega_k \sin (\phi_k)}{s^2 + \omega_k^2},$$

where $g_k$, $\phi_k$ and $\omega_k$ are, respectively, the gain, the angle and the frequency of the resonator $k$.

Some authors work with discrete-time resonators by discretizing the transfer function (1). As it was aforementioned,
any approximation during the discretization process will lead to undesired dynamical behaviors. Consequently, only an exact transformation is valid for the discrete-time resonator, that is, the impulse invariance\(^1\),

\[
R_k(z) = Z\{R_k(s)\},
\]

where \(R_k(z)\) is the corresponding discrete-time version of the continuous-time resonator \(R_k(s)\).

It can be shown that this result is only a particular case of the discrete-time resonator diagram depicted in Fig. 2. First, using the \(z\)-transform properties, the system can be expressed as a LTI system \[5\]

\[
R_k(z) = \frac{1}{2} g_k \left[ H(ze^{-j\omega_k T}e^{-j\phi_k}) + H(ze^{j\omega_k T}e^{j\phi_k}) \right],
\]

where \(H(z) = \frac{h_k(z)}{h_0(z)}\) denotes a discrete-time integrator. And second, if a backward Euler integrator is chosen, i.e., \(H(z) = \frac{z}{z-1}\), then, the \(z\)-domain transfer function of the resonator becomes

\[
R_k(z) = g_k \frac{\cos(\phi_k)z^2 - \cos(\omega_k T + \phi_k)z}{z^2 - 2\cos(\omega_k T)z + 1},
\]

which is exactly the same transfer function that one obtains by performing the transform in (2) with the transfer function (1). This result shows that it is worth working directly in discrete-time instead of translating a transfer function from continuous time; observe that the \(H(z)\) block can be any kind of integrator, giving different properties to the resonator. The \(H(z)\) block may even be a transfer function with finite gain at 0Hz.

Analyzing (4), it can be deduced that the transfer function has two poles and two zeros: both poles are marginally stable placed at \(z = e^{\pm j\omega_k T}\), one zero is placed at \(z = 0\) and the second one is placed at \(z = \frac{\cos(\phi_k)}{\cos(\omega_k T + \phi_k)}\). This last zero is a function of \(\phi_k\), and as it will be explained later, it may be (and usually is) out of the unit circle, that is, it may be non-minimum phase (NMP). However, even if it is contrary to the

\(1Z\{G(s)\} = \frac{3}{3} L^{-1}\{G(s)(t)\}_{t=nT}\}, \text{ where } T \text{ is the sampling period, } L \text{ is the Laplace transform and } \frac{3}{3} \text{ is the } z\text{-transform.}

conventional wisdom about placement of zeros, the situation is the same as in continuous time, where this NMP-zero plays an important role for the controller robustness \[10\].

### III. Control Structure

Let’s suppose the discrete-time control diagram depicted in Fig. 3, where the plant to be controlled is \(P(z)\) and the AFC controller is \(K(z)\). The usual structure of \(K(z)\) is the addition of \(N\) resonators in parallel, \(\sum_{k=1}^{N} R_k(z)\), giving infinite gain to the open-loop transfer function \(L(z) = K(z)P(z)\) at the fundamental frequency and the harmonics of the reference signals to be tracked and the disturbance signals to be rejected\(^2\).

The plant \(P(z)\) may be a closed-control loop, as in \[8\] to reduce the control effort of resonators.

Each resonator has two parameters to be tuned: the gain \(g_k\) and the angle \(\phi_k\). Two possible criteria for tuning the resonator’s parameters are examined: when gains are low and when gains are large.

#### A. AFC Control Design for Low Gains

When all \(g_k\) tend to 0, the closed-loop poles tend to the open-loop transfer function \(L(z)\) ones, i.e., all the resonators’ poles tend to a position on the unit circle. Therefore, the departure angle is crucial for stability.

In continuous time, it has been shown with several different techniques that each resonator angle \(\phi_k\) must coincide with

\(^2\)Each resonator increases by 2 the order of the controller and needs only 5 products and 3 additions (two from the discrete-time integrators). The necessary operations needed to compute the carriers of each frequency are carried out recursively from the signal produced by a PLL.
the phase of the plant at each resonator frequency $\angle P(j\omega_k)$ in order to maximize the phase margin. Hence, when $g_k$ increases, all the poles move into the open left half-plane with a perpendicular trajectory with respect to the imaginary axis [10]. Furthermore, one can see that phases of the open-loop transfer function, $\angle L(j\omega_k)$, at the resonator’s frequencies $\omega_k$, are zero, what is a sign of robustness [8].

The question is now if one can obtain some kind of equivalence of these design rules for a discrete-time system using discrete-time resonators. In order to answer this question, the point is to force all resonators’ poles to follow the most robust possible trajectory when $g_k$ increases, that is, the perpendicular line to the unit circle, and analyze what happens to the resonators’ angles.

The closed-loop transfer function is $M(z) = \frac{K(z)P(z)}{1+K(z)P(z)}$, so the closed-loop poles are the roots of

$$1 + K(z)P(z) = 0.$$  

Moreover, one can state that $z$ is a function of each $g_k$, i.e., $z(g_k)$. Computing the implicit derivative of $1 + K(z(g_k))P(z(g_k)) = 0$ with respect to each $g_k$ when $g_k = 0$, it is possible to obtain the poles departure angles, since the place of the poles when $g_k = 0$ are known (those of the open-loop transfer function $L(z)$).

Next, setting $z(g_k) = \rho(g_k)e^{j\theta(g_k)}$, and replacing $z(0)$ by $e^{j\omega_k T}$ and $z'(0)$ by $e^{j\omega_k T}\rho'(0) + j e^{j\omega_k T}\theta'(0)$, one obtains the expression

$$H_n(1)P(e^{j\omega_k T}) + 2H'_d(1)\left[\rho'(0) + j\theta'(0)\right]e^{j\phi_k} = 0$$  

Setting $P(e^{j\omega_k T}) = |P_k|e^{j\psi_k}$ (the module and the angle of the plant $P(z)$ at the frequency $\omega_k T$) and separating this equation into its real and imaginary parts, the resulting two-equation system can be solved in order to find the unknowns $\rho'(0)$ and $\theta'(0)$

$$\rho'(0) = -\frac{|P_k|H_n(1)}{2H'_d(1)} \cos(\psi_k - \phi_k)$$

$$\theta'(0) = \frac{|P_k|H_n(1)}{2H'_d(1)} \sin(\psi_k - \phi_k).$$

Therefore, forcing the departure angles to be perpendicular to the unit circle, $\theta'(0) = 0$, one obtains the relation between the resonators’ angles $\phi_k$ and the plant’s angles $\psi_k$; if it is chosen $\psi_k = \phi_k$, it can be observed that, indeed, trajectories for low gains will be perpendicular to the unit circle. Now, those trajectories will move towards the origin if $\rho'(0) < 0$, that is, if $\frac{H_n(1)}{P_k H'_d(1)} > 0$ ($|P_k|$ is always positive), what is always true for an integrator.

This result shows that in discrete time, one arrives at the same conclusion as in continuous time: for low gains, the resonators’ angles $\phi_k$ must coincide with the plant’s angles $\psi_k$ at the frequencies $\omega_k T$ to obtain the maximum phase margin. In other words, the estimated angle of the plant can be incorrect by as much $\pm \frac{\pi}{2}$ rad, exactly as in [10], and the poles will still move into the unit circle.

On the other hand, another property follows from the previous demonstration. The addition of a resonator, causes a loss of $\pi$ rad at the frequency which it is tuned. Choosing the angles for the resonators equal to those to the plant $\psi_k = \phi_k$ at the working frequencies $\omega_k T$, it can be shown from (3) that the open-loop transfer function $L(z)$ presents a null phase at each resonator’s frequency $\omega_k T$. This can be explained since the phase of a resonator at $\omega_k$ is $-\phi_k$ rad, exactly as shown in continuous time in [8].

Fig. 4 depicts the regions where the resonator zero is or not minimum phase in function of the frequency $\omega_k T$ and the angle $\phi_k$. Since $\phi_k = \psi_k$, the usual working region has been marked, showing that probably, the zero will be out of the unit circle. Normally, plants’ phases starts at $0$ rad and diminish up to $-\frac{\pi}{2}$. On the other hand, the frequency $\omega_k T$ depends on the control system specifications, but in general remains in the first quadrant $[0, \frac{\pi}{2}]$.

**B. AFC Control Design for Large Gains**

The design method presented for low gains assures the maximum phase margin, but this is not the solution which minimizes the transient’s duration. Nevertheless, this solution corresponds to an specific $\phi_k$ and a gain $g_k \gg 0$.

Note that the gains $g_k$ cannot be increased indefinitely since the closed-loop poles tend to the open-loop transfer function $L(z)$ zeros, and they are usually NMP by two reasons:

- first, because the angles $\phi_k$ may place the resonators’ zeros outside the unit circle as previously explained,

$$\text{At the } \omega_k T \text{ frequency, the resonator’s phase presents a discontinuity of } \pi \text{ rad, but the average is } -\phi_k, \text{ see [5].}$$
• and second, because most likely, the plant \( P(z) \), has been obtained from a continuous-time plant \( P(s) \) using a zoh and, then, in general it has relative degree 1. Moreover, this fact implies, in some cases, the appearance of sampling zeros and, usually, the second one is NMP [11].

The slowest time constant of a system is related to the farthest pole w.r.t. the origin. Therefore, the aim of this technique is to minimize the transient of the system response by choosing the optimum angles \( \phi_k \) and a gains \( g_k \) for resonators which minimizes the biggest pole modulus. Unfortunately, the approach presented in the previous section is not valid anymore since the poles are not moving near the unit circle \((g_k \approx 0, g_k > 0)\). Then, a non-linear optimization approach can be used, but the problem will be nonconvex.

For a fixed angle \( \phi_k \), when the gain of a resonator \( g_k \) increases, the closed-loop poles start to follow the system’s root locus branches inside the unit circle. For a large enough gain \( g_k \), the system will become unstable either because a resonator’s closed-loop pair of poles go out the unit circle, or because a closed-loop pant's pole does. In between, there is a gain \( g_k \) that minimizes the biggest pole modulus. This modulus can be also minimized w.r.t. the angle \( \phi_k \).

### IV. APPLICATION EXAMPLE

This example has been borrowed and adapted from [10] Fig. 5 shows the plant to be controlled by means of a resonator. Being \( P(s) = \frac{1}{(s+10)(s+1)} \) the plant,

\[
R_1(z) = g_1 \frac{\cos(\omega_1 T)}{z^2 + 2\cos(\omega_1 T)z + 1}
\]

is the resonator and \( r(n) = \sin(\omega_1 n T) \) the signal reference. The working frequency is \( \omega_1 = \frac{1}{2} \text{ rad} \).

\[
\omega_1 T = \frac{\pi}{4}, \text{ i.e., } \psi_1 = \angle P(j \omega_1 T) = -0.9708 \text{ rad}. \text{ The gain } g_1 \text{ is a positive value starting at 0.}
\]

Indeed, for low gains, the departure angle of the pole is perpendicular to the unit circle and it moves towards the origin. Note also that the zero of the resonator is NMP: \( z = \frac{\cos(\omega_1 T + \phi_1)}{\cos(\phi_1)} = 1.7542 \). Nevertheless, this is the angle \( \phi_1 \) that maximizes the phase margin.

On the other hand, it can be found a set of phase \( \phi_1 \) and gain \( g_1 \) values that minimizes the transient of the system. Fig. 7 describe how have they been found.

![Control diagram of the sampled-data control system. The plant \( P(s) \) is continuous-time and the control system works in discrete-time.](image)

![System root locus for low \( g_k \) in the neighborhood of one resonator’s pole: \( z = e^{j\frac{\pi}{4}} \).](image)

For a determined value of \( \phi_1 \), when increasing the gain
$g_1$, resonator’s poles first move towards the origin and the dominant pole of the plant moves the opposite. Therefore, the optimum point occurs when the three poles are at the same distance from the origin. This distance from the origin is a function of $\phi_1$, and it can also be minimized. The found values minimizing the transient’s duration are $g_1 = 5.815$ and $\phi_1 = -1.505$ rad.

Obviously, the faster response is provided by the one that has been chosen to minimize the transient, but the most robust is the one that maximizes the phase margin. An indicator of robustness is the inverse of infinity norm of the sensitivity transfer function $d = \|S(z)\|_\infty^{-1}$, where $S(z) = (1 + L(z))^{-1}$. The value of $d$ represents the minimum distance from the open-loop transfer function polar plot, $L(e^{j\omega T})$, to the critical point $-1$. For instance, a low gain of $g_1 = 2$, in the case where the phase margin is maximum, gives $d = 0.856$, but the settling time is around 60 s. On the other hand, in the case where the transient has been minimized, $d = 0.319$, but the settling time reduces to 15 s approximately. These results can be observed in simulations depicted in Figs. 8 and 9.

![Figure 8](image)

**Fig. 8.** Comparison of both transients: reference (blue), maximum phase margin (red) and optimized transient (green).

The dashed lines in Fig. 9 represent the $\pi$ rad leap with infinite gain produced at the resonator’s frequency. For the case where the phase margin is maximum, the asymptote produced by the discontinuity is perpendicular to the real axis, giving the maximum robustness to the system. Any deviation in this angle will move the polar plot closer to $-1$.

V. Conclusion

In this article, a review of discrete-time resonators has been presented. Also, some guidelines for discrete-time AFC controllers design have been demonstrated w.r.t. the existing ones in continuous time: if the resonators’ gains are small, their angles must coincide with those of the plant at the tuning frequencies to maximize the phase margin. Nevertheless, even if they are not easy to compute, there are other possible designs like the one that minimizes the transient response. It is important to point out that working directly in discrete time is necessary when dealing with sampled-data control systems if one wants to take into account the sampling effects existing in sampled-data control systems.

VI. Appendix

When the gains $g_k$ are null, all the closed-loop resonator’s poles are on the unit circle. In order to evaluate their departure angle when the gain $g_k$ increases, it is necessary to compute the derivative of the $z$ variable in function of $g_k$ at $g_k = 0$. Resonators are going to be used in the general form presented in (3). Nevertheless, to avoid any indeterminate form, one can split them in numerator and denominator as $R_k(z) = \frac{1}{2} g_k R_{nk}(z)$. Therefore, equation (5) can be written as

$$2 \prod_{k=1}^{N} R_{dk}(z) + P(z) \sum_{k=1}^{N} g_k R_{nk}(z) \prod_{i=1, i \neq k}^{N} R_{di}(z) = 0. \ (8)$$

Note that the variable $z$ is a function of $g_k$. Without loss of generality, the implicit derivative is going to be carried out with respect to the gain $g_1$ (the same process could be carried out again with any $g_k$). Therefore, equation (8) can be rewritten as

$$2 R_{d1}(z(g_1)) \prod_{k=2}^{N} R_{dk}(z(g_1)) + P(z(g_1)) g_1 R_{n1}(z(g_1)) \prod_{k=2}^{N} R_{dk}(z(g_1)) + \prod_{k=2}^{N} R_{di}(z(g_1)) = 0. \ (9)$$

$$P(z(g_1)) \sum_{k=2}^{N} g_k R_{nk}(z(g_1)) \prod_{i=1, i \neq k}^{N} R_{di}(z(g_1)) = 0.$$
The derivative of the first summand with respect $g_1$ when $g_1 = 0$ is $2R'_d(\epsilon^{j\omega_1T})z'(0) \prod_{k=2}^{N} R_{dk}(\epsilon^{j\omega_1T})$, since $z(0) = e^{j\omega_1T}$ and $R_{d1}(\epsilon^{j\omega_1T}) = 0$. The derivative of the second summand when $g_1 = 0$ is $P(\epsilon^{j\omega_1T})R_{n1}(\epsilon^{j\omega_1T}) \prod_{k=2}^{N} R_{dk}(\epsilon^{j\omega_1T})$. And finally, the derivative of the third summand is null, since again, $R_{d1}(\epsilon^{j\omega_1T}) = 0$ and all the gains $g_k$ are supposed to be null. Consequently, the derivative of (9) turns out to be

$$2R'_d(\epsilon^{j\omega_1T})z'(0) + P(\epsilon^{j\omega_1T})R_{n1}(\epsilon^{j\omega_1T}) = 0.$$ 

Replacing $R_{n1}(\epsilon^{j\omega_1T})$ by $H_n(1)H_d(\epsilon^{j2\omega_1T})e^{-j\phi_1}$, $R_{d1}(\epsilon^{j\omega_1T})$ by $H_n'(1)e^{-j\omega_1T}H_d(\epsilon^{j2\omega_1T})$ and simplifying, one arrives to the equation

$$2H_n'(1)e^{-j\omega_1T}z'(0) + P(\epsilon^{j\omega_1T})H_n(1)e^{-j\phi_1} = 0,$$

where it is only necessary to replace $z'(0)$ by $e^{j\omega_1T}p'(0) + je^{j\omega_1T}h'(0)$ to obtain the expression (6).

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6Due to the root locus symmetry, only positive frequencies are used.

7$R_{d1}(\epsilon^{j\omega_1T}) = H_d(1)H_d(\epsilon^{j2\omega_1T})$ and $H_d(1) = 0$ since the denominator of an integrator is $z - 1$.

8The numerator $R_{n1}(\epsilon^{j\omega_1T})$ is $H_n(1)H_d(\epsilon^{j2\omega_1T})e^{-j\phi_1} + H_n(\epsilon^{j2\omega_1T})H_d(1)e^{j\phi_1}$, that is, $H_n(1)H_d(\epsilon^{j2\omega_1T})e^{-j\phi_1}$. 

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