A New Procedure for Discretization and State Feedback Control of Uncertain Linear Systems

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Abstract—This paper addresses the problem of constant sampling discretization of uncertain time-invariant continuous-time linear systems in polytopic domains. To circumvent the difficulty of dealing with the exponential of uncertain matrices, a new discretization method, based on Taylor series expansion, is proposed. The resulting discrete-time uncertain system is described in terms of homogeneous polynomial matrices with parameters lying in the unit simplex and an additive norm-bounded uncertainty which represents the discretization residual error. As a second contribution, linear matrix inequality (LMI) based conditions for the synthesis of a stabilizing state feedback control for discrete-time linear systems with polynomial dependence on the uncertain parameters and an additive norm-bounded uncertainty are proposed. Numerical experiments illustrate the discretization technique advantages of using higher orders in the Taylor series expansion to obtain more precise approximations. The examples also show that, at the price of simple line searches in a scalar parameter, and using Lyapunov functions of higher degrees, less conservative results for robust state feedback control design of discretized uncertain systems can be obtained.

I. INTRODUCTION

Although in control theory most of the signals of interest, like command inputs, actuator outputs, sensor readings, are generally continuous-time, the use of digital controllers is widely disseminated due to the advance of digital technology and of computers [1]. As a matter of fact, digital controllers have several advantages when compared to the analog ones, like low costs of implementation and replication, flexibility, expansibility, and simplicity in programming. Therefore, there exists a growing interest in system modeling and control design using discrete-time based methodologies.

Additionally, any realistic modeling or control design strategy must take into account the presence of uncertainties in the physical processes. Uncertainties arise from parameter variations, external perturbations, noise associated with the collected information or measurements, due to the accuracy of sensors and actuators, or can be related to hidden dynamics [2].

Usually, the synthesis of digital controllers is performed in three different ways [3]: i) emulation design or continuous-time based synthesis, where the controller is designed directly in the continuous-time domain, being converted to the discrete-time using, for instance, a Tustin transformation; ii) direct discrete-time design, where an approximate discretized model of the plant is used for discrete-time control synthesis, ignoring the inter-sampling behavior; iii) sampled-data design, where the controller is designed in the discrete-time domain taking into account the inter-sampling behavior. This paper is more concerned with case ii), since a more accurate discretized model for the original continuous-time uncertain system is proposed.

In the literature, there are only a few methods that cope with the discretization of uncertain systems. In [4], the Chebyshev quadrature formula and interval arithmetic are used for the digital modeling of a continuous-time uncertain system whose state space matrices are assumed to be interval matrices. Another example, in this case for a switched system affected by networked induced delay, can be found in [5]. In the authors’ knowledge, all methods in literature for discretization of uncertain systems are only numerical approximations [4], [5]. In fact, exact discretization of uncertain systems is still an open problem due to the difficulty of dealing with the exponential of an uncertain matrix. To circumvent this drawback some works use a first order Taylor series expansion approach [6]–[8] leading, however, to inaccurate discrete-time models for large values of sampling time. Moreover, in this case, there is no guarantee that the digital control law stabilizes the original continuous-time system.

The aim of this paper is to propose a new discretization procedure with constant sampling time for time-invariant continuous-time uncertain systems in polytopic domains. The approach, based on an extension of the Taylor series expansion of an arbitrary degree ℓ, converts the continuous-time polytopic system into an equivalent discrete-time system whose matrices are homogeneous polynomials of degree ℓ on the uncertain parameters, with additive norm-bounded terms. The norm-bounded terms represent the discretization residual error, which depends on the degree used in the series expansion, on the sampling time, and on the continuous-time uncertainty domain. The upper bound of the discretization residual error is numerically computed through an exhaustive grid in the uncertain domain. The proposed procedure systematically provides more accurate approximations of arbitrary precision as the degree of the Taylor series increases at the price of more complex discrete-time polynomial systems. It is worth to say that, differently of some approaches in the literature, the controller designed in the discrete-time domain guarantees the stability of the original continuous-time system since the error of the discretization procedure is taken...
into account. This discretization technique is more general than the one presented in [4], since it can cope with polytopes of any number of vertices. As another contribution, a new condition of robust state feedback control design for discrete-time polynomial systems with additive norm-bounded uncertainty is proposed. The conditions are formulated as linear matrix inequalities (LMIs) with a scalar parameter $\xi$ in the interval $(-1,1)$, i.e., the conditions become LMIs for fixed values of $\xi$. If verified, the conditions provide a robust gain and a homogeneous polynomially parameter-dependent Lyapunov matrix of arbitrary degree that certifies the closed-loop stability of both discrete and continuous-time systems. Numerical examples illustrate the advantages of the proposed discretization technique. The experiments also show that the proposed discretization technique leads to improvements on the project of digital controllers for polytopic continuous-time systems. Moreover, less conservative synthesis results can be provided by the proposed conditions at the price of line searches in the scalar parameter $\xi \in (-1,1)$ and of higher degrees in the Lyapunov function.

II. PRELIMINARIES

Consider the continuous-time uncertain linear system

$$\dot{x}(t) = E(\alpha)x(t) + F(\alpha)u(t)$$

(1)

where $x(t) \in \mathbb{R}^{n_x}$ is the system state and $u(t) \in \mathbb{R}^{n_u}$ is the control signal. Moreover, $E(\alpha) \in \mathbb{R}^{n_x \times n_x}$ and $F(\alpha) \in \mathbb{R}^{n_x \times n_u}$ are the uncertain system matrices belonging to a polytopic domain, i.e., they can be written as a convex combination of the $N$ known vertices

$$(E,F)(\alpha) = \sum_{i=1}^{N} \alpha_i (E_i,F_i)$$

(2)

and $\alpha$ is a vector of time-invariant parameters belonging to the unit simplex, given by

$$\Lambda_N = \left\{ (\lambda_1,...,\lambda_N) \in \mathbb{R}^N : \sum_{i=1}^{N} \lambda_i = 1, \lambda_i \geq 0, \ i = 1,...,N \right\}.$$

The aim of this paper is to obtain an uncertain discrete-time equivalent model for system (1), as accurate as possible, for the design of a digital controller that reads the state $x(t)$ at sampling instants $kT$, $k = 1,2,...$, where $T$ is a constant sampling period, providing a control signal $u(t)$. The proposed discretization procedure yields the following discrete-time model

$$x((k+1)T) = A(\alpha)x(kT) + B(\alpha)u(kT)$$

(3)

where $A(\alpha)$ and $B(\alpha)$ are the uncertain system matrices of the discretized system and $\alpha \in \Lambda_N$.

III. DISCRETIZATION OF UNCERTAIN SYSTEMS

This section proposes a new discretization procedure, based on Taylor series expansion, for a polytopic uncertain continuous-time system using a constant sampling period. The resulting discrete-time system has state matrices represented by homogeneous polynomials of degree $\ell$ on uncertain parameters belonging to the unit simplex, plus a norm-bounded term. The additional term, related to the residue of the approximation, depends on the degree $\ell$ of the series expansion, on the sampling period and on the uncertain parameters as well.

Therefore, the system matrices of (3) can be written as

$$A(\alpha) = A(\alpha) + \Delta A(\alpha) \quad \text{and} \quad B(\alpha) = B(\alpha) + \Delta B(\alpha)$$

(4)

with

$$A(\alpha) = \sum_{j=0}^{\ell} \frac{E(\alpha)^{T}}{j!} \quad \text{and} \quad B(\alpha) = \sum_{j=1}^{\ell} \frac{E(\alpha)^{j-1}}{j!} T F(\alpha)$$

(5)

and

$$\Delta A(\alpha) = e^{E(\alpha)T} A(\alpha)$$

$$\Delta B(\alpha) = \left( \int_{0}^{T} e^{E(\alpha) \xi} T F(\alpha) \right)$$

(6)

where $\Delta A(\alpha)$ and $\Delta B(\alpha)$ are the residues of the $\ell$-order Taylor series expansion, and $T$ is the sampling time.

Since $E(\alpha) \in \mathbb{R}^{n_x \times n_x}$, and, in the matrix case, products in multinomial series are non commutative, one has

$$E(\alpha)^q = \sum_{p \in \mathcal{P}(q)} \alpha_{p} E_{p}$$

(7)

where $\mathcal{P}(q)$ is the set of $q$-tuples obtained as all possible combinations of non-negative integers $k_i, i = 1,...,N$, such that $k_1 + k_2 + \cdots + k_N = q$, that is

$$\mathcal{P}(q) \triangleq \{ k = (k_1,k_2,\cdots,k_N) \in \mathbb{N}^N : \sum_{j=1}^{N} k_j = q, \ k_j \geq 0 \}.$$

$\mathcal{P}(q)$ is the set of $q$-tuples obtained as all possible combinations of non-negative integers $q_i, i = 1,...,q$, such that $p_i \in \{1,...,N\}$, that is

$$\mathcal{P}(q) \triangleq \{ p = (p_1,\cdots,p_q) \in \mathbb{N}^q : p_i \in \{1,...,N\}, \ i = 1,...,q \}$$

and $\mathcal{R}(k), k \in \mathcal{K}(q)$, is the subset of all $q$-tuples $p \in \mathcal{P}(q)$ such that the elements $j$ of $p$ have multiplicity $k_j$, for $j = 1,...,N$, that is

$$\mathcal{R}(k) \triangleq \{ p = (p_1,\cdots,p_q) \in \mathbb{N}^q : m_p(j) = k_j, \ j = 1,...,N \}$$

where $m_p(j)$ denotes the multiplicity of the element $j$ in $p$.

To illustrate the definitions, consider $q = 4$ and $N = 2$. In this case, the set $\mathcal{K}(4)$ is given by

$$\mathcal{K}(4) = \{ (40),(31),(22),(13),(04) \}$$

(8)
and, for instance, choosing \( k = 31 \), \( \alpha^k = \alpha_1^3\alpha_2 \), the set \( \mathcal{R}(k) \) is
\[
\mathcal{R}(31) = \{(11112), (11121), (11211), (12111), (21111)\}. \tag{11}
\]

Finally, one has,
\[
\sum_{p \in \mathcal{R}(31)} E_p = E_1^3 E_2 + E_1^3 E_2 E_1 + E_1 E_2 E_1^2 + E_2 E_1^2. \tag{12}
\]

By definition, for \( N \)-tuples \( k \) and \( k' \), one writes \( k \geq k' \) if \( k_i \geq k'_i \), \( i = 1, \ldots, N \). Operations of summation \( k + k' \) and subtraction \( k - k' \) (whenever \( k' \leq k \)) are defined componentwise. Consider, also, the \( N \)-tuple \( e_i \) defined as a null vector with \( N \) elements with a \( i \)-th component equal to 1.

Using the definitions presented above, one can write (5) as
\[
A_{\ell}(\alpha) = I + T E(\alpha) + \frac{T^2}{2} E(\alpha)^2 + \cdots + \frac{T^\ell}{\ell!} E(\alpha)^\ell
= \left( \sum_{i=1}^N \alpha_i \right)^\ell + T \sum_{i=1}^N \alpha_i \left( \sum_{k \geq i} \frac{(\ell-1)!}{k!} E_i + \cdots + \frac{T^j}{j!} \sum_{k \geq i} \frac{(\ell-j)!}{k!} E_p \right)
\]
\[
\triangleq \sum_{k \in \mathcal{R}(\ell)} \alpha^k A_k \tag{13}
\]
and matrix (6) can also be written as
\[
B_{\ell}(\alpha) = T F(\alpha) + \frac{T^2}{2} E(\alpha) F(\alpha) + \cdots + \frac{T^\ell}{\ell!} E(\alpha)^{\ell-1} F(\alpha)
= T \left( \sum_{i=1}^N \alpha_i \right)^{\ell-1} F(\alpha) + \frac{T^2}{2} \left( \sum_{i=1}^N \alpha_i \right)^{\ell-2} F(\alpha) + \cdots + \frac{T^\ell}{\ell!} E(\alpha)^{\ell-1} F(\alpha)
\]
\[
= \sum_{k \in \mathcal{R}(\ell)} \alpha^k \left( T \sum_{k \geq i} \frac{(\ell-1)!}{k!} F_i + \cdots + \frac{T^j}{j!} \sum_{k \geq i} \frac{(\ell-j)!}{k!} E_p F_i \right)
\]
\[
\triangleq \sum_{k \in \mathcal{R}(\ell)} \alpha^k B_k \tag{14}
\]
where \( ! = k_1 k_2 \cdots k_N \) and \( A_k \) and \( B_k \) are the coefficients of the discretized system polynomial matrices \( A_\ell(\alpha) \) and \( B_\ell(\alpha) \).

This paper proposes a generic degree discretization for uncertain systems motivated by the fact that a first-order approximation sometimes cannot be able to reproduce in a satisfactory manner the behavior of the original uncertain continuous-time system. In fact, this aspect becomes more evident as the sampling time increases. As an example, consider the stable uncertain continuous-time system, \( \dot{x}(t) = E(\alpha)x(t) \), with three vertices, given by
\[
\begin{bmatrix}
E_1 & E_2 & E_3 \\
4.0 & 3.0 & -12.1 \\
-8.6 & -4.2 & 2.3 \\
-3.6 & 0.6 & -6.1 \\
-20.9 & 5.9 \\
\end{bmatrix} \tag{15}
\]

Figure 1 illustrates the fact that, by increasing the degree \( \ell \) of approximation of the discretized system, \( x(k+1) = A_\ell(\alpha)x(k) \), the discrete-time state trajectories become tighter to the corresponding continuous-time ones. An arbitrary \( \alpha = [0.5998 \ 0.3665 \ 0.0337] \) has been chosen for the simulation, with \( x_0 = [1 \ -2]^T \) and a sampling time \( T = 0.15s \).

IV. STABILIZATION

In this section, a new LMI based control synthesis condition for polynomially parameter-dependent discrete-time systems with additive norm-bounded uncertainties is proposed.

Following the lines used in [5], where a similar uncertain discrete-time model is obtained from a precisely known continuous-time system with uncertain sampling periods, the discretized system (3), with \( A(\alpha) \) and \( B(\alpha) \) given in (4), is rewritten as the augmented system
\[
z((k+1)T) = (\tilde{A}_\ell(\alpha) + \Theta_\ell(\alpha)) z(kT) + \mathbb{B} v(kT) \tag{16}
\]
where
\[
z(kT) = \begin{bmatrix} x(kT) \\ u(kT) \end{bmatrix}, \quad v(kT) = u((k+1)T), \quad \mathbb{B} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad \tilde{A}_\ell(\alpha) = \begin{bmatrix} A_\ell(\alpha) & B_\ell(\alpha) \\ 0 & 0 \end{bmatrix}, \quad \Theta_\ell(\alpha) = \begin{bmatrix} \Delta A_\ell(\alpha) & \Delta B_\ell(\alpha) \end{bmatrix}.
\]

The additive term \( \Theta_\ell(\alpha) \) represents the discretization residual error, and can be bounded by \( \delta = \sup_{\alpha \in \Lambda_N} ||\Theta_\ell(\alpha)|| \), i.e., \( ||\Theta_\ell(\alpha)|| \leq \delta \). An estimate of the bound \( \delta \) can be computed by performing a search in a grid of values of \( \alpha \in \Lambda_N \). In
this case, the computational burden increases as the number of the continuous-time vertices (dimension of $\alpha$) and the number of tested values in the grid augments.

Defining a state feedback control law as

$$v(kT) = u(k+1)T = Kz(kT)$$

(17)

or

$$u(kT) = K[x((k-1)T)'] - u((k-1)T)']'$$

(18)

the following theorem can be stated for the stabilization of system (16), and, consequently, for (3).

**Theorem 1** If there exists a degree $g > 0$, $g \in \mathbb{N}$, symmetric matrices $W_k \in \mathbb{R}^{(n_x+n_u) \times (n_x+n_u)}$, $k \in \mathcal{X}(g)$, with $g \in \mathbb{N}$, matrices $G \in \mathbb{R}^{(n_x+n_u) \times (n_x+n_u)}$ and $Z \in \mathbb{R}^{n_x \times (n_x+n_u)}$, a Pólya’s relaxation degree $d \in \mathbb{N}$, and a given scalar parameter $\xi \in (-1,1)$, such that the following LMIs hold

$$S_k = \sum_{k \in \mathcal{X}(g+d)} \frac{d!}{k!} W_{k-k} > 0, \quad \forall k \in \mathcal{X}(g+d)$$

(19)

$$M_k + \sum_{k \in \mathcal{X}(w-\ell)} M_{k-k} + \sum_{k \in \mathcal{X}(w-g)} M_{k-k} < 0, \quad \forall k \in \mathcal{X}(w)$$

(20)

where

$$M_k = \frac{w!}{k!} \begin{bmatrix} \delta^2 I - \xi (\mathbb{B}Z + \mathbb{Z}'B') & * & * \\ -\zeta G + \mathbb{Z}'B' & G - G' & * \\ \xi G + \mathbb{Z}'B' & G - G' & G + G' - I \end{bmatrix}$$

$$M_{k-k} = \frac{(w-\ell)!}{k!} \begin{bmatrix} -\zeta (\bar{A}_{k-k}G + G'\bar{A}_{k-k}) & * & * \\ G\bar{A}_{k-k} & * & * \\ -W_{k-k} & * & * \end{bmatrix}$$

$$M'_{k-k} = \frac{(w-g)!}{k!} \begin{bmatrix} -W_{k-k} & * & * \\ 0 & W_{k-k} & * \\ 0 & 0 & 0 \end{bmatrix},$$

where $\bar{A}_k$ are the coefficients of matrix $\bar{A}(\alpha)$ with approximation degree $\ell \in \mathbb{N}$, $w \in \mathbb{N}$, $w \in \{g+d, \ell+d\}$, and $\delta$ given by $\sup_{\alpha \in \Lambda_N} \Vert \Theta(\alpha) \Vert$, then $K = 2G^{-1}$ is a robust stabilizing state feedback gain for system (16) and, consequently, (3).

**Proof:** First, note that $(\alpha_1 + \cdots + \alpha_N)^d = 1$ for any $d \in \mathbb{N}$, matrix $W(\alpha)$ can be rewritten as

$$\left( \sum_{i=1}^{N} \alpha_i \right)^d W(\alpha) = \sum_{k \in \mathcal{X}(g+d)} \alpha_k S_k.$$  

Thus, if $S_k > 0$, $k \in \mathcal{X}(g+d)$, then, $W(\alpha) > 0$ hold $\forall \alpha \in \Lambda_N$. Now, defining the closed-loop matrix $A_{cl}(\alpha) = \bar{A}(\alpha) + \Theta(\alpha) + \mathbb{B}ZG^{-1}$, and choosing

$$Q = \begin{bmatrix} \delta^2 I - W(\alpha) & 0 & 0 \\ * & W(\alpha) & 0 \\ * & * & I \end{bmatrix}, \quad U = \begin{bmatrix} \xi I & -\xi I' \\ -I & I' \end{bmatrix},$$

$$V = \begin{bmatrix} \Theta(\alpha) & A_{cl}(\alpha) \\ 0 & -I \\ -I & 0 \end{bmatrix}, \quad N_U = \begin{bmatrix} I & 0 \\ \xi I & -I' \\ 0 & I \end{bmatrix}, \quad N_V = \begin{bmatrix} I & A_{cl}(\alpha)' \\ \Theta(\alpha)' \end{bmatrix}$$

where $U$ and $V$ denote arbitrary bases of the null space of $U$ and $V$ respectively, one has

$$Q + U'G'V + V'GU < 0$$

(22)

which is (20) multiplied by $\alpha^k$, summed up for $k \in \mathcal{X}(w)$. Such conditions are equivalent, by the Projection Lemma [9]–[11], to

$$N_U'QN_U = A_{cl}(\alpha)W(\alpha)A_{cl}(\alpha)' - W(\alpha) - \Theta(\alpha)\Theta(\alpha)' + \delta^2 I < 0$$

(23)

and

$$N_V'QN_V = \begin{bmatrix} \delta^2 I - W(\alpha) + \xi^2 W(\alpha) & -\xi W(\alpha) \\ * & W(\alpha) - I \end{bmatrix} < 0$$

(24)

Knowing that $||\Theta(\alpha)|| < \delta$, (23) implies

$$-W(\alpha) + A_{cl}(\alpha)W(\alpha)A_{cl}(\alpha)' < 0$$

which, along with $W(\alpha) > 0$, certifies the closed-loop stability of the system (16) and, from (24), one can notice that $\xi \in (-1,1)$.\]

Note that $\xi \in (-1,1)$ represents a degree of freedom to be exploited in the search for a feasible solution. For instance, a line search on $\xi$ can be performed, or simply a set of given values can be tested.

**Remark 1** Although the stabilizability of an uncertain discrete-time system, obtained from any discretization procedure, does not imply, in the general case, that the original continuous-time system is also stabilizable, in the proposed approach the state feedback control law

$$u(t) = Kz((k-1)T), \quad t \in [kT, (k+1)T]$$

(25)

guarantees the stability of the closed-loop uncertain continuous-time system (1), since the approximation error of the discretization procedure is taken into account in the discrete-time model.

Following the lines used in [5], for any arbitrary $\alpha \in \Lambda_N$ and a given sampling period $T$, as $z(kT) \to 0$ and $k \to \infty$, the solution of the linear system (1) over the interval $[kT, (k+1)T]$ can always be bounded by a quantity $\tilde{x}(kT)$, that is

$$\sup_{t \in [kT, (k+1)T]} \Vert x(t) \Vert \leq \tilde{x}(kT)$$

with $\tilde{x}(kT)$ given by

$$\sup_{t \in [0,T]} \left\Vert e^{E(\alpha)t}x(0) \right\Vert + \int_0^{T-kT} e^{E(\alpha)t}F(\alpha)u(t)dt$$

As both $x(kT)$ and $u(kT)$ converge to zero as $k \to \infty$, then $x(t) \to 0$ as $t \to \infty$, and, consequently, as this is valid for any arbitrary $\alpha \in \Lambda_N$, then the asymptotic closed-loop stability of the uncertain system (1) with the feedback control law (25) is assured.
V. NUMERICAL EXAMPLES

Numerical comparisons between the approach proposed in this paper and other methods in the literature for state feedback control of uncertain linear systems are presented next. All the routines were implemented in Matlab, version 7.10 (R2010a) using Yalmip [12] and SeDuMi [13], in an AMD Phenom (TM) II X6 1090T (3.2GHz), 4GB RAM, computer.

Example 1 Consider a mass, spring and damper continuous-time system given in [14], [15]

\[ \dot{x}(t) = E(c_1,c_2) x(t) + Fu(t) \]  \hspace{1cm} (26)

where

\[ c_1 \in [1.6, 4.2], \quad c_2 \in [6.4, 9.6] \]

\[ E(c_1,c_2) = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -c_1 & c_1 & -0.2 & 0.2 \\ c_1/2 & -c_1 + c_2/2 & 0.1 & -0.15 \end{bmatrix}, \quad F = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}. \]

By evaluating the dynamic matrix of the system at the extreme values of the uncertain parameters \((c_1, c_2)\), a four vertices polytope is obtained. The aim is to design a robust state feedback digital controller that assures the stability of the uncertain continuous-time system. For this purpose, state feedback gains are obtained by using the condition presented in [16] and Theorem 1 for the discrete-time models obtained from (13) and (14) with \(\ell = 1, 2, 3\) and \(T = 0.3s\). Note that the condition in [16] does not take into account the additive uncertainty in the discrete-time model, and can deal only with polytopic systems, i.e., \(\ell = 1\). In all tests, Lyapunov matrices \(W(\alpha)\) with affine \((g = 1)\) dependence on the uncertain parameter \(\alpha\) are employed.

In the literature, when dealing with polytopic continuous-time systems controlled by a digital computer, the polytope vertices are discretized using a first order Taylor approximation generating an affine polytope and a discrete-time condition for control design is applied, as can be seen in the examples presented in [6]–[8]. Applying [16, Theorem 3] in the first order discrete-time model, a robust state feedback gain is found. However, the obtained controller cannot stabilize system (26), since [16, Theorem 3] did not take into account the approximation error. To illustrate that, Figure 1 shows a time-simulation for the continuous-time system (26) performed in Simulink Matlab with \(\alpha = [0.3660 \ 0.0909 \ 0.2718 \ 0.2713]\), \(x_0 = [1 \ -2 \ 3 \ -1]'\) and the control gain obtained by [16, Theorem 3].

At this point, it is important to emphasize the crucial role of the bound on the discretization error presented in this paper. For low degrees of Taylor approximation, the value of \(\delta\) is high and the state feedback control synthesis condition from Theorem 1 does not provide a stabilizing gain. By increasing the degree \(\ell\) of the Taylor series expansion, the bound is reduced as follows:

\[ \delta_{\ell=1} = 0.4385, \quad \delta_{\ell=2} = 0.1924, \quad \delta_{\ell=3} = 0.0224. \]

In this example, only for \(\ell \geq 3\) the value of \(\delta\) was sufficiently small such that Theorem 1 provided a feasible solution. Figure 1 shows the Simulink Matlab time-responses for the same values for \(\alpha\) and \(x_0\) as in Figure 1 with the robust state feedback gain from Theorem 1 for \(\ell = 3, g = 1, d = 0, \delta = 0.0224\). In this case, the designed controller stabilizes the continuous-time system.

Example 2 Consider the continuous-time polytopic system whose system matrix vertices are:

\[ [E_1 \ E_2 \ F_1 \ F_2] = \begin{bmatrix} 1.80 & -0.80 & -a & -1.12 & -0.27 & -b \\ 3.10 & -2.15 & 4.34 & -3.01 & 1.80 & -2.4 \end{bmatrix} \]

The system is sampled with \(T = 0.1s\) and a 1st to 9th order Taylor approximation, in (13) and (14), is applied. The aim of this example is to show that the increase in the degree \(\ell\) combined with a simple line search in the scalar parameter
\( \xi \in (-1,1) \) in Theorem 1 can reduce the conservatism of the results, that is, a larger family of uncertain continuous-time systems can be stabilized.

![Graph](a)

![Graph](b)

Fig. 3. Stabilizable region for the Example 2 provided by Theorem 1 with \( g = 1, d = 0 \) and: (a) \( \xi = 0 \) with different degrees (\( \ell \)) of Taylor series approximation (higher degrees also stabilize the lower ones); (b) \( \ell = 5 \). First case: \( \xi = 0 \) (●); second case: \( \xi \in [-0.9,0.9] \) (● and ▲).

Two tests were performed. In the first case, Theorem 1, with \( g = 1, d = 0 \) and \( \xi = 0 \) has been used. Figure 2 shows that with the increase of the approximation degree (\( \ell \)) a larger set of continuous-time systems can be stabilized. In the second case, for a fixed discretization degree (\( \ell = 5 \)), \( g = 1 \) and \( d = 0 \), a line search in the scalar parameter has been performed, where 19 equally spaced values in the interval \([-0.9,0.9]\) were tested. Figure 2 demonstrates the advantage of employing the scalar parameter. As can be seen, a wider range of values of \( a \) and \( b \) can be stabilized by a search in \( \xi \) for constant values of \( \ell \).

Larger domains of parameters \( a \) and \( b \) can also be obtained using Lyapunov functions of degree \( g \geq 1 \) in Theorem 1. For instance, with \( \ell = 7, \xi = 0, a = 8, g = 1 \) and \( d = 0 \) the stabilizable range for \( b \) is \([0.2,0.6]\) (see Figure 2). By simply increasing the degree of the Lyapunov function to \( g = 2 \), the range of \( b \) is extended to \([0.2,0.8]\).

VI. Conclusion

This paper proposed a new discretization procedure for uncertain time-invariant linear systems based on Taylor series expansion. Using an appropriate degree of approximation, the resulting discrete-time system reproduces accurately the dynamic behavior of the continuous-time system. Additionally, new conditions to synthesize robust state feedback controllers for uncertain discrete-time systems with polynomial dependence on the parameters and an additive norm bounded term have been proposed in terms of LMI relaxations. Numerical experiments show that, differently from the usual approaches based on first order Taylor approximation, the proposed method guarantees the stabilization of the uncertain continuous-time system.

References