Convex Relaxations of a Probabilistically Robust Control Design Problem

A. M. Jasour and C. Lagoa

Abstract—In this paper, we address the problem of designing probabilistic robust controllers for discrete-time systems whose objective is to reach and remain in a given target set with high probability. More precisely, given probability distributions for the initial state, uncertain parameters and disturbances, we develop algorithms for designing a control law that (i) maximizes the probability of reaching the target set in \( N \) steps and (ii) makes the target set robustly positively invariant. As defined, the problem is nonconvex. To solve this problem, a sequence of convex relaxations is provided, whose optimal value is shown to converge to solution of the original problem. In other words, we provide a sequence of semidefinite programs of increasing dimension and complexity which can arbitrarily approximate the solution of the probabilistic robust control design problem addressed in this paper. Two numerical examples are presented to illustrate preliminary results on the numerical performance of the proposed approach.

I. INTRODUCTION

In this paper we provide results aimed at designing robust controllers that maximize the probability of reaching a given target set. More precisely, we start with an uncertain polynomial system subject also to external perturbations and for which we know the probability distribution of the initial state, the uncertainty and the disturbances. Then, given a target set defined by polynomial inequalities and number of steps \( N \), we provide algorithms for designing a nonlinear state feedback control law that (i) makes the target set robustly positively invariant and (ii) maximizes the probability of reaching the target set in \( N \) steps. Throughout this paper, it is assumed that a static polynomial state feedback control law exists that makes the target set robustly invariant.

Probabilistic robust formulations such as the one above are used in different areas to deal with uncertain systems in order to ensure that the probability of failure/success is minimized/maximized. A few examples are minimization of probability of obstacle collision in motion planning of robotic systems under environment uncertainty, risk minimization problem in the area of economy, finance, and many other areas that can be formulated as instances of chance control problem. Although, in some particular cases probabilistic control problems are convex (e.g., see [1], [2]), in general, chance control problems are not convex. In this paper, we use results on the chance constrained optimization and measure theory (e.g., see [3], [4], [5]) to develop a sequence of semidefinite problems whose solution converges to the solution of the probabilistic robust control problem mentioned above.

A. Previous Work

Several approaches have been proposed to involve statistics of uncertainty in control procedure of uncertain systems. The main approaches are (i) adding probabilistic constraints on states and inputs of system (e.g., see [1], [2]), (ii) minimizing the expected value of the objective function (e.g., see [1], [2]). The main problem in the formulation of chance constrained control is the efficient evaluation of the probabilistic constraints. Hence, several approaches have been provided to propose tractable approximations of the chance constraints involved in a probabilistic control design problem.

One of such methods is the so-called randomized approach; see [6-10] and references therein. In this case, the probabilistic constraint is replaced by a (large) number of deterministic constraints obtained by drawing iid samples of the random parameters. Being a randomized approach, there is always a (perhaps small) probability of failure of the algorithm.

In [11-15], an alternative approach is proposed where one analytically determines an upper bound on the probability of constraint violation. In [16] expected value of uncertain objective function is proposed using the notion of particles. It tries to approximate the distribution of the system state using a finite number of particles.

In this paper, we take a different approach. We incorporate the probability directly in the objective function and aim at maximizing the probability of desired defined control objectives. The proposed method is based on results on semialgebraic chance optimization in [3], where one aims at maximizing the probability of a set defined by polynomial inequalities. Being, in general, a non-convex problem, a hierarchy of semidefinite relaxations for the approximation of the solution was proposed. These results provide the main motivation for the approach taken in this paper.

B. The Sequel

The outline of the paper is as follows. In section II, the notation used in this paper as well as preliminary results on measures theory and chance optimization are presented. In section III, an explicit definition of chance robust control problem is given. In sections IV and V, a sequence of convergent convex relaxations is provided. Numerical examples

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are given in the section VI and conclusion is stated in section VII.

II. NOTATION AND PRELIMINARY RESULTS

A. Notation and Definitions

Let $\mathbb{R}[x]$ be the ring of real polynomials in the variables, $\mathbb{R}[x]_d \subset \mathbb{R}[x]$ be the set of polynomials of degree at most $d$ and $\mathbb{N}_d^\alpha = \{ \alpha \in \mathbb{N}^n : \|\alpha\|_1 \leq d \}$, where $n \in \mathbb{N}$, and $\Sigma^2[x] \subset \mathbb{R}[x]$ be the subset of sums of squares (SOS) polynomials. $\sigma(x)$ is a SOS polynomial if it takes the form $\sigma(x) = \sum_{i=1}^d h_i(x)^2$, for finitely many polynomials $h_i(x)$ [5]. Given two square symmetric matrices $A$ and $B$, the notation $A \succ 0$ denotes positive semidefiniteness of $A$ and $A \succeq B$ stands for $A - B$ being positive semidefinite. Given a measure $\mu$, $\text{supp}(\mu)$ denotes the support of the measure $\mu$; i.e., the smallest set that contains all sets with strictly positive $\mu$ measure.

A sequence $y = (y_\alpha)$, where $y_\alpha \in \mathbb{R}$, is said to have a representing finite Borel measure $\mu$ if $y_\alpha = \int x^\alpha \, d\mu$ for every $\alpha \in \mathbb{N}^n$ [3], [4]. In this case, $y$ is the moment sequence of the measure $\mu$. Given two measures $\mu_1$ and $\mu_2$ on a Borel $\sigma$-algebra $\Sigma$, the notation $\mu_1 \ll \mu_2$ means $\mu_1(S) \leq \mu_2(S)$ for any set $S \in \Sigma$. Given measures $\mu_1$ with $\text{supp}(\mu_1) \in \Sigma_{\mu_1}$ and $\mu_2$ with $\text{supp}(\mu_2) \in \Sigma_{\mu_2}$, $\mu = \mu_1 \times \mu_2$ is the measure satisfying $\mu(S_1 \times S_2) = \mu_1(S_1)\mu_2(S_2)$ for any measurable sets $S_1 \in \Sigma_{\mu_1}$, $S_2 \in \Sigma_{\mu_2}$ [4].

Moment matrix: To build the moment matrix associated with sequence $y = (y_\alpha)$, that contains all the moments up to order $2d$, we arrange the moments according to a graded reverse lexicographic order (grevlex) of the corresponding monomials so that we have $0 = \alpha^{(1)} < \ldots < \alpha^{(2d)}$, where $S_d = \{\alpha^{(i)}\}$ is the number of moments in $\mathbb{R}^n$ up to order $d$. The moment matrix $M_d(y)$ is the symmetric matrix as follows [3], [4]:

$$M_d(y)(i,j) = y_{\alpha^{(i)} + \alpha^{(j)}}, \quad \forall i,j \leq S_d \quad (1)$$

Localizing matrix: Given a polynomial $g \in \mathbb{R}[x]$ with coefficient vector $g = \{g_j\}$ and degree $\delta$, localizing matrix $M_d(gy)$ with respect to $y$ and $g$ is as follows [3], [4]:

$$M_d(gy)(i,j) = \sum_{\gamma \in \mathbb{N}^n} g_{\gamma} y_{\gamma + \alpha^{(i)} + \alpha^{(j)}}, \quad \forall i,j \leq S_d - \lfloor \frac{d}{2} \rfloor \quad (2)$$

B. Preliminary Results

In this section we review some basic results on positive polynomials and measures. For an in depth exposition we refer the reader to [3,4,5].

Consider compact semi-algebraic set:

$$K = \{ x \in \mathbb{R}^n : g_j(x) \geq 0, \quad j = 1,2,\ldots, l \} \quad (3)$$

for some polynomials $g_j \in \mathbb{R}[x]$.

(i) If $f \in \mathbb{R}[x]$ is strictly positive on $K$, then:

$$f = \sigma_0 + \sum_{j=1}^l \sigma_j g_j \quad (4)$$

for some $\sigma_j \in \mathbb{R}^2[x]$.

(ii) The sequence $y = (y_\alpha)$ has a representing finite Borel measure $\mu$ on $K$ if:

$$M_d(y) \succ 0, \quad M_d(g_j y) \succ 0, \quad j = 1,\ldots, m \quad (5)$$

for every $d \in \mathbb{N}^n$; e.g. see [3], [4].

(iii) Given two measures $\mu_1$ and $\mu_2$ on $K$, with moment sequences $y_1 = (y_1\alpha)$ and $y_2 = (y_2\alpha)$, we have $\mu_1 \ll \mu_2$ if [4]:

$$M_d(y_2 - y_1) \succ 0, \quad M_d(g_j (y_2 - y_1)) \succ 0, \quad j = 1,\ldots, l \quad (6)$$

for every $d \in \mathbb{N}^n$.

C. Relaxation of Chance Constrained Problems

Finally, we provide a review of some of the results in [3] which are used in this paper. Consider the chance constrained problem defined as:

$$P_1^* = \max_{\mu, \mu_x} \left\{ \text{Prob}_{\mu_x} \{ g(x,q) \geq 0 \} \right\} \quad (7)$$

where $q \in \mathbb{R}^m$ is a random variable vector with probability measure $\mu_q$ with $\text{supp}(\mu_q) \subseteq Q$, and $x \in \chi \subseteq \mathbb{R}^n$ is our decision variable and $g : \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}$, is polynomial. This problem is equivalent to the following problem in the space of measures

$$P_2^* = \max_{\mu, \mu_x} \int d\mu \quad (8)$$

subject to

$$\mu \ll \mu_x \times \mu_q \quad (9)$$

$\mu_x$ is a probability measure

$$\text{supp}(\mu_x) \subseteq \chi \quad (10)$$

$$\text{supp}(\mu) \subseteq \Sigma_K \subseteq \mathbb{R}^n \quad (11)$$

where $\Sigma_K = \{(x,q) : g(x,q) \geq 0 \} \subseteq (\chi \times Q)$. This problem is equivalent to the Problem 1 in Eq. (7) in the following sense: (i) The optimal values are the same, (ii) If $\mu_x^*$ is a solution of Problem, then, any $x^* \in \text{supp}(\mu_x^*)$ is a solution of Problem 1 in Eq. (7), (iii) If $x^*$ be a solution of Problem 1, then $\mu_x^* = \delta_{x^*}$ is a solution of Problem 2 in Eq. (8).

III. PROBLEM STATEMENT

Consider the following discrete-time stochastic dynamic system

$$x(k + 1) = f(x(k), u(k), \delta, \omega(k)) \quad (12)$$

where $f : \mathbb{R}^{n+2m+p} \rightarrow \mathbb{R}^n$ is a polynomial function, $x(k) \in \chi \subseteq \mathbb{R}^n$ is the system state, $u(k) \in \psi \subseteq \mathbb{R}^p$ is the control input, $\delta \in \Delta \subseteq \mathbb{R}^p$ is the uncertain model parameter and $\omega(k) \in \Omega \subseteq \mathbb{R}^m$ is the disturbance, at time step $k$.

The initial state $x(0) \in \chi_0 \subseteq \chi$, model parameter $\delta$, and disturbance $\omega(k)$ at time $k$ are independent random variables having probability measure $\mu_{\chi_0}$, $\mu_\delta$, and $\mu_\omega$, with compact supports $\text{supp}(\mu_{\chi_0}) \subseteq \chi_0$, $\text{supp}(\mu_\delta) \subseteq \Delta$ and $\text{supp}(\mu_\omega) \subseteq \Omega$, respectively. We assume that $\chi_0, \Delta, \Omega$ are...
compact semialgebraic sets of the form $\chi_0 = \{x : g_0(x) \geq 0\}$, $\Delta = \{\delta : g_\delta(\delta) \geq 0\}$, $\Omega = \{\omega : g_\omega(\omega) \geq 0\}$ for given polynomials $g_0, g_\delta, g_\omega$. Although each of these sets is defined by just one polynomial, the approach proposed in this paper can be extended to more complex semialgebraic sets. This assumption is only done to simplify the exposition.

Let $N$ be a given integer. The desired terminal set at time step $N$ is defined as the compact semialgebraic set

$$\chi_N = \{x : g_N(x) \geq 0\}.$$  

We aim at finding a polynomial state feedback control input

$$u(x) = \sum_{i \in N_n^d} b_i x^i$$

where $u : R^n \rightarrow R^m$ is polynomial of order no more that $n_u$ and $b \in B$ is a vector of coefficients $b_i$, such that $\chi_N$ is an invariant set and maximizes the probability of reaching $\chi_N$ in $N$ steps. Terminal set $\chi_N$ is invariant under control law if

$$f(x, u(x), \delta, \omega) \in \chi_N$$

for all $x \in \chi_N, \ \delta \in \Delta, \ \omega \in \Omega$. Under the definitions provided above, the stochastic control problem can be stated as follows

**Problem 1:** Solve,

$$P^{1*} = \max_{\mathbf{b}} \text{Prob}_{\mu_{x_0}, \mu_\delta, \mu_\omega} \{g_N(x(N)) \geq 0\}$$

subject to,

$$x(k+1) = f(x(k), u(k), \delta, \omega(k))$$

$$u(k) = \sum_{i \in N_n^d} b_i x^i(k)$$

$$x_0 \sim \mu_{x_0}, \delta \sim \mu_\delta, \omega(k) \sim \mu_\omega$$

$$f(x, u(x), \delta, \omega) \in \chi_N$$

for all $x \in \chi_N, \ \delta \in \Delta, \ \omega \in \Omega$.

### IV. AN EQUIVALENT PROBLEM

As mentioned before we address this problem in two steps. First we determine a set of control laws that renders the set $\chi_N$ robustly positively invariant. Then, we search for a control law in this set that maximizes the probability of reaching $\chi_N$ in $N$ steps.

**A. Set Invariant Control Laws**

In first step, we are looking for a set of parameters of control laws that render desired terminal set $\chi_N$ invariant. In this paper, we approximate this set by a semialgebraic set $P_d \subseteq B$ of the form

$$P_d = \{\mathbf{b} : p_d(\mathbf{b}) \geq 0\}$$

where the $p_d$ is a polynomial of order $d$ of the form

$$p_d(\mathbf{b}) = \sum_{j \in N_n^d} \gamma_j b_j \in \mathbb{R}[\mathbf{b}]_d.$$  

To determine $\lambda_j$ the coefficients of this polynomial, one needs to solve the following optimization problem involving SOS polynomials, which can be easily done using semidefinite programming.

$$\min_{\lambda_j, \sigma_0, \sigma_1, \sigma_2, \sigma_3} \sum_{j \in N_n^d} \gamma_j \lambda_j$$

subject to

$$g_N(f(x, \sum_{i=0}^{m} b_i x^i, \delta, \omega)) - \sum_{j \in N_n^d} \lambda_j b_j = \sigma_0(x, \mathbf{b}, \delta, \omega)$$

$$+ \sigma_1(x, \mathbf{b}, \delta, \omega) g_N(x) + \sigma_2(x, \mathbf{b}, \delta, \omega) g_\omega(\omega) + \sigma_3(x, \mathbf{b}, \delta, \omega) g_\delta(\delta)$$

where $\gamma_j$ is $j$-th moment of uniform probability measure over the set $B$ of parameters of the control law and $\sigma_0, \sigma_1, \sigma_2, \sigma_3 \subseteq \mathbb{R}[x, \mathbf{b}, \delta, \omega]$, are finite degree SOS polynomials such that $\deg(\sigma_0) \leq d$, $\deg(\sigma_1 g_N) \leq d$, $\deg(\sigma_2 g_\omega) \leq d$, $\deg(\sigma_3 g_\delta) \leq d$.

We then have the following result:

**Theorem 1:** Let $p_d(\mathbf{b})$ be a polynomial constructed by solution of the optimization problem (11). Then, for any $\mathbf{b} \in P_d = \{\mathbf{b} : p_d(\mathbf{b}) \geq 0\}$ the corresponding control law

$$u(x) = \sum_{i \in N_n^d} b_i x^i$$

renders the set $\chi_N$ positively invariant. Moreover, define the set

$$P_{\text{total}} = \{u(x) \text{ renders the set } \chi_N \text{ positively invariant}\}.$$  

Then

$$\lim_{d \rightarrow \infty} \mu_B(P_{\text{total}} - P_d) = 0$$

where $P_{\text{total}} - P_d$ denotes the elements of $P_{\text{total}}$ not in $P_d$.

**Proof:** Recall that

$$\chi_N = \{x : g_N(x) \geq 0\}.$$  

Define function

$$p^*(\mathbf{b}) = \min_{x \in \chi_N, \delta \in \Delta, \omega \in \Omega} \left\{g_N(f(x, \sum_{i} b_i x^i, \delta, \omega))\right\}$$

Then,

$$u(x) = \sum_{i} b_i x^i$$

renders the set invariant if and only if

$$p^*(\mathbf{b}) \geq 0.$$  

Now, the results on robust polynomial optimization in [17-19] show that

$$p_d(\mathbf{b}) \leq p^*(\mathbf{b}) \text{ for all } \mathbf{b} \in B.$$  

Hence, for any $\mathbf{b} \in P_d$, we have

$$0 \leq p_d(\mathbf{b}) \leq p^*(\mathbf{b})$$
and the corresponding control law makes the set $\chi_N$ invariant. This proves the first part of the theorem.

The second part of the theorem is a consequence of the fact that

$$\int |p_d(b) - p^*(b)| d\mu_B(b) \to 0 \text{ as } d \to \infty$$

a result that has also been proven in [17-19].

Therefore, $b$ parameters of control law should belong to the semialgebraic set $P_d$, otherwise the trajectories of system may not remain inside the desired terminal set.

B. Maximizing Probability of Reaching

B. Maximizing Probability of Reaching $\chi_N$

Now that we have an estimate of the set of control laws that render the set $\chi_N$ positively invariant, we can now address the problem of maximizing the probability of reaching the target set in at most $N$ steps. Note that this is equivalent to maximizing the probability of $x(N) \in \chi_N$ since we have restricted the control laws to those that make the set $\chi_N$ invariant.

Define the function $h$ as

$$x(N) = h(x_0, b, \delta, \omega)$$

as the value of the state at time $N$ when the value of the uncertain parameters is $\delta$, the disturbances are

$$\omega = [\omega_0, ..., \omega_N],$$

the control has coefficients $b$ and the initial condition is $x_0$. Note that since one has a polynomial system and a polynomial control law, $h$ is a polynomial. Additionally define the semialgebraic set

$$K_1 = \{(x_0, b, \delta, \omega) : g_N(h(x_0, b, \delta, \omega) \geq 0)\}$$

which represents all the values of the variables that will result in $x(N) \in \chi_N$ and the semialgebraic set

$$K_2 = P_d(b) \cap B$$

of control laws that render the set $\chi_N$ invariant. Define the following problem

**Problem 2:** Solve

$$P^2* = \max_{\mu, \mu_b} \int d\mu$$

subject to

$$\mu \leq \mu_b \times \mu_{x_0} \times \mu_\delta \times \prod_{k=0}^{N-1} \mu_{\omega_k}$$

$$\mu_b$$ is a probability measure

$$\text{supp}(\mu) \subseteq K_1$$

$$\text{supp}(\mu_b) \subseteq K_2$$

This problem is equivalent to the problem addressed in this paper in the following sense.

**Theorem 2:** Problem 2 is equivalent to Problem 1 in the following sense: Let’s restrict our attention to control laws that have

$$b \in P_d$$

Under this addition restriction one has

1) The optimal values are the same.
2) If $\mu_b^*$ be a solution of Problem 2, then, any $b^* \in \text{supp}(\mu_b^*)$ is a solution of Problem 1.
3) If $b^*$ be a solution of Problem 1, then $\mu_b^* = \delta_b^*$ is a solution of Problem 2

**Remark:** The restriction $b \in P_d$ becomes “inactive” for large values of $d$, in which case both problems are equivalent.

**Proof:** Problem 1 is a semialgebraic chance constrained optimization, where $[x_0, \delta, \omega_0, ..., \omega_N]$ is a random vector with probability measure $[\mu_{x_0}, \mu_\delta, \mu_{\omega_0}, ..., \mu_{\omega_N}]$, and $[b_0, ..., b_m] \in K_2$ is our decision variable vector, and $g_N(h(x_0, b, \delta, \omega))$ is a polynomial. Therefore, based on Eq.(7) and Eq.(8), the problem of maximizing probability of reaching the target set in $N$ steps is equivalent to problem (13). This is a consequence of the results in [3].

V. SEMIDFINITE RELAXATIONS

In this section, a sequence of semidefinite programs is provided which can arbitrarily approximate the optimal solution of Problem 2. Unlike Problem 2 in which we are looking for a measure, in the provided semidefinite program, we aim at finding a sequence of moments of a measure that satisfies the criteria of Problem 2. One should note that looking for a sequence of moments associated with one measure, is equivalent to looking for the measure itself. Proceeding as in [3], this leads to the following finite dimensional approximation.

**Problem 3:** Let $y = (y_0), y_b = (y_{b_0})$ be a sequence with appropriate dimension, and defined semialgebraic sets $K_1, K_2$. Consider the sequence of semidefinite programs as:

$$P^{3i} = \sup_{y, y_b} y_0$$

subject to

$$M_i(y) \geq 0, M_{i-r_j}(g(h(.)|y) \geq 0$$

$$M_i(y_b) \geq 0, M_{i-r_j}(p(.)|y_b) \geq 0$$

$$M_i(\tilde{y} - y) \geq 0$$

where $\tilde{y} = (y_{b_0})$ are the moments of measure $\tilde{\mu} = \mu_b \times \mu_{x_0} \times \mu_\delta \times \prod_{k=0}^{N-1} \mu_{\omega_k}$, if one assumes that $y_b$ are the moments of a measure $\rho_b$. Given the fact that $y_{x_0}, y_{\delta}, y_{\omega_k}$, sequence of moments of measures $\mu_{x_0}, \mu_\delta, \mu_{\omega_k}$, are given $\tilde{y}$ is a linear transformation of $y_b$. The sequence of programs provided above converges to the solution of the original problem. More precisely, we have the following result.

**Theorem 3:** Optimal value of problem $P^{3i}$ converges to optimal value of problem $P^2$ as $i \to \infty$. 

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Sketch of proof: It can be shown that $y, y_b$ in Problem 3 are bounded and converge to the sequence of moments of measures $\mu$ and $\mu_0$ satisfying the optimal value of Problem 3 in the weak * topology $\sigma(l_\infty, l_1)$ sense; see [3].

VI. NUMERICAL RESULTS

We now present two numerical examples that illustrate that the proposed semidefinite relaxations are effective in finding an appropriate control input even with lower order relaxations. Further research is needed to develop more efficient numerical implementations and study their behavior. Matlab toolboxes Yalmip [20] and Gloptipoly [21] are employed to solve the semidefinite programs (11) and (14), respectively. See the Appendix for code that implements the approach provided in this paper.

Example 1: Consider the uncertain systems as:

$$x_1(k+1) = \delta x_2(k)$$
$$x_2(k+1) = x_1(k) + 2x_2(k) + u(k) + \omega(k)$$

$$x_0 \sim U[-10, 10]^2, \delta \sim U[-0.5, 0.5], \omega(k) \sim U[-0.4, 0.4]$$

where, uncertain initial state $x_0$, model parameter $\delta$, and disturbance $\omega$ has uniform probability distribution $U$. We aim at leading the system using state feedback control $u(k) = -b_1x_1(k) - b_2x_2(k)$ to the unit circle centered at the origin in at most 2 steps, in presence of uncertainties. Solving the semidefinite program (11), the semialgebraic set $P_d$ for $d = 2$ is obtained as (Fig. 1):

$$P_d = \{ b : -6.46 + 2.45b_1 + 5.35b_2 - 1.22b_1^2 - 0.099b_1b_2 - 1.33b_2^2 \geq 0 \}$$

Solving semidefinite program (14) with relaxation order $i = 6$, the obtained optimal probability is 1. Using the obtained optimal $y, y_b$, the control is

$$u(k) = -0.99x_1(k) - 1.99x_2(k)$$

Where, $[0.99, 1.99]$ are the moments order one, from the moments sequence $y_i$. Applying the obtained control input to the uncertain system (15), with probability one, the trajectories of the system for all initial states from the box $[-10, 10]^2$ will reach and remain in a unit ball, in presence of model uncertainty and disturbances; see Fig. 2.

Example 2: Consider the uncertain systems as:

$$x_1(k+1) = x_2(k)$$
$$x_2(k+1) = x_1(k)x_2(k) + u(k) + \omega(k)$$

$$x_0 \sim U[-5, 5]^2, \omega(k) \sim U[-0.5, 0.5]$$

where, uncertain initial state $x_0$, model parameter $\delta$, and disturbance $\omega$ has uniform probability distribution $U$. We aim at leading the state of the system to unit box $[-1, 1]^2$ at most in 2 steps, in presence of uncertainties. Since, the system consists of polynomial with order two, we use a state feedback control of the form $u(k) = b_1x_1(k)^2 + b_2x_1(k)x_2(k) + b_3x_2(k)^2$ to control the system. Solving the semidefinite programs (11) and (14), with relaxation order $i = 6$, the obtained optimal probability is 1. Using the obtained optimal $y, y_b$, the control input is:

$$u(k) = 0.98x_1(k)^2 - 0.94x_1(k)x_2(k) - 0.98x_2(k)^2$$

State response of the system under obtained control input is provided in Fig 3.
VII. CONCLUSION

In this paper, we present a novel approach to the chance constrained controller design when the objective is to reach a given target set with high probability. A sequence of semidefinite relaxations is provided whose solution converge to the optimum of the original problem. Two examples are provided that show that, even for low order relaxations, one obtains a good approximation of the optimum value of the design parameter. Further research effort is now being put in two areas: i) chance robust control for continuous-time uncertain polynomial systems and ii) development of more efficient algorithms to solve the semidefinite problems involved in estimating probabilities.

VIII. APPENDIX

The Yalmip and Gloptipoly codes for example 1:

```matlab
% Example 1: Chance Constrained Controller Design
% Authors: C. Lagoa, X. Li, M. Sznaier
%
% Problem Statement:
% Design a controller for a linear system such that the probability of
% staying within a target set is at least 95%.
% The system is subject to uncertain disturbances.
% % PROBLEM STATEMENT
% % Design a controller for a linear system such that the probability of
% % staying within a target set is at least 95%.
% % The system is subject to uncertain disturbances.
% % The controller design problem is formulated as a chance constrained
% % optimization problem.
% % % SOLUTION
% % The solution is obtained using semidefinite relaxations.
% % The feasibility of the relaxed problem is checked numerically.
% % % REFERENCES
% % [1] C. M. Lagoa, X. Li, and M. Sznaier, "Probabilistically constrained
% % linear programs and risk-adjusted controller design", SIAM J. Optim.,
% % 1995.
% % Constrained Algebraic Problems", 51st IEEE Conference on Decision
% % and Control, Maui, Hawaii, 2012.
% % [4] Jean B. Lasserre, "Global optimization with polynomials and the
% % [5] Jean B. Lasserre, Moments Positive Polynomials and Their Applications,
% % Imperial College Press, 2010.
% % [6] Yasunawa Fujisaki,Yasuaki Kozawa, "Probabilistic Robust Controller
% % Design: Probable Near Minimax Value and Randomized Algorithms",
% Constrained Controller in Uncertain Environments", 51st IEEE Conference
% on Decision and Control, Maui, Hawaii, 2012.
% Uncertain Systems", Springer Journal on Probabilistic and Randomized
% [10] Giuseppe C. Calafiore, Fabrizio Dabbeni, and Roberto Tempo, "Randomized
% Algorithms for Probabilistic Robustness with Real and Complex
% [12] Daniel Lyons, Jan-P. Calliess, and Uwe D. Hanebeck, "Chance
% Constrained Model Predictive Control for Multi-Agent Systems with
% Coupling Constraints", American Control Conference, Canada, 2012.
% [13] Masahiro Ono, Lars Blackmore, and Brian C. Williams, "Chance
% Constrained Infinite Horizon Optimal Control with Nonconvex Constraints",
% American Control Conference, Canada, 2012.
% [16] Lars Blackmore, Masahiro Ono, Askar Bektassov and Brian C.
% Williams, "A Probabilistic Particle Control Approximation of Chance
% [18] Rida Laraki, Jean B. Lasserre, "Semidefinite programming for min-
% [19] Jean B. Lasserre, Tung Phan Thanh, "A "joint + marginal" heuristic
% [22] D. Henrion, J. B. Lasserre, J. Loefberg, "GloptiPoly 3: moments,
```