Robust observer design for a class of stochastic nonlinear systems
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Abstract—This contribution deals with the robust observer design of some nonlinear stochastic systems with multiplicative noises. The drift part of the considered systems contains a linear part, a bilinear one and a Lipschitz nonlinearity. The uncertainties that affect the system concern all the matrices of the dynamics of this one. The parametric uncertainties under consideration are unstructured and norm-bounded. The observer design is formulated in terms of LMI to be satisfied in order to obtain the observer gains. Our approach ensures the almost sure exponential stability of the estimation error which is less conservative than the usual mean-square exponential stability.

I. INTRODUCTION

This paper is devoted to the robust observer design for a class of uncertain nonlinear Itô stochastic systems controlled by multiplicative noises which is studied in [1]. These noises are Brownian motions, also called Wiener processes. The dynamics of the stochastic systems is modeled by a Stochastic Differential Equation (SDE) with a random term describing the randomness within the systems to describe. The model of the considered systems contains two parts: the drift one which corresponds to the dominant action of the system and the diffusion one representing randomness along the dominant behavior. The stochastic modelling plays a great role in engineering and sciences (see [2], [3], [4] and references therein). In addition, modelling systems with stochastic differential equations is more realistic when the deterministic description is not satisfactory.

The drift of the considered stochastic systems has three parts: a linear and bilinear ones and a Lipschitz nonlinearity, while the Brownian motion is multiplied with the state in the diffusion. The parametric uncertainties under consideration are unstructured and norm-bounded. These uncertainties should result from system identification, model reduction, …, and affect the system matrices. The interest of building robust observer is to get observer that still ensures a good estimation of the state even if there are parameter uncertainties. We propose to design a full order robust observer using gain matrices that are obtained by the resolution of some LMI. These LMI are obtained by using Lyapunov method which permits to ensure the almost sure exponential stability of the observation error dynamics. Indeed this notion of stability should be less restrictive than the usual mean-square stability: the equilibrium point of a SDE can be mean-square exponentially unstable whereas it is almost surely exponentially stable [1]. It should be noticed that, in our knowledge, almost all works dedicated to the observer design for stochastic systems with multiplicative noise are based on the mean-square exponential stability of the observation error [5], [6], [7], [8], [9], [10], [11], [12], [13], [14], [15], [16], [17].

This paper is organised as follows. Section II states the robust observer problem where some preliminaries results on stochastic systems are given in section II-A, while the robust design problem is presented in section II-B. The analysis of the almost sure exponential stability of the filtering error is given in section III. The section IV concerns the observer design via the resolution of LMI for unstructured parametric uncertainties.

Notations. \(\mathbb{R}^n\) denote the n-dimensional euclidean space. \(\|A\| = \sqrt{\sum_{i,j} A_{i,j}^2}\) is the Euclidean norm of the matrix A, while \(\|x\| = \sqrt{x^T x}\) is the Euclidean norm of the vector x. \(a \vee b\) is the maximum of reals a and b. For a symmetric matrix \(A, A > 0\) means that the matrix A is positive definite. Symbols \(<\), \(\leq\) and \(\geq\) for matrices are defined similarly. “a.s.” means almost surely.

II. PROBLEM STATEMENT

We consider the following uncertain stochastic system

\[
\begin{align*}
\dot{x}(t) &= ((A_{0} + \Delta A_{0}) x + \ell(x) + \sum_{i=1}^{m} u_{i}(t)(A_{i} + \Delta A_{i}) x) dt \\
&\quad + (A_{w0} + \Delta A_{w0}) x d w_{x}(t) \\
\dot{y}(t) &= C x(t) dt + D x(t) d w_{y}(t)
\end{align*}
\]

where \(x(t) \in \mathbb{R}^{n}\) is the state vector, \(u(t) \in \mathbb{R}^{m}\) is a known input vector, \(y(t) \in \mathbb{R}^{p}\) is output vector and \(w_{x}(t) \in \mathbb{R}\) and \(w_{y}(t) \in \mathbb{R}\) are independent brownian motions.

The function \(\ell(x(t))\) verifies the following Lipschitz condition

\[
\|\ell(x(t)) - \ell(x(t))\| \leq \kappa\|x(t) - x(t)\|
\]

with \(\kappa > 0\) and with \(\ell(0) = 0\). We also suppose that the control u(t) is bounded as follows

\[
\Omega_{u} = \{u(t) \in \mathbb{R}^{m} \mid u_{i,\text{min}} \leq u_{i}(t) \leq u_{i,\text{max}}, \; i = 1, \ldots, m\}.
\]

The matrices \(\Delta A_{0}, \Delta A_{i}\) and \(\Delta A_{w0}\) represent the parametric uncertainties that affect the stochastic nonlinear system and verify the following constraint \((i = 1, \ldots, m)\)

\[
[\Delta A_{0} \Delta A_{i} \Delta A_{w0}] = F(t)[H_{A_{0}} \ H_{A_{i}} \ H_{A_{w0}}]
\]

where \(\Delta(t) \in \mathbb{R}^{r \times j}\) is an unknown matrix with Lebesgue measurable elements satisfying \(\Delta(t)\Delta^{T}(t) \leq {I}_r\), and where

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F, \( H_{A_i0} \), \( H_{A_j} \), and \( H_{A_{w0}} \) are known given constant matrices of appropriate dimensions.

### A. Preliminaries results

In this section, we recall some results on almost sure exponential stability that will be useful in the sequel. For this, let us consider the following general nonlinear stochastic systems which is similar to (1)

\[
\begin{align*}
\dot{x}(t) &= f(x(t)) dt + g(x(t)) dw(t) \\
\dot{y}(t) &= h(x(t)) dt + q(x(t)) dw(t)
\end{align*}
\]

(5a)

(5b)

The functions \( f(x(t)) \) and \( h(x(t)) \) are Lebesgue integrable and the functions \( g(x(t)) \) and \( q(x(t)) \) are Lebesgue square-integrable as it is needed for Itô calculus [1], [4].

To guarantee the existence and the uniqueness of the solution \( x(t) \) of the SDE (1a), the functions \( f(x(t)) \) and \( g(x(t)) \) satisfy the following relations for all \( t \in \mathbb{R}^n \) and \( \forall \tau(t) \in \mathbb{R}^m \) (see [1], [4])

\[
\|f(x(t))\|^2 + \|g(x(t))\|^2 \leq k_1 (1 + \|x(t)\|^2), \\
\|f(x(t)) - f(\tau(t))\| + \|g(x(t)) - g(\tau(t))\| \leq k_2 \|x(t) - \tau(t)\|
\]

where \( k_1 \) and \( k_2 \) are given strictly positive reals.

The concept of almost sure exponential stability is defined as follows.

**Definition 1:** ([1], [18]) The equilibrium point of the EDS (5) is almost surely exponentially stable if \( \exists \beta > 0 \) such that

\[
\limsup_{t \to +\infty} \frac{1}{t} \ln(\|x(t)\|) < -\beta \quad a.s.
\]

(6)

for any \( x_0 \in \mathbb{R}^n \), where \( \limsup_{t \to +\infty} \frac{1}{t} \ln(\|x(t)\|) \) is the Lyapunov exponent of the solution \( x(t) \).

Let the function \( V(x(t)) \) from \( \mathbb{R}^n \) to \( \mathbb{R}^+ \) be a Lyapunov function candidate. To analyze the almost sure exponential stability by using a Lyapunov approach, we apply the Itô formula to \( V(x(t)) \) and we obtain [1], [4]

\[
dV(x(t)) = \mathbb{D}V(x(t)) dt + 2\mathbb{V}(x(t)) dw(t)
\]

(7)

where

\[
\begin{align*}
\mathbb{D}V(x(t)) &= \frac{\partial V}{\partial x}(x(t)) f(x(t)) + \frac{1}{2} g^T(x(t)) \frac{\partial^2 V}{\partial x^2}(x(t)) g(x(t)), \\
2\mathbb{V}(x(t)) &= \frac{\partial V}{\partial x}(x(t)) g(x(t)).
\end{align*}
\]

The following theorem give sufficient conditions for almost sure exponential stability.

**Theorem 1:** ([1], [18]) If it exist a positive definite Lyapunov function \( V(x(t)) \) and some constants \( c_0 > 0 \), \( c_1 > 0 \), \( c_2 \in \mathbb{R} \) and \( c_3 \geq 0 \) such that

\[
\begin{align*}
&c_1 \|x(t)\|^{c_0} \leq V(x(t)), \quad &\text{(8a)} \\
&2\mathbb{V}(x(t)) \leq c_2 V(x(t)), \quad &\text{(8b)} \\
&\|2\mathbb{V}(x(t))\| \geq c_3 V^2(x(t)), \quad &\text{(8c)}
\end{align*}
\]

then

\[
\limsup_{t \to +\infty} \frac{1}{t} \ln(\|x(t)\|) \leq \frac{2c_2 - c_3}{2c_0} \quad a.s. \quad \forall x_0 \in \mathbb{R}^n
\]

(9)

and the equilibrium point of (5) is almost surely exponentially stable if

\[
c_3 > 2c_2.
\]

(10)

The constant \( c_2 \) in the theorem 1 can be negative, positive or zero, i.e. \( \mathcal{L}V(x(t)) \) can be non definite or positive definite. In the other hand, for another types of stochastic stability (such mean-square exponential stability), \( \mathcal{L}V(x(t)) \) must be negative definite, but it is not the case for the almost sure exponential stability (see (8b)). In addition, under some usual conditions [1], the mean-square exponential stability implies the almost sure exponential stability.

### B. The robust observer design problem

In this paper, our goal is to design a robust observer given by the following equation

\[
\begin{align*}
\tilde{d}x(t) &= \left( A_{\delta} \tilde{x}(t) + \sum_{i=1}^{m} u_i(t)A_i \tilde{x}(t) + \ell(\tilde{x}(t)) \right) dt \\
&\quad + \sum_{i=1}^{m} \left[ K_0 + \sum_{i=1}^{m} K_i u_i(t) \right] (dy(t) - C \tilde{x}(t)) dt
\end{align*}
\]

(11)

such that the filtering error is almost surely exponentially stable in spite of the presence of the uncertainties. Notice that \( K_i \) are the gain matrices to determine in order to fulfill this objective.

### III. Analysis of the almost sure exponential stability of the filtering error

The filtering error \( e(t) = x(t) - \tilde{x}(t) \) is driven by the following SDE

\[
\begin{align*}
\dot{e}(t) &= \left( A_{\delta} e(t) + \sum_{i=1}^{m} (A_i - K_i C) u_i(t) \right) e(t) \\
&\quad + \ell(x(t)) - \ell(x(t) - e(t))) dt + (A_{\delta} e(t) + \sum_{i=1}^{m} \Delta A_i(t) x(t) dt \\
&\quad + (A_{w0} + \Delta A_{w0}) x(t) dw(t) - \left[ K_0 + \sum_{i=1}^{m} K_i u_i(t) \right] D x(t) dw(t) \quad \text{(12)}
\end{align*}
\]

We define the following expressions with \( u_0 = 1 \) (i = 0, \ldots, m)

\[
\mathcal{A}_0(u(t)) = \begin{bmatrix} (A_i + \Delta A_i) & 0 \\ 0 & A_i - K_i C \end{bmatrix} u_i(t), \\
\mathcal{A}_i(u(t)) = \sum_{i=0}^{m} \mathcal{A}_i(u(t)) + \mathcal{H}(u(t)) = \sum_{i=0}^{m} \begin{bmatrix} 0 & 0 \\ K_i D \end{bmatrix} u_i(t), \\
\mathcal{A}_{w0} = \begin{bmatrix} (A_{w0} + \Delta A_{w0}) & 0 \\ 0 & (A_{w0} + \Delta A_{w0}) \end{bmatrix}, \quad L(X)(t) = \begin{bmatrix} \ell(x(t)) \\ \ell(x(t) - e(t)) \end{bmatrix}, \quad X(t) = \begin{bmatrix} x(t) \\ e(t) \end{bmatrix}
\]

and the SDE (1a) and (12) can be written in the following compact form

\[
\begin{align*}
\dot{d}X(t) &= \left( \mathcal{A}_0(u(t)) X(t) + \mathcal{H}(u(t)) \right) dt \\
&\quad + \mathcal{A}_{w0} X(t) dw_x(t) - \mathcal{H}(u(t)) X(t) dw_y(t) \quad \text{(13)}
\end{align*}
\]

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With the quadratic Lyapunov function \( V(X(t)) = X^T(t)PX(t) \) where \( P = PT > 0 \), we obtain
\[
\mathcal{B}V(X(t)) = X^T(t)(A^T_wP + \mathcal{P}A_{w0})X(t)
\] (14)
where \( \mathcal{B}V(X(t)) \in \mathbb{R}^{2n \times 2n(d+1)} \).

With the form of the structure of the terms involved in the SDE (13), the relation (14) gives
\[
\frac{\partial V(X(t))}{\partial X} = \left[ \begin{array}{c} 0 \\ \mathcal{B}V(X(t)) \end{array} \right]
\] (15)
and the condition (8c) of theorem 1 implies that \( c_3 = 0 \). So, if \( 2V(x(t)) \geq 0 \) (then \( c_2 > 0 \) in (8b)), the condition \( c_3 > 2c_2 \) in this theorem will not be verified.

To resolve this problem, we are going to exploit the triangular structure of the SDE (13) using a theorem we proposed by Barbata et al. [19] (see theorem 2).

If we put
\[
f_1(x(t), u(t)) = \left( A_{w0} + \Delta A_{w0} + \sum_{i=1}^{m} u_i(t)(A_{i} + \Delta A_{i}) \right) x(t) + \ell(x(t)),
\]
\[
g_1(x(t)) = \left[ (A_{w0} + \Delta A_{w0})x(t) \right],
\]
\[
f_2(x(t), e(t), u(t)) = \left( A_{i} - K_iC + \sum_{i=1}^{m} (A_{i} - K_iC)u_i(t) \right) e(t) + \ell(x(t)) - \ell(x(t) - e(t))
\]
\[
+ (\Delta A_{i0} + \sum_{i=1}^{m} \Delta A_{i})x(t),
\]
\[
g_2(x(t), u(t)) = \left[ (A_{w0} + \Delta A_{w0})x(t) \right],
\]
\[
w(t) = \left[ w_1(t) \\ w_2(t) \right],
\]
with
\[
g_2(x(t), u(t)) = K_0x(t) + \sum_{i=1}^{m} K_iu_i(t)x(t),
\]
the SDE (13) becomes
\[
dx(t) = f_1(x(t), u(t)) dt + g_1(x(t)) dw(t) \quad (16a)
\]
\[
de(t) = f_2(x(t), e(t), u(t)) dt + g_2(x(t), u(t)) dw(t) \quad (16b)
\]

In association to the triangular SDE (16), we consider the following “decoupled” SDE
\[
dx(t) = f_1'(x(t), u(t)) dt + g_1'(x(t)) dw(t) \quad (17a)
\]
\[
d\tilde{x}(t) = f_2'(0, \tilde{x}(t), u(t)) dt \quad (17b)
\]
and we make the following assumption.

**Assumption 1:** It exists a real \( k > 0 \) such that, \( \forall t \geq 0 \),
\[
\|f_2(x(t), e(t), u(t)) - f_2(0, \tilde{x}(t), u(t))\| \\
\leq k(\|x(t)\| + \|e(t) - \tilde{x}(t)\|),
\] (18)
\[
\text{tr}\left((g_1(x(t)) - g_1(\tilde{x}(t)))(g_1(x(t)) - g_1(\tilde{x}(t)))^T\right) \\
\leq k\|x(t) - \tilde{x}(t)\|^2, \quad (19)
\]
\[
\text{tr}\left((g_2(x(t), u(t)) - g_2(\tilde{x}(t), u(t))) \\
\times (g_2(x(t), u(t)) - g_2(\tilde{x}(t), u(t)))^T\right) \\
\leq k\|x(t) - \tilde{x}(t)\|^2. \quad (20)
\]

**Theorem 2:** ([19]) With assumption 1, the equilibrium point of SDE (16) is almost surely exponentially stable if and only if the equilibrium point of SDE (17) is almost sure exponentially stable.

Since \( u(t) \) is bounded (see (3)) and due to assumptions done for the system (1), the assumption (1) is satisfied.

Theorem 2 can be applied to the full-order observer design as follows.

**Theorem 3:** If assumption 1 holds with SDE (13), then the system (11) is a full-order observer for the stochastic system (1) guaranteeing the almost sure exponential stability of the filtering error if the SDE (1a) is almost surely exponentially stable and if there exist gain matrices \( K_0, \ldots K_m \) such that the ODE
\[
e(t) = \left( A_{0} - K_0C + \sum_{i=1}^{m} (A_{i} - K_iC)u_i(t) \right) e(t) - \ell(-e(t)) \quad (21)
\]
is exponentially stable.

**Proof:** As the SDE (13) and (16) have the same structure, the proof is reduced to apply theorem 2 to the SDE (13).

To reduce the conservatism in the determination of the stability conditions, the control input \( u(t) \) is replaced by another variable \( e(t) \) which range of variation is between \(-1 \) and \(1\).

So, using \( \Omega_u \) given by (3), we define \( e(t) \) like this
\[
u_i(t) = \alpha_i + \sigma_i e_i(t) \quad (22)
\]
where
\[
\alpha_i = \frac{u_{i,max} + u_{i,min}}{2} \quad \text{and} \quad \sigma_i = \frac{u_{i,max} - u_{i,min}}{2}
\]
for \( i = 1, \ldots, m \), with \( \alpha_0 = 1 \) and \( \sigma_0 = 0 \). The “new” control input \( e(t) \) is bounded as follows
\[
\Omega_e = \{ e(t) \in \mathbb{R}^m \mid -1 \leq e_i(t) \leq 1, \ i = 1, \ldots, m \} \quad (23)
\]
and the matrix \( \Delta_e(t) \) defined by
\[
\Delta_e(t) = \text{bdiag}(e_1(t)I_n(t), \ldots, e_m(t)I_n(t)) \quad (24)
\]
verifies
\[
\Delta_e(t) \Delta_e(t)^T \leq I_{mn}. \quad (25)
\]

Using the notations introduced in (22)-(24), the SDE (1a) and ODE (21) are rewritten as follows
\[
dx(t) = \left( \sum_{i=0}^{m} \alpha_i(A_{i} + \Delta A_{i})x(t) + H_1x(t)H_2 + \ell(x(t)) \right) dt \\
+ (A_{w0} + \Delta A_{w0})x(t) dw(t) \quad (26)
\]
and
\[
e(t) = \left( \sum_{i=0}^{m} (A_{i} - K_iC)\alpha_i + (H_1 - \mathcal{K}e')\Delta_e(t)H_2 \right) e(t) \\
- \ell(-e(t)) \quad (27)
\]
with
\[ H_1 = [\sigma_1 A_{i1}, \ldots, \sigma_m A_{im}], \quad H_2 = [I_n] \]
\[ \mathcal{K} = [K_1, \ldots, K_m], \quad \mathcal{C} = \text{bdig}(\sigma_1 C, \ldots, \sigma_m C). \]

Remark 1: In the sequel, the new control input \( e(t) \) is treated as an uncertainty.

IV. ROBUST OBSERVER DESIGN VIA LMI

In this part, our purpose is to design a robust observer (11) for system (1) when it is subjected to parametric unstructured uncertainties. The following theorem gives the solution of the observer design problem for the stochastic system (1) such that the estimation error is almost sure exponential stable.

Theorem 4: Consider that assumption 1 is verified. Then the system (11) is an observer for the uncertain system (1) such that the observation error is almost surely exponentially stable if it exist matrices \( P = P^T > 0 \), \( Q = Q^T > 0 \) and \( Y_i \) (for \( i = 0, \ldots, m \)), and reals \( \mu_1 > 0 \), \( \mu_2 > 0 \), \( \mu_3 > 0 \), \( \mu_4 > 0 \), \( \mu_5 > 0 \), \( \mu_6 > 0 \) and \( \mu_7 > 0 \) such that the following LMI

\[
\begin{bmatrix}
(1,1)_{\alpha} & \sigma_1 P A_{i1} & \ldots & \sigma_m P A_{im} & P & PF & A_{i1}^T P & 0 \\
(1,2)^T & -\mu_1 I_n & 0 & 0 & 0 & 0 & 0 & 0 \\
\vdots & 0 & \ddots & 0 & 0 & 0 & 0 & 0 \\
(1,m)^T & 0 & \ldots & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \mu_1 I_n & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\end{bmatrix} < 0
\]

(28)

are verified and if one of the following LMI

\[
\begin{bmatrix}
A_{i0}^T P + PA_{i0} + \mu_2 H_{i0}^T A_{i0} H_{i0} + \sqrt{2} P P & PF & A_{i0}^T P \\
(1,1)_{\alpha} & -\mu_2 I_n & 0 & 0 & 0 & 0 & 0 \\
(1,2)^T & 0 & \ddots & 0 & 0 & 0 & 0 \\
(1,m)^T & 0 & \ldots & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\end{bmatrix} < 0
\]

(29)

is satisfied where \( \rho > 0 \) is a given real, \( \alpha = \sum_{i=0}^{m} \alpha_i \) with \( \alpha_0 = 1 \), and

\[
(1,1)_m = \sum_{i=0}^{m} \alpha_i (A_{i0} + A_{i0}^T P) H_{i0}^T H_{i0} + \mu_3 \sum_{i=0}^{m} \alpha_i (H_{i0})^T (H_{i0}) \\
+ \mu_4 I_n + \mu_5 \kappa^2 I_n + \mu_6 H_{i0}^T A_{i0} H_{i0} - \rho P, \\
(1,1)_b = \sum_{i=0}^{m} \alpha_i ((Q A_{i0} - Y_i C) + (Q A_{i0} - Y_i C)^T) \\
+ (\mu_6 + \mu_7 \kappa^2) I_n.
\]

The gains \( K_i \) are then given by \( K_i = Q^{-1} Y_i \) for \( i = 0, \ldots, m \).

Proof: Using theorem 3, the proof can be splitted into two parts: first the almost sure exponential stability of the SDE (1a) is shown in the part 1, second the exponential stabilization of the ODE (21) is given in part 2.

Part 1: We are going to show that the equilibrium point of the SDE (1a) is almost surely exponentially stable by using the Lyapunov function candidate \( v_i(x(t)) = x(t)^T P x(t) \) with \( P = P^T > 0 \).

The inequality (8a) holds if we choose \( c_0 = 2 \) and \( c_1 = \lambda_{\text{min}}(P) \) where \( \lambda_{\text{min}}(P) \) is the smallest eigenvalue of \( P \).

The Itô formula applying to the SDE (1a) with the Lyapunov function \( V(x(t)) \) gives

\[
\mathcal{L} V_i(x(t)) = x(t)^T (\sum_{i=0}^{m} \alpha_i (PA_{i0} + \Delta A_{i0}) + (A_{i0} + \Delta A_{i0})^T P) \\
+ (A_{i0} + \Delta A_{i0})^T P (A_{i0} + \Delta A_{i0})) x(t) \\
+ 2x(t)^T P H_i A_{i0} H_i x(t) + 2x(t)^T P e(x(t)).
\]

(32)

Using inequalities given by (2), (25) and (41), the relation (32) becomes

\[
\mathcal{L} V_i(x(t)) \leq x(t)^T (\sum_{i=0}^{m} \alpha_i (PA_{i0} + \Delta A_{i0}) + (A_{i0} + \Delta A_{i0})^T P) \\
+ (A_{i0} + \Delta A_{i0})^T P (A_{i0} + \Delta A_{i0}) + \mu_4^{-1} P H_i H_i^T P \\
+ \mu_4 H_i^2 H_i + \mu_2 \kappa^2 I_n + \mu_2 PP - \rho P < 0
\]

(33)

is satisfied.

Since we have \( H_i^2 H_i = I_n \), \( \Delta A_{i0} = F \Delta(t) H_{i0} \) and \( \Delta A_{i0} = F \Delta(t) H_{i0} \), the inequality (34) becomes

\[
\sum_{i=0}^{m} \alpha_i (PA_{i0} + \Delta A_{i0}) + \sum_{i=0}^{m} \alpha_i (P F \Delta(t) H_{i0}) + (F \Delta(t) H_{i0})^T P) \\
+ (A_{i0} + F \Delta(t) H_{i0})^T P (A_{i0} + F \Delta(t) H_{i0}) + \mu_1 I_n \\
+ \mu_4^{-1} \kappa^2 I_n + \mu_2 PP - \rho P < 0
\]

(35)

Using the inequalities (41) and (42) yields

\[
2x(t)^T \sum_{i=0}^{m} \alpha_i (F \Delta(t) H_{i0})^T P) x \leq \mu_3 \sum_{i=0}^{m} \alpha_i (H_{i0})^T (H_{i0}) x \\
+ x(t)^T \mu_4 \sum_{i=0}^{m} \alpha_i P FF^T P x, \\
(A_{i0} + \Delta A_{i0})^T P (A_{i0} + \Delta A_{i0}) \leq A_{i0}^T (P - \mu_4 FF^T) A_{i0} \\
+ \mu_4^{-1} H_{i0}^T H_{i0},
\]

(36)

\[ \mathcal{L} v_i(x(t)) \geq \mu_5 \sum_{i=0}^{m} \alpha_i (PA_{i0} + \Delta A_{i0}) + (A_{i0} + \Delta A_{i0})^T P) \\
+ (A_{i0} + \Delta A_{i0})^T P (A_{i0} + \Delta A_{i0}) + \mu_4^{-1} P H_i H_i^T P \\
+ \mu_4 H_i^2 H_i + \mu_2 \kappa^2 I_n + \mu_2 PP - \rho P < 0
\]

(34)
and \( \sum_{i=0}^{m} \sum_{j=0}^{m} \alpha_i \frac{A_{ij}}{H_{ij}^T} + P(A_{w0} + H_{w0}) \) is equivalent to

\[
\sum_{i=0}^{m} \alpha_i (PA_{ti} + A_{ti}^T P) + \mu_1 H_{1i}^T P + \mu_1 I_n + \mu_2^{-1} \kappa_2 I_n \\
+ \mu_2 PP + \mu_3 \sum_{i=0}^{m} \alpha_i (H_{ti}) (H_{ti})^T \\
+ \mu_4 \sum_{i=0}^{m} \alpha_i (H_{ti})^T (H_{ti}) \\
+ \mu_5 \sum_{i=0}^{m} \mu_4^{-1} H_{ti}^T H_{w0} - \rho P < 0 \quad (36)
\]

Applying the Schur lemma [20] on inequality (36) gives the LMI (28) with \( \alpha = \sum_{i=0}^{m} \alpha_i \). If the LMI (28) holds, then the condition (8b) is satisfied.

The application of \( \mathcal{B}_i \) to the stochastic differential equation (26) gives

\[
\mathcal{B}_i = x^T(t)(A_{w0} + \Delta A_{w0})^T P + P(A_{w0} + \Delta A_{w0}) x(t)
\]

Since we have

\[
[\mathcal{B}_i(x(t))]^2 - c_i V_i(x(t)) = x^T(t)(A_{w0} + \Delta A_{w0})^T P + P(A_{w0} + \Delta A_{w0}) + \sqrt{2\rho} P < 0,
\]

the condition (8c) is verified if it exists a real \( c_3 = 2\rho \) such that one of the two following LMI

\[
(A_{w0} + \Delta A_{w0})^T P + P(A_{w0} + \Delta A_{w0}) + \sqrt{2\rho} P < 0 \quad (37a)
\]

\[
(A_{w0} + \Delta A_{w0})^T P + P(A_{w0} + \Delta A_{w0}) - \sqrt{2\rho} P > 0 \quad (37b)
\]

is satisfied.

The inequality (37a) can be rewritten as

\[
A_{w0}^T P + PA_{w0} + \Delta A_{w0}^T P + P\Delta A_{w0} + \sqrt{2\rho} P < 0.
\]

Using the inequality (41), the term \( x^T(t)(\Delta A_{w0})^T P + P\Delta A_{w0}) x(t) \) can be bounded as follows

\[
2x^T(t)\Delta A_{w0} x(t) = 2x^T(t)P\Delta A_{w0} x(t) \\
\leq x^T(t)\left(\mu_5 PFF^T P + \mu_5^{-1} H_{w0}^T H_{w0} \right)x(t)
\]

with \( \mu_5 > 0 \) and the inequality (37a) holds if we have

\[
A_{w0}^T P + PA_{w0} + \mu_5 PFF^T P + \mu_5^{-1} H_{w0}^T H_{w0} + \sqrt{2\rho} P < 0.
\]

Applying the Schur lemma to the previous inequality leads to the LMI (30).

If we put \( \Delta(t) = -\Delta(t) \) and \( \Delta(t)^T \Delta(t) \leq I_2 \), then the inequality (41) gives

\[
2x^T(t)\Delta A_{w0} x(t) = 2x^T(t)P\Delta A_{w0} x(t) \\
\leq x^T(t)\left(\mu_5 PFF^T P + \mu_5^{-1} H_{w0}^T H_{w0} \right)x(t)
\]

Since \( \Delta A_{w0} = -F\Delta(t) H_{w0} \), the previous inequality can be rewritten as

\[
2x^T(t)\Delta A_{w0} x(t) \leq x^T(t)\left(-\mu_5 PFF^T P - \mu_5^{-1} H_{w0}^T H_{w0} \right)x(t)
\]

where \( \mu_5 > 0 \) and the inequality (37b) holds if we have

\[
A_{w0}^T P + PA_{w0} - \mu_5 PFF^T P - \mu_5^{-1} H_{w0}^T H_{w0} - \sqrt{2\rho} P < 0
\]

or equivalently

\[
-(A_{w0}^T P + PA_{w0}) + \mu_5 PFF^T P + \mu_5^{-1} H_{w0}^T H_{w0} + \sqrt{2\rho} P < 0.
\]

Applying the Schur lemma to the previous inequality leads to the LMI (31).

Since \( c_2 < \rho \) and \( c_3 = 2\rho \), the inequality (10) is satisfied and the almost sure exponential stability of the equilibrium point of the SDE (1a) is proved.

Part 2: Following theorem 3, the second part of the proof is devoted to the exponential stability of the ODE (21) by using a Lyapunov function candidate \( V_\epsilon(t) = e^T(t)Qe(t) \) with \( Q = Q^T > 0 \).

By rewriting the ODE (21) as in equation (27), the time derivative of \( V_\epsilon(t) \) along the trajectory of the ODE (27) is given by

\[
\dot{V}_\epsilon(t) = e^T(t) \sum_{i=0}^{m} \alpha_i ((A_i - K_i)C) + Q(A_i - K_i)C(t) + 2e^T(t)Q(H_1 - K_i)C) + e^T(t)Q\ell(-e(t)).
\]

Using inequalities given by (2), (25) and (41), the inequality (38) becomes

\[
\dot{V}_\epsilon(t) = e^T(t) \sum_{i=0}^{m} \alpha_i ((A_i - K_i)C) + Q(A_i - K_i)C(t) + 2e^T(t)Q(H_1 - K_i)C) + e^T(t)Q\ell(-e(t)).
\]

Then the equilibrium point of the ODE (21) is exponentially stable if the following inequality

\[
\sum_{i=0}^{m} \alpha_i ((A_i - K_i)C) + Q(A_i - K_i)C(t) + 2e^T(t)Q(H_1 - K_i)C) + e^T(t)Q\ell(-e(t)).
\]

is verified where \( Y_i = QK_i \) for \( i = 0, \ldots, m \).

Applying the Schur lemma on inequality (40) leads to the LMI (29).

\[-(A_{w0}^T P + PA_{w0}) + \mu_5 PFF^T P + \mu_5^{-1} H_{w0}^T H_{w0} + \sqrt{2\rho} P < 0.
\]

\[\Box\]

V. Conclusion

In this paper, we proposed a solution to the robust filtering problem for a class of stochastic bilinear systems in controls with multiplicative noises and an additional Lipschitz non-linearity. The system matrices are affected by parametric unstructured norm-bounded uncertainties. The synthesis method developed guarantees the almost sure exponential stability of the equilibrium point of the observation error and is based on the resolution of LMI in spite of the uncertainties. The advantage of our approach is to propose an alternative to the use of exponential stability in mean square by using a less restrictive type of stability. The use of the triangular structure of the filtering problem allowed to decouple the stability of the error dynamics from the stability of the considered stochastic system.
VI. APPENDIX

Lemma 1: (21) We consider three matrices $A \in \mathbb{R}^{n \times q}$, $B \in \mathbb{R}^{p \times n}$ and $C \in \mathbb{R}^{q \times p}$ with $C^T C \preceq I_p$, then, for all real $\mu > 0$, we have

$$2x^T ACBx \leq \mu x^T A^T A x + \frac{1}{\mu} x^T B^T B x$$

(41)

Lemma 2: (22) We consider four matrices $A \in \mathbb{R}^{n \times q}$, $B \in \mathbb{R}^{p \times n}$, $C \in \mathbb{R}^{q \times p}$, $D \in \mathbb{R}^{r \times q}$ and $E \in \mathbb{R}^{r \times p}$ with $C^T C \preceq I_p$, $\det(E) \neq 0$ and $E - \mu A A^T > 0$, then, for all real $\mu > 0$, we have

$$(D + ACB)^T E^{-1} (D + ACB)$$

$$\leq D^T (E - \mu A A^T)^{-1} D + \frac{1}{\mu} B^T B$$

(42)

REFERENCES


