Multi-Input Control-Affine Systems Linearizable via One-Fold Prolongation and Their Flatness

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Abstract—We study flatness of multi-input control-affine systems, defined on an \( n \)-dimensional space. We give a complete geometric characterization of systems that become static feedback linearizable after a one-fold prolongation of a suitably chosen control. They form a particular class of flat systems, that is of differential weight equal to \( n + m + 1 \), where \( m \) is the number of controls. We illustrate our results by an example: the quadrotor helicopter.

I. INTRODUCTION

In this paper, we study flatness of nonlinear control systems of the form
\[
\Sigma : \dot{x} = F(x, u),
\]
where \( x \) is the state defined on an open subset \( X \) of \( \mathbb{R}^n \) and \( u \) is the control taking values in an open subset \( U \) of \( \mathbb{R}^m \) (more generally, an \( n \)-dimensional manifold \( X \) and an \( m \)-dimensional manifold \( U \), respectively). The dynamics \( F \) are smooth and the word smooth will always mean \( C^\infty \)-smooth.

The notion of flatness has been introduced in control theory in the 1990’s, by Fliess, Lévine, Martin and Rouchon [1], [2], see also [3], [4], [5] and has attracted a lot of attention because of its multiple applications in the problem of constructive controllability and motion planning (see, e.g. [6], [7], [8], [9], [10], [11]). Flat systems form a class of control systems, whose set of trajectories can be parametrized by \( m \) functions and their time-derivatives, \( m \) being the number of controls. More precisely, the system \( \Sigma : \dot{x} = F(x, u) \) is flat if we can find \( m \) functions, \( \phi_i(x, u, \ldots, u^{(r)}) \), for some \( r \geq 0 \), called flat outputs, such that
\[
x = \gamma(\phi_1, \ldots, \phi^{(s-1)}) \text{ and } u = \delta(\phi_1, \ldots, \phi^{(s)}),
\]
for a certain integer \( s \), where \( \phi = (\phi_1, \ldots, \phi_m) \). Therefore the evolution in time of all state and control variables can be determined from that of flat outputs without integration and all trajectories of the system can be completely parameterized. A similar notion, of systems of undetermined differential equations integrable without integration, has been studied by Hilbert and Cartan [12], [13], see also [8], where connections between Cartan prolongations and flatness were studied.

Flatness is closely related to the notion of feedback linearization. It is well known that systems linearizable via invertible static feedback are flat. Their description (1) uses the minimal possible, which is \( n + m \), number of time-derivatives of the components of flat outputs \( \phi_i \). In general, a flat system is not linearizable by static feedback, with the exception of the single-input case where flatness reduces to static feedback linearization, see [14]. For any flat system, that is not static feedback linearizable, the minimal number of time-derivatives of \( \phi_i \) needed to express \( x \) and \( u \) (which will be called the differential weight [11]) is thus bigger than \( n + m \) and measures actually the smallest possible dimension of a precompensator linearizing dynamically the system. Therefore the simplest systems for which the differential weight is bigger than \( n + m \) are systems linearizable via one-dimensional precompensator, thus of differential weight \( n + m + 1 \). They form the class that we are studying in the paper: our goal is to give a geometric characterization of control-affine systems that become static feedback linearizable after a one-fold prolongation of a suitably chosen control.

The paper is organized as follows. In Section II, we recall the definition of flatness and define the notion of differential weight of a flat system. In Section III, we give our main results: we characterize control-affine systems that become static feedback linearizable after one-fold prolongation. They form a particular class of flat systems, that is, flat systems of differential weight \( n + m + 1 \). We illustrate our results by an example in Section IV and provide proofs in Section V.

II. FLATNESS

The fundamental property of flat systems is that all their solutions may be parametrized by a finite number of functions and their time-derivatives. Fix an integer \( l \geq -1 \) and denote \( X_l = X \times U \times \mathbb{R}^m \) and \( \bar{u}^l = (u, \bar{u}, \ldots, u^{(l)}) \). For \( l = -1 \), we put \( X^{-1} = X \) and \( \bar{u}^{-1} \) is empty.

Definition 1: The system \( \Sigma : \dot{x} = F(x, u) \) is flat at \( (x_0, \bar{u}^0) \in X^l \), for \( l \geq -1 \), if there exists a neighborhood \( \mathcal{O}^l \) of \( (x_0, \bar{u}^0) \) and \( m \) smooth functions \( \phi_i = \phi_i(x, u, \bar{u}, \ldots, u^{(l)}) \), \( 1 \leq i \leq m \), defined in \( \mathcal{O}^l \), having the following property: there exist an integer \( s \) and smooth functions \( \gamma_i, 1 \leq i \leq n \), and \( \delta_j, 1 \leq j \leq m \), such that
\[
x_i = \gamma_i(\phi, \bar{u}, \ldots, \phi^{(s-1)}) \text{ and } u_j = \delta_j(\phi, \bar{u}, \ldots, \phi^{(s)})
\]
along any trajectory \( x(t) \) given by a control \( u(t) \) that satisfies \( (x(t), u(t), \ldots, u^{(l)}(t)) \in \mathcal{O}^l \), where \( \varphi = (\varphi_1, \ldots, \varphi_m) \) and is called a flat output.
In the particular case $\phi_i = \phi_i(x)$, we will say that the system is $x$-flat. In our study, the flat outputs depend on $x$ only, i.e., we will deal with $x$-flat systems, and $l$ is -1 or 0.

The minimal number of derivatives of components of a flat output, needed to express $x$ and $u$, will be called the differential weight of that flat output and is formalized as follows. By definition, for any flat output $\varphi$ of $\Xi$ there exist integers $s_1, \ldots, s_m$ such that

$$x = \gamma(\phi_1, \phi_1, \ldots, \phi_{s_1}, \phi_{s_1}, \phi_{s_1}, \ldots, \phi_{s_1}, \phi_{s_1}, \phi_{s_1}, \phi_{s_1}),$$

$$u = \delta(\phi_1, \phi_1, \ldots, \phi_{s_1}, \phi_{s_1}, \phi_{s_1}, \phi_{s_1}, \phi_{s_1}, \phi_{s_1}, \phi_{s_1}, \phi_{s_1}),$$

Moreover, we can choose $(s_1, \ldots, s_m)$ such that (see [11]) if for any other $m$-tuple $(\tilde{s}_1, \ldots, \tilde{s}_m)$ we have

$$x = \gamma(\phi_1, \phi_1, \ldots, \phi_{\tilde{s}_1}, \phi_{\tilde{s}_1}, \phi_{\tilde{s}_1}, \ldots, \phi_{\tilde{s}_1}, \phi_{\tilde{s}_1}, \phi_{\tilde{s}_1}),$$

$$u = \delta(\phi_1, \phi_1, \ldots, \phi_{\tilde{s}_1}, \phi_{\tilde{s}_1}, \phi_{\tilde{s}_1}, \phi_{\tilde{s}_1}, \phi_{\tilde{s}_1}, \phi_{\tilde{s}_1}, \phi_{\tilde{s}_1}),$$

then $s_i \leq \tilde{s}_i$, for $1 \leq i \leq m$. We will call $\sum_{i=1}^m (s_i + 1) = m + \sum_{i=1}^m s_i$ the differential weight of $\varphi$. A flat output of $\Xi$ is called minimal if its differential weight is the lowest among all flat outputs of $\Xi$. We define the differential weight of a flat system to be equal to the differential weight of a minimal flat output.

Consider a control-affine system

$$\Sigma : \dot{x} = f(x) + \sum_{i=1}^m u_i g_i(x),$$

where $f$ and $g_1, \ldots, g_m$ are smooth vector fields on $X$. The system $\Sigma$ is linearizable by static feedback if it is equivalent via a diffeomorphism $z = \phi(x)$ and an invertible feedback transformation, $u = \alpha(x) + \beta(x)v$, to a linear controllable system $\Lambda : \dot{z} = Az + Bv$. The problem of static feedback linearization was solved by Jakubczyk and Respondek [15] and Hunt and Su [16] who gave the following geometric necessary and sufficient conditions. Define the distributions $D^0 = D^0 + f, D^1$, where $D^0 = \text{span}\{g_1, \ldots, g_m\}$. The system $\Sigma$ is locally static feedback linearizable if and only if for any $i \geq 0$, the distributions $D^i$ are of constant rank, involutive and $D^{n-1} = TX$. Therefore the geometry of static feedback linearizable systems is given by the following sequence of nested involutive distributions:

$$D^0 \subset D^1 \subset \cdots \subset D^{n-1} = TX.$$

A feedback linearizable system is static feedback equivalent to the Brunovsky canonical form

$$(Br) \quad \begin{align*}
\bar{z}_i^j &= \bar{z}_i^{j+1}, \\
\bar{z}_{\rho_i}^\rho &= \nu_i
\end{align*}$$

where $1 \leq i \leq m$, $1 \leq j \leq \rho_i - 1$, and $\sum_{i=1}^m \rho_i = n$, see [17], and is clearly flat with $q = (z_1^1, \ldots, z_m^n)$ being a minimal flat output (of differential weight $n + m$). In fact, an equivalent way of describing static feedback linearizable systems is that they are flat systems of differential weight equal to $n + m$.

In general, a flat system is not linearizable by static feedback, with the exception of the single-input case. Any single-input system is flat if and only if it is static feedback linearizable (and thus of differential weight $n+1$), see [5], [14]. Flat systems can be seen as a generalization of linear systems. Namely they are linearizable via dynamic, invertible and endogenous feedback, see [2], [1], [4], [7]. Our goal is thus to describe the simplest flat systems that are not static feedback linearizable: control-affine systems that become static feedback linearizable after one-fold prolongation, which is the simplest dynamic feedback. They are flat systems of differential weight equal to $n + m + 1$. In this paper, we will completely characterize them and show how their geometry differs and how it reminds that given by the involutive distributions $D^i$ for static feedback linearizable systems.

### III. MAIN RESULTS

Throughout, we deal only with systems that are not static feedback linearizable. This occurs if there exists an integer $k$ such that $D^k$ is not involutive. Suppose that $k$ is the smallest integer satisfying that property and assume $\text{rk} D^k - \text{rk} D^{k-1} \geq 2$ (see Proposition 3, in Section $V$, asserting that the latter is necessary for dynamic linearizability and thus for flatness with differential weight $n + m+1$). We also assume $m \geq 3$. The case $m = 2$ will be briefly discussed at the end of this section.

From now on, unless stated otherwise, we assume that all ranks involved are constant in a neighborhood of a given $x_0 \in X$. All results presented here are valid in an open and dense subset of $X \times U \times \mathbb{R}^{ml}$ (the integer $l$ being large enough) and hold locally, around a given point $(x_0, \tilde{u}_0)$ of that set.

**Proposition 1:** Consider a control-affine system $\Sigma$ given by (2). The following conditions are equivalent:

(i) $\Sigma$ is flat at $(x_0, \tilde{u}_0)$, of differential weight $n + m + 1$;

(ii) $\Sigma$ is $x$-flat at $(x_0, u_0)$, of differential weight $n + m + 1$;

(iii) There exists, around $x_0$, an invertible static feedback transformation $u = \alpha(x) + \beta(x)\tilde{u}$, bringing the system $\Sigma$ into the form $\Sigma : \dot{x} = f(x) + \sum_{i=1}^m \tilde{u}_i \tilde{g}_i(x)$, such that the prolongation

$$\Sigma^{(1,0,\ldots,0)} : \begin{align*}
\dot{x} &= \tilde{f}(x) + y_1 \tilde{g}_1(x) + \sum_{i=2}^m v_i \tilde{g}_i(x) \\
y_1 &= v_1
\end{align*}$$

is locally static feedback linearizable, with $y_1 = \tilde{u}_1, v_i = \tilde{u}_i$, for $2 \leq i \leq m$, $\tilde{f} = f + a\tilde{g}$ and $\tilde{g}_i = g_i\beta$, where $g = (g_1, \ldots, g_m)$ and $\tilde{g}_i = (\tilde{g}_1, \ldots, \tilde{g}_m)$.

A system $\Sigma$ satisfying (iii) will be called dynamically linearizable via invertible one-fold prolongation. Notice that $\Sigma^{(1,0,\ldots,0)}$ is, indeed, obtained by prolonging the control $\tilde{u}_1$ as $v_1 = \tilde{u}_1$. The above results asserts that for systems of differential weight $n + m + 1$, flatness and $x$-flatness coincide and that, moreover, these properties are equivalent to linearizability via the simplest dynamic feedback, namely one-fold preintegration.
Let $A$ and $B$ be two distributions of constant rank and $f$ a vector field. Denote $[A, B] = \{[a, b] : a \in A, b \in B\}$ and $[f, B] = \{[f, b] : b \in B\}$. Clearly, $[A, B] = [A, B] + A + B$. We will use this notation throughout. If $A \subset B$, we will write $\text{cork}(A \subset B)$ to denote $\text{rk}(B/A)$. So, frequently used $\text{cork}(D^k \subset [D^k, D^k])$ simply means $\text{rk}((D^k, D^k)/D^k) = \text{rk}((D^k, D^k) + D^k/D^k)$.

Our main result describing flat systems of differential weight $n + m + 1$ is given by two following theorems corresponding to the first noninvolutive distribution $D^k$, being either $D^0$, i.e., $k = 0$ (Theorem 2) or $D^k$, for $k \geq 1$ (Theorem 1). For both theorems, we assume that $\text{cork}(D^k \subset [D^k, D^k]) \geq 2$. The particular case $\text{cork}(D^k \subset [D^k, D^k]) = 1$ will be discussed at the end of this section (Theorem 3).

**Theorem 1:** Assume $k \geq 1$ and $\text{cork}(D^k \subset [D^k, D^k]) \geq 2$. A control system $\Sigma$ given by (2), is $x$-flat at $x_0$, with the differential weight $n + m + 1$, if and only if it satisfies around $x_0$:

(A1) There exists an involutive distribution $H^k \subset D^k$, of corank one;

(A2) The distributions $H^i$, for $i \geq k + 1$, are involutive, where $H^i = H^{i-1} + [f, H^{i-1}]$;

(A3) There exists $\rho$ such that $H^\rho = TX$.

The geometry of systems described by the previous theorem can be summarized by the following sequence of inclusions:

$D^0 \subset \cdots \subset D^{k-1} \subset H^k \subset D^k \subset H^{k+1} \subset \cdots \subset H^\rho = TX$

where all the distributions, except $D^k$, are involutive and the integer below $\subset$ means that the inclusion is of corank one. The main structural condition is the existence of a corank one involutive subdistribution $H^k$ in $D^k$. Under the hypotheses $\text{cork}(D^k \subset [D^k, D^k]) \geq 2$, the subdistribution $H^k$ is unique and can be explicitly calculated, [18], [19]. Its construction will be described after stating Theorem 2. Moreover, under the assumption $\text{cork}(D^k \subset [D^k, D^k]) \geq 2$, the condition (A1) implies (via the Jacobi identity) the inclusion $D^{k-1} \subset H^k$. The latter yields $D^k \subset H^{k+1}$ which gives $D^k \subset H^{k+1}$ (since $H^{k+1}$ is involutive by (A2)). It is clear that in the particular case $D^k = TX$, where $D^k$ is the involutive closure of $D^k$, we have $\rho = k + 1$.

The previous theorem enables us to define, up to a multiplicative function, the **characteristic control** $u_c$, i.e., the control to be prolonged in order to obtain $\Sigma^{(1, \ldots, 0)}$ locally static feedback linearizable. According to Proposition 4 (ii) in Section V, to $H^k$ we can associate a unique corank one subdistribution $\mathcal{H}$ in $D^0$ such that $H^0 = D^{k-1} + ad_f\mathcal{H}$. Since $\text{rk} \mathcal{H} = m - 1$, we can find $m$ functions $\beta_1, \ldots, \beta_m$ (not vanishing simultaneously) such that $u_c(x) = u_1(x)\beta_1(x) + \cdots + u_m(x)\beta_m(x) = 0$ if and only if $\sum u_i(x)\delta_i(x) \in H(x)$. It is the characteristic control $u_c$ (becoming $\bar{u}_1$ after feedback) that needs to be preintegrated in order to dynamically linearize the system, that is, we put $v_1 = \frac{d}{dt}(u_1(x)\beta_1(x) + \cdots + u_m(x)\beta_m(x)) = \frac{d}{dt}\bar{u}_1$.

If $k = 0$, i.e., the first noninvolutive distribution is $D^0$, then a similar result holds, but in the chain of involutive subdistributions $H^0 \subset H^1 \subset H^2 \subset \cdots$ (playing the role of $H^k \subset H^{k+1} \subset H^{k+2} \subset \cdots$) the distribution $H^1$ is not defined as $H^{k+1}$, but as $H^1 = G^1 + [f, H^0]$ where $G^1 = D^0 + [D^0, D^0]$, (compare (A2) and (A2)) and satisfies an additional nonsingularity condition (CR). In fact, flat systems with $k = 0$ exhibit a singularity in the control space (created by one-fold prolongation of the characteristic control) defined by $U_{\text{sing}}(x) = \{u(x) \in \mathbb{R}^m : \text{rk span} \{g_1, h_i, [f + u_1g_1 + \sum_i u_ih_i, 2 \leq j \leq m]\} < \text{rk} H^1(x)\}$ and excluded by (CR), where $H^0 = \text{span} \{h_2, \ldots, h_m\}$ and $D^0 = \text{span} \{g_1, h_2, \ldots, h_m\}$.

**Theorem 2:** Assume $k = 0$ and $\text{cork}(D^0 \subset [D^0, D^0]) \geq 2$. A system $\Sigma$ given by (2), is $x$-flat at $(x_0, u_0)$, with the differential weight $n + m + 1$, if and only if it satisfies:

(A1)' There exists an involutive distribution $H^0 \subset D^0$, of corank one;

(A2)' The distributions $H^i$, for $i \geq 1$, are involutive, where $H^i = G^1 + [f, H^0]$ and $H^{i-1} = H^i + [f, H^{i-1}]$, for $i \geq 2$;

(A3)' There exists $\rho$ such that $H^\rho = TX$;

(CR) $u_0 \notin U_{\text{sing}}(x_0)$.

Similarly to Theorem 1, if $D^0 = TX$, then $\rho = 1$.

In order to verify the conditions of Theorem 1 (respectively Theorem 2), we have to check whether the distribution $D^k$ (respectively $D^0$) contains an involutive subdistribution $H^k$ (respectively $H^0$) of corank one. Now we will explain how to do it. Consider a distribution $D$ of rank $d$, defined on a manifold $X$ of dimension $n$ and define its annihilator $D^\perp = \{\omega \in \Lambda^l(X) : \omega|_D = 0, \forall f \in D\}$. Let $\omega_1, \ldots, \omega_s$, where $s = n - d$, be differential 1-forms locally spanning the annihilator of $D$, that is $D^\perp = I = \text{span} \{\omega_1, \ldots, \omega_s\}$. The Engel rank of $D$ equals 1 at $x$ if and only if $D$ is non involutive and $(d\omega_1 \wedge d\omega_j)(x) = 0 \mod I$, for any $1 \leq i, j \leq s$. For any $\omega \in I$, we define $W(\omega) = \{f \in D : f \cdot d\omega \in D^\perp\}$, where $\cdot$ is the interior product. The characteristic distribution $C = \{f \in D : [f, D] \subset C\}$ of $D$ is given by

$$C = \bigcap_{i=1}^s W(\omega_i).$$

It follows directly from the Jacobi identity that the characteristic distribution is always involutive. Let $\text{rk}[D, D] = d + r$. Choose the differential forms $\omega_1, \ldots, \omega_r, \ldots, \omega_s$ such that $I = \text{span} \{\omega_1, \ldots, \omega_s\}$ and $I^\perp = \text{span} \{\omega_{r+1}, \ldots, \omega_s\}$, where $I^\perp$ is the annihilator of $[D, D]$. Define the distribution

$$V = \sum_{i=1}^r W(\omega_i).$$

We have the following result proved by Bryant [18], see also [19].
Proposition 2: Consider a distribution $D$ of rank $d$ and let $\text{rk} [D, D] = d + r$.

(i) Assume $r \geq 3$. The distribution $D$ contains an involutive subdistribution $H$ of corank one if and only if it satisfies

(ISD1) The Engel rank of $D$ equals one;
(ISD2) The characteristic distribution $C$ of $D$ has rank $d - r - 1$.

Moreover, that involutive subdistribution is unique and is given by $H = \mathcal{V}$.

(ii) Assume $r = 2$. The distribution $D$ contains a corank one subdistribution $L$ satisfying $[L, L] \subset D$ if and only if it verifies (ISD1)-(ISD2).
In that case, $L$ is unique and given by $L = \mathcal{V}$.
Moreover, $L = \mathcal{V}$ is the involutive distribution $H$ of corank one in $D$ if and only if $L = \mathcal{C}$.

(iii) Assume $r = 1$. The distribution $D$ contains an involutive subdistribution of corank one if and only it satisfies the condition (ISD2). In the case $r = 1$, if an involutive subdistribution of corank one exists, it is never unique.

The above conditions are easy to check and a unique involutive subdistribution of corank one can be constructed if $r \geq 2$. As a consequence, the conditions of Theorem 1 (resp. Theorem 2) are verifiable, i.e., given a control-affine system, we can verify whether it is flat with the differential weight $n + m + 1$ and verification involves differentiation and algebraic operations only, without solving PDE’s or bringing the system into a normal form.

Let us now consider the case $r = 1$, that is, $\text{cork}(D^k \subset [D^k, D^k]) = 1$. If the distribution $D^k$ contains a corank one involutive subdistribution, the latter is no longer unique (see (iii) of Proposition 2).

The involutivity of $D^k$ can be lost in two different ways: either $[D^{k-1}, D^k] \not\subset D^k$ or $[D^{k-1}, D^k] \subset D^k$ and there exist $1 \leq i, j \leq m$ such that $[ad^i_{g1}, ad^j_{g2}] \notin D^k$.

As asserts Theorem 3 below, in the case $[D^{k-1}, D^k] \not\subset D^k$, the corank one involutive subdistribution $H^k$ can be uniquely identified by another argument. To this end, we introduce the characteristic distribution $C^k$ (defined above) of $D^k$. The subdistribution $H^k$ has to verify some additional conditions analogous to those of Theorem 1. If $[D^{k-1}, D^k] \subset D^k$ and there exist $1 \leq i, j \leq m$ such that $[ad^i_{g1}, ad^j_{g2}] \notin D^k$, any corank one involutive subdistribution $H^k$ may serve to define a control (different distributions yield different controls) whose prolongation gives a static feedback linearizable system.

Theorem 3: Assume cork $(D^k \subset [D^k, D^k]) = 1$ and $[D^{k-1}, D^k] \not\subset D^k$. A control system $\Sigma$, given by (2), is $x$-flat at $x_0$, with the differential weight $n + m + 1$, if and only if the following conditions are satisfied:

(C1) $\text{rk} C^k = \text{rk} D^k - 2$, where $C^k$ is the characteristic distribution of $D^k$;
(C2) $\text{rk} (C^k \cap [D^{k-1}, D^k]) = \text{rk} D^{k-1} - 1;$
(C3) The distributions $H^i_j$, for $i \geq k$, are involutive, where $H^k = C^k + D^{k-1}$ and $H^{i+1} = H^i + [f, H^i]$;
(C4) There exists $\rho$ such that $H^\rho = TX$.

It is clear that the above result can be applied only for $k \geq 1$, otherwise $[D^{k-1}, D^k] \not\subset D^k$ would not have any sens. It can be shown that in the case $[D^{k-1}, D^k] \not\subset D^k$ (no mater the value of cork $(D^k \subset [D^k, D^k])$), the involutive subdistribution $H^k$ can always be defined as above, i.e., the computation of $H^k$ using the procedure given by Proposition 2 and that provided by conditions (C1)-(C3) of the above theorem are equivalent if $[D^{k-1}, D^k] \not\subset D^k$. This is not valid anymore if $[D^{k-1}, D^k] \subset D^k$; indeed, in that case $D^{k-1} \subset C^k$, the condition (C2) is not verified and (C3) would give $H^k = C^k$. Notice that in the case $[D^{k-1}, D^k] \subset D^k$, the inclusion $C^k \subset H^k$ is always satisfied and is of corank one if additionally cork $(D^k \subset [D^k, D^k]) = 1$, i.e., $H^k = C^k + \{g\}$, where $g$ is a vector field belonging to $D^k$, but not to $D^{k-1}$.

Let us now consider the two-input control-affine system $\Sigma$, i.e., $m = 2$. Any corank one involutive subdistribution $H^k$ of $D^k$ satisfies cork $(D^k \subset H^{k+1}) = 1$ therefore, $D^k = H^{k+1}$ and we necessarily have cork $(D^k \subset [D^k, D^k]) = 1$. Thus, neither Theorem 1 (if $k \geq 1$) nor Theorem 2 (if $k = 0$) applies to the case $m = 2$. On the other hand, Theorem 3 covers the case $m = 2$ but only if $[D^{k-1}, D^k] \not\subset D^k$. In [20], we treat the case $m = 2$ in its full generality. Namely, we define (by another method) the involutive subdistribution $H^k$ in all cases satisfying $D^k \neq TX$ (no mater whether $[D^{k-1}, D^k] \not\subset D^k$ or $[D^{k-1}, D^k] \subset D^k$ and $[ad^i_{g1}, ad^j_{g2}] \notin D^k$). Moreover, in the particular case $D^k = TX$ and $[D^{k-1}, D^k] \not\subset D^k$, the subdistribution $H^k$ is defined as in Theorem 3. Finally, if $D^k = TX$ and $[D^{k-1}, D^k] \subset D^k$, we have shown, in [20], that the system is flat of differential weight $n + 3$ without any additional condition.

IV. EXAMPLE: QUADROTOR HELICOPTER

A quadrotor is a four rotor helicopter. Assume that a body frame is fixed at the center of gravity of the quadrotor, with the $z$-axis pointing up-wards. The body frame is related to the inertial frame by a position vector $(x, y, z)$ and 3 angles $(\theta, \psi, \varphi)$ representing pitch, roll and yaw, respectively. The equations of motion are given by the following control system on $X = \mathbb{R}^6 \times S^1 \times S^1 \times S^1$ (see [21], [22]):

\[
\begin{align*}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= u_1 (\cos \varphi \sin \theta \cos \psi + \sin \varphi \sin \psi) \\
\dot{y}_1 &= y_2 \\
\dot{y}_2 &= u_1 (\sin \varphi \sin \theta \cos \psi - \cos \varphi \sin \psi) \\
\dot{z}_1 &= z_2 \\
\dot{z}_2 &= -g + u_1 (\cos \varphi \cos \psi) \\
\dot{\theta} &= u_2 \\
\dot{\psi} &= u_3 \\
\dot{\varphi} &= u_4
\end{align*}
\]
The control $u_1$ represents the total thrust on the body in the $z$-axis, $u_2$ and $u_3$ are the pitch and roll inputs and $u_4$ is the yawing moment. The quadrotor helicopter has been shown to be flat, with $(x_1, y_1, z_1, \phi)$ a flat output (see [22]). The system is not static feedback linearizable, however, it becomes static feedback linearizable after a one fold prolongation. To illustrate our results, fix $\xi_0 \in X$ such that $(\cos \theta \cos \psi \cos \phi \cos \sin \theta \cos \phi + \sin \phi \sin \phi)(\xi_0) \neq 0$. Applying the invertible feedback transformation $\tilde{u}_1 = u_1(\cos \phi \sin \theta \cos \psi + \sin \phi \sin \phi)$ and $\tilde{u}_i = u_i$, for $2 \leq i \leq 4$, we get:

$$
\tilde{\Sigma}_{QH} = \begin{cases}
\dot{x}_1 = x_2 \\
\dot{x}_2 = \tilde{u}_1 \\
\dot{\tilde{z}}_2 = -\tilde{g} + \tilde{u}_1 b(\theta, \psi, \phi) \\
\dot{\psi} = \tilde{u}_3
\end{cases}
$$

where

$$a = \frac{\cos \phi \sin \theta \cos \psi - \cos \phi \sin \phi}{\cos \phi \sin \phi + \cos \phi \sin \phi} \quad \text{and} \quad b = \frac{\cos \phi \cos \phi}{\cos \phi \sin \phi + \cos \phi \sin \phi}.
$$

The distribution

$$D^0 = \text{span} \left\{ \frac{\partial}{\partial \phi} \right\}
$$

is not involutive. Indeed, the vector fields $g_i$, $1 \leq i \leq 4$, $[g_1, g_2]$ and $[g_1, g_3]$ are independent at $\xi_0$ (provided that $\cos \theta_{\xi_0} \cos \psi_{\xi_0} \cos \phi_{\xi_0} \neq 0$, which is verified according to our assumption), thus we obtain

$$G^1 = D^0 + [D^0, D^0] = \text{span} \left\{ \frac{\partial}{\partial \phi}, \frac{\partial}{\partial \theta} \right\}.
$$

Here $k = 0$ and cork $(D^0 \subset [D^0, D^0]) = 2$, consequently we are in the case of Theorem 2. It is immediate to identify the unique corank one involutive subdistribution of $D^0$, that is: $H^0 = \text{span} \left\{ \frac{\partial}{\partial \theta}, \frac{\partial}{\partial \phi} \right\}$.

We have $H^1 = G^1 + [f, H^0] = G^1$ (since $[f, g_i] = 0$, for $2 \leq i \leq 4$), which is clearly involutive, and $H^2 = TX$. The system $\Sigma_{QH}^{(1,0,0)}$ satisfies all conditions of Theorem 2, hence the corresponding prolongation given by

$$\Sigma_{QH}^{(1,0,0)} = \begin{cases}
\dot{x}_1 = x_2 \\
\dot{x}_2 = \tilde{u}_1 \\
\dot{\tilde{z}}_2 = -\tilde{g} + \tilde{u}_1 b(\theta, \psi, \phi) \\
\dot{\psi} = v_3
\end{cases}
$$

where $v_i = \tilde{u}_i$, for $2 \leq i \leq 4$, is locally static feedback linearizable. Indeed, applying the following change of coordinates $\tilde{\theta} = \tilde{u}_1 a(\theta, \psi, \phi)$ and $\tilde{\psi} = -\tilde{g} + \tilde{u}_1 b(\theta, \psi, \phi)$ (which is valid in a neighborhood of $\xi_0$ and for $\tilde{u}_{10} \neq 0$) and a suitable feedback transformation, we get the Brunovsky canonical form with $(x_1, y_1, z_1, \phi)$ playing the role of the top variables. From this, it is obvious that $(x_1, y_1, z_1, \phi)$ is a minimal flat output, i.e. of differential weight $n + m + 1 = 14$.

### V. Proofs

#### A. Notations and useful results

Consider a control system of the form $\Sigma : \dot{x} = f(x) + u_1 g_1(x) + \sum_{i=2}^{m} u_i h_i(x)$. By $\Sigma^{(1,0,\ldots,0)}$ we will denote the system $\Sigma$ with one-fold prolongation, that is

$$\Sigma^{(1,0,\ldots,0)} : \begin{cases}
\dot{x} = f(x) + u_1 g_1(x) + \sum_{i=2}^{m} v_i h_i(x) \\
\dot{y}_1 = v_1
\end{cases}
$$

with $y_1 = u_1$ and $v_j = u_j$, for $2 \leq j \leq m$. Throughout this section,

$$F = \sum_{i=1}^{n} (f_i + y_1 g_{i1}) \frac{\partial}{\partial x_i}
$$

de note the control vector fields of the prolonged system. To $\Sigma^{(1,0,\ldots,0)}$, we associate the distributions $D^0_p = \text{span} \{ g_1, H_{2} \cdots H_m \}$ and $D^{0+1}_p = D^0_p + [F, D^0_p]$, for $i \geq 0$; the subindex $p$ referring to the prolonged system $\Sigma^{(1,0,\ldots,0)}$.

In our proofs we will need the following technical results. Consider a control system $\Sigma$, given by (2), and let $D^k$ be the first noninvolutive distribution.

**Proposition 3:** Assume that $\Sigma$ is dynamically linearizable via invertible one-fold prolongation. If $k \geq 1$, then $\text{rk} D^k - \text{rk} D^{k-1} \geq 2$.

**Proposition 4:** Assume $k \geq 1$ and suppose that $D^k$ contains an involutive subdistribution $H^k$, of corank one. Then

(i) If $\text{cork} (D^k \subset [D^k, D^k]) \geq 2$, then $H^k$ satisfies $D^{k-1} \subset H^k$.

(ii) If $H^k$ satisfies $D^{k-1} \subset H^k$, then there exists a distribution $H_{\ell}$, uniquely associated to $H^k$, such that $H \subset D^0$, of corank one, and $H^\ell = D^{k-1} + dH^\ell$. Moreover, all distributions $H^\ell = D^{i-1} + dH^\ell$, for $0 \leq i \leq k-1$, are involutive.

#### B. Proof of Theorem 1

In this section, we will prove Theorem 1, which is a general result, whereas Theorems 2 and 3 deal with the particulars cases $k = 0$ and cork $(D^k \subset [D^k, D^k]) = 1$.

**Necessity:** Let us consider a flat control system $\Sigma : \dot{x} = f(x) + \sum_{i=1}^{m} u_i g_i(x)$, of weight $n + m + 1$. According to Proposition 1, there exists an invertible feedback transformation $u = a(x) + \beta(x) \tilde{u}$, bringing $\Sigma$ into the form $\Sigma : \dot{x} = \tilde{f}(x) + \tilde{u}_1 \tilde{g}_1(x) + \sum_{i=1}^{m} \tilde{u}_i \tilde{h}_i(x)$, such that the prolongation

$$\Sigma^{(1,0,\ldots,0)} : \begin{cases}
\dot{x} = \tilde{f}(x) + y_1 \tilde{g}_1(x) + \sum_{i=2}^{m} v_i \tilde{h}_i(x) \\
\dot{y}_1 = v_1
\end{cases}
$$

The control $u_1$ represents the total thrust on the body in the $z$-axis, $u_2$ and $u_3$ are the pitch and roll inputs and $u_4$ is the yawing moment.
with $y_1 = \tilde{u}_1$ and $v_j = \tilde{u}_j$, for $2 \leq j \leq m$, is locally static feedback linearizable. For simplicity of notation, we will drop the tilde, we will keep distinguishing $g_j$ from $h_j$ (which could also be denoted $g_{ji}$, $2 \leq j \leq m$) whose controls are not preintegrated. Since $\Sigma^{(1,0,\ldots,0)}$ is locally static feedback linearizable, $D^0_p$ are involutive, of constant rank, for any $i \geq 0$, and there exists an integer $\rho$ such that $\text{rk } D^\rho_p = n + 1$. We have

$$D^1_p = \text{span } \{ \frac{\partial}{\partial y_1}, h_j, 2 \leq j \leq m \},$$

where $\Sigma = \text{span } \{ \frac{\partial}{\partial y_1}, g_1, h_j, \text{ad } j h_j + y_1 g_1 \}$, $2 \leq j \leq m$. Since $k \geq 1$, the distribution $D^0 = \text{span } \{ g_1, h_j, 2 \leq j \leq m \}$ is involutive, thus $[g_1, h_j] \in D^0$ and hence $D^1_p = \text{span } \{ \frac{\partial}{\partial y_1}, g_1, h_j, \text{ad } j h_j, 2 \leq j \leq m \}$. It is easy to prove (by an induction argument) that, for $1 \leq i \leq k$,

$$D^i_p = \text{span } \{ \frac{\partial}{\partial y_1}, g_1, \ldots, \text{ad } j^{-1} g_1, h_j, \ldots, \text{ad } j^{-1} h_j, 2 \leq j \leq m \}.$$ 

Since the intersection of involutive distributions is an involutive distribution, $D^i_p \cap TX = \text{span } \{ g_1, \ldots, \text{ad } j^{-1} g_1, h_j, \ldots, \text{ad } j^{-1} h_j, 2 \leq j \leq m \}$ is involutive, for $1 \leq i \leq k$. We deduce that

$$H^k = \text{span } \{ g_1, \ldots, \text{ad } j^{-1} g_1, h_j, \ldots, \text{ad } j^{-1} h_j, 2 \leq j \leq m \}$$

is involutive. It is immediate that $D^{k-1} \subset H^k \subset D^k$, where the second inclusion is of corank one, otherwise $H^k = D^k$ and $D^0$ would be involutive, which contradicts our hypothesis. Recall that $H^i = H^{i-1} + \{ f, H^{i-1} \}$, for $i \geq k + 1$. We have

$$D^{k+1}_p = \text{span } \{ \frac{\partial}{\partial y_1} \} + H^k + \{ f, H^k \} = \text{span } \{ \frac{\partial}{\partial y_1} \} + \text{H}^k+1$$

and by an induction argument

$$D^{k+i}_p = \text{span } \{ \frac{\partial}{\partial y_1} \} + H^{k+i}, i \geq 2.$$ 

Consequently, the involutivity of $D^{k+i}_p$ implies that of $H^{k+i}$, for $i \geq 1$. Moreover, $\text{rk } D^0_p = n + 1$, proving that $\text{rk } H^0 = n$, i.e., $H^0 = TX$. Sufficiency: Consider a control system $\Sigma : \dot{x} = f(x) + u_1 g_1(x) + \sum_{i=2}^m u_i h_i(x)$ satisfying $(A1) - (A3)$, where $H = \text{span } \{ h_j, 2 \leq j \leq m \}$ is defined by Proposition 4(ii). By the same proposition, the involutivity of $H^i = D^{i-1} + \text{ad } j h_i H$ follows for $0 \leq i \leq k - 1$. It is immediate to see that the prolongation

$$\Sigma^{(1,0,\ldots,0)} : \begin{cases} \dot{x} = f(x) + y_1 g_1(x) + \sum_{i=2}^m v_i h_i(x) \\ \dot{y}_1 = v_1 \end{cases}$$

with $y_1 = u_1$ and $v_j = u_j$, for $2 \leq j \leq m$, is locally static feedback linearizable. Indeed, the linearizability distributions $D^{\rho}_p$, associated to $\Sigma^{(1,0,\ldots,0)}$, are of the form

$$D^i_p = \text{span } \{ \frac{\partial}{\partial y_1} \} + H^i, i \geq 0,$$

and the involutivity of $H^i$ implies that of $D^{\rho}_p$, because $H^i$ does not depend on $y_1$. Moreover, $\text{rk } H^0 = n$, thus $\text{rk } D^0_p = n + 1$ and $\Sigma^{(1,0,\ldots,0)}$ is locally static feedback linearizable. By Proposition 1, the system $\Sigma$ is flat of differential weight $n + 1$.

References


