Robust output feedback control of uncertain switched Euler-Lagrange systems

Teng-Hu Cheng1, Ryan J. Downey1, and Warren E. Dixon1

Abstract—Output feedback (OFB) controllers for switched systems have been typically developed for linear systems. Of the OFB controllers that have been designed for switched nonlinear systems, at least partial knowledge of the system parameters is required, or the system results in local stability. In this paper, an OFB, time-dependent, switched controller is developed for an Euler Lagrange system with parametric uncertainty and exogenous disturbances. The time-dependent switching signal is designed using an average dwell-time scheme based on multiple Lyapunov functions such that the switched system achieves semi-global uniformly ultimately bounded (UUB) tracking with arbitrary switching sequences.

I. INTRODUCTION

Euler-Lagrange dynamics are commonly used to model various systems. Some Euler Lagrange systems exhibit discontinuous behaviors (e.g., a robot that transitions from a non-contact to contact state in force control applications). To account for both continuous and discontinuous behaviors, hybrid or switched systems methods have been developed. In addition to discontinuous behaviors, systems often lack sensing required for full state feedback, motivating the need for output feedback (OFB) controllers. Researchers have designed OFB controllers for switched systems when measurements are limited, but these controllers have been predominantly designed for linear, switched systems. In [1], an OFB controller was designed based on a fuzzy model for a switched linear system to overcome uncertainty in the system. In [2]–[9], gain conditions for the OFB controller were obtained by solving a LMI (linear matrix inequality) for a switched linear system. In [10]–[12], gain conditions were developed for an OFB controller designed for a linear, switched delay system by solving a LMI.

In [13], the average dwell-time scheme for linear systems was extended to nonlinear systems for a supervisory control algorithm. In [14], the average dwell-time concept was extended to nonlinear integral input-to-state stable systems. In [15], full state feedback robust controllers were designed for switched systems to compensate for system uncertainties and disturbances. In [16], an OFB controller for a switched nonlinear system was designed under the assumption of exact model knowledge. A T-S fuzzy method was used in [17] to address the uncertainties of the switched nonlinear system, but the controller was designed based on a locally linear time-varying system. In [18], an OFB controller was developed for switched nonlinear systems with parameter uncertainty, where a series of local OFB robust controllers were used.

In [19], a robust OFB controller was developed that only requires position measurements for a continuous nonlinear system with parametric uncertainties and bounded disturbances. The contribution in this paper is to show how the result in [19] can be extended as an OFB controller with a time-dependent switching signal for a switched, nonlinear, Euler-Lagrange system with parametric uncertainties and bounded exogenous disturbances. The switched controller is designed based on multiple Lyapunov functions and ensures that the position tracking error is semi-global uniformly ultimately bounded (UUB). The paper is organized as follows. Section II represents the dynamics of each Euler Lagrange subsystem. In Section III, the controller for each subsystem is designed. Based on the system dynamics and the controller development, the stability analysis is provided in Section IV, and the average dwell-time is designed in Section V.

II. DYNAMIC MODEL

Consider $N$ distinct Euler-Lagrange subsystems where the dynamics of each subsystem is defined as

$$M_i(q)\ddot{q} + V_m,i(q, \dot{q})\dot{q} + G_i(q) + F_i\dot{q} + \tau_{d,i}(t) = \tau_i(t),$$

(1)

where $i \in \mathbb{S}$ denotes the $i^{th}$ subsystem, $\mathbb{S} \triangleq \{1, 2, \ldots, N\}$ denotes finite index sets of all subsystems, $M_i : \mathbb{R}^n \rightarrow \mathbb{R}^{n \times n}$ denotes the inertia matrix, $V_m,i : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}^{n \times n}$ denotes the centripetal-Coriolis matrix, $G_i : \mathbb{R}^n \rightarrow \mathbb{R}^n$ denotes the gravity vector, $F_i : \mathbb{R}^n \rightarrow \mathbb{R}^n$ denotes the constant, diagonal, positive-definite, viscous friction matrix, $\tau_{d,i} : [0, \infty) \rightarrow \mathbb{R}^n$ denotes the generalized bounded disturbance, $\tau_i : [0, \infty) \rightarrow \mathbb{R}^n$ denotes the control input, and $q, \dot{q}, \ddot{q} \in \mathbb{R}^n$ denote the generalized states. The states $q$ are measurable but $\dot{q}$ and $\ddot{q}$ are not. The functions $M_i, V_m,i, G_i, F_i$, and $\tau_{d,i}$ are considered to be unknown.

Property 1. [20] The inertia matrix $M_i, \forall i \in \mathbb{S}$, is symmetric, positive definite, and satisfies the following inequality:

$$m_1 \|\varsigma\|^2 \leq \varsigma^T M_i \varsigma \leq m_2 \|\varsigma\|^2, \quad \forall \varsigma \in \mathbb{R}^n, \quad i \in \mathbb{S},$$

(2)

where $m_1, m_2 \in \mathbb{R}$ denote two known positive constants.
Property 2. [20] The inertia and the centripetal-Coriolis matrices satisfy the following skew-symmetric relationship:

\[ \zeta^T \left( \frac{1}{2} \mathbf{M}_i - V_{m,i} \right) \zeta = 0, \quad \forall \zeta \in \mathbb{R}^n, \quad i \in \mathcal{S}. \quad (3) \]

Property 3. [20] The centripetal-Coriolis matrix satisfies the following relationship

\[ V_{m,i}(q, \omega)\eta = V_{m,i}(q, \eta)\omega, \quad \forall \eta, \omega \in \mathbb{R}^n, \quad i \in \mathcal{S}. \quad (4) \]

Property 4. [20] The Euler-Lagrange system from (1) can be linearly parametrized as

\[ Y\dot{\theta}_i = M_i(q)\ddot{q} + V_{m,i}(q, \dot{q})\dot{q} + G_i(q) + F_i\dot{q}, \quad i \in \mathcal{S}, \quad (5) \]

where \( \theta_i \in \mathbb{R}^p \) denotes the vector including all the unknown system parameters of \( i \)-th subsystem, and \( Y : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}^{n \times p} \) denotes the regression matrix which is a function of \( q, \dot{q}, \dot{q} \).

By utilizing the desired trajectory, (5) can be rewritten as

\[ Y_\delta \theta_i = M_i(q_d)\ddot{q} + V_{m,i}(q_d, \dot{q})\dot{q} + G_i(q_d) + F_i\dot{q}, \quad (6) \]

where \( i \in \mathcal{S} \), and \( Y_\delta : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}^{n \times p} \) denotes the desired regression matrix as a function of \( q_d, \dot{q}_d, \dot{q}_d \), which denote the desired position, velocity, and acceleration, respectively. By design, \( q_d, \dot{q}_d, \dot{q}_d \) are bounded in \( \mathcal{L}_\infty \).

Property 5. [20] The centripetal-Coriolis and friction matrices, and the gravity, disturbance, and unknown parameter vectors can be upper bounded as

\[ \|V_{m,i}\| \leq \xi_{c1,i} \|\dot{q}\|, \quad \|F_i\| \leq \xi_{f,i}, \]

\[ \|G_i\| \leq \xi_{g,i}, \quad \|r_{d,i}\| \leq \xi_{d,i}, \quad \|\theta_i\| \leq \xi_{\theta_l,i}, \quad i \in \mathcal{S}, \]

where \( \xi_{c1,i}, \xi_{f,i}, \xi_{g,i}, \xi_{d,i}, \xi_{\theta_l,i} \in \mathbb{R} \) are positive constants of \( i \)-th subsystem.

To facilitate further analysis, the following vector function \( \text{Tanh}(\cdot) \in \mathbb{R}^n \) and matrix function \( \text{Cosh}(\cdot) \in \mathbb{R}^{n \times n} \) are defined as

\[ \text{Tanh}(\zeta) \triangleq \left[ \tanh(\zeta_1), \ldots, \tanh(\zeta_n) \right]^T \]

and

\[ \text{Cosh}(\zeta) \triangleq \text{diag} \{ \cosh(\zeta_1), \ldots, \cosh(\zeta_n) \}, \]

where \( \zeta = [\zeta_1, \ldots, \zeta_n]^T \in \mathbb{R}^n \), and \( \text{diag} \{ \cdot \} \) denotes a diagonal matrix. Based on (8), the following inequalities hold [19]

\[ \frac{1}{2} \tanh^2(\|\zeta\|) \leq \ln(\cosh(\|\zeta\|)) \leq \sum_{j=1}^{n} \ln(\cosh(\zeta_j)) \leq \|\zeta\|^2, \]

\[ \tanh^2(\|\zeta\|) \leq \|\text{Tanh}(\zeta)\|^2 = \text{Tanh}^T(\zeta)\text{Tanh}(\zeta). \quad (9) \]

III. CONTROL DEVELOPMENT

Robust OFB controllers are developed for each uncertain subsystem under the constraint that the only available measurement for feedback is the position variable \( q \). To quantify the objective, the position error \( e \in \mathbb{R}^n \) is defined as

\[ e = q_d - q, \quad (10) \]

where \( q_d \) is the desired trajectory. The difference between the actual system parameters and the estimated parameters for each subsystem is defined as

\[ \hat{\theta}_i \triangleq \theta_i - \hat{\theta}_i, \quad i \in \mathcal{S}, \]

where \( \hat{\theta}_i \in \mathbb{R}^p \) denotes the parameter estimation error, and \( \theta_i \in \mathbb{R}^p \) denotes the constant best-guess estimates of \( \theta_i \) for the \( i \)-th subsystem. In addition, the estimate error can be upper bounded as

\[ \|\hat{\theta}_i\| \leq \xi_{\theta_2,i}, \quad i \in \mathcal{S}, \]

where \( \xi_{\theta_2,i} \in \mathbb{R} \) denotes a known positive constant for the \( i \)-th subsystem.

A. Robust output feedback tracking controller

Based on the subsequent development, and stability analysis, the following control input is designed [19]

\[ r_i = Y_\delta \hat{\theta}_i - k_i \Gamma^{-1} y + \text{Tanh}(e), \quad i \in \mathcal{S} \]

where \( k : [0, \infty) \rightarrow \mathbb{R} \) is a positive time-varying, differentiable control gain, \( \Gamma : \mathbb{R}^n \rightarrow \mathbb{R}^{n \times n} \) is

\[ \Gamma \triangleq \text{diag} \{ (a - y_1^2)^2, (a - y_2^2)^2, \ldots, (a - y_n^2)^2 \}, \]

where \( a \in \mathbb{R} \) is an adjustable positive constant, and \( y \in \mathbb{R}^n \) denotes an auxiliary signal for the velocity tracking error defined as [19]

\[ y_j \triangleq p_j - ke_j. \]

In (15), \( p_j \in \mathbb{R} \) denotes the solution to [19]

\[ \dot{p}_j = - (a - (p_j - ke_j)^2)(p_j - ke_j - \text{Tanh}(e_j)) - k(\text{Tanh}(e_j) + p_j - ke_j) + \dot{e}_j \]

where \( j \in \{1, 2, \ldots, n\} \) denotes the \( j \)-th element of the vector variable, and the initial conditions for \( p_j(0) \) are selected as

\[ -\frac{a}{\sqrt{n}} + k(0)e_j(0) < p_j(0) < \frac{a}{\sqrt{n}} + k(0)e_j(0). \]

Provided that the initial condition \( p_j(0) \) is selected based on (17), then (15) can be used to show that

\[ |y_j(0)| < \frac{a}{\sqrt{n}}, \]

which is independent of the magnitude of \( e_j(0) \). Following the development in [19], the continuous time-varying control
gain is designed as
\[ k = \max_{i \in S} \left\{ \frac{1}{m_1} \left( k_{n_1,i} \sum_{j=1}^{n} \sum_{p=1}^{p} (Y_{djk}^2 \xi_{djk}) + k_{n_2,i} \sum_{j=1}^{k_{n_2,i}} (Y_{djk}^2 \xi_{djk}) + k_{n_3,i} \xi_{k,i} + k_{n_4,i} \xi_{k,i}^2 \right) \right\}. \]

In (18), \( \xi_{k,i} \in \mathbb{R} \) denotes a known positive function defined as
\[ \xi_{k,i} = m_2 \| \dot{q}_d \| + \xi_{c_1,i} \| \dot{q}_d \|^2 + \xi_{f,i} \| \ddot{q}_d \| + \xi_{g,i} + \xi_{d,i}, \quad i \in S, \]
where \( m_1 \) and \( m_2 \) were defined in (2), \( k_{n_1,i}, k_{n_2,i}, k_{n_3,i}, k_{n_4,i} \) denote positive constant control gains, \( \xi_{c_1,i} \) was defined in (7), \( \xi_{g,i} \) was defined in (12), and \( \xi_{d,i}, h = 1 - 6 \) denote some positive constants that upper bound the parameters of the system dynamics and the desired trajectory. To facilitate the subsequent stability analysis, the control gains \( k_{n_1,i}, k_{n_2,i}, k_{n_3,i}, \) and \( k_{n_4,i} \) are selected based on the sufficient condition
\[ \epsilon_i < \frac{1}{2}, \quad k_{n_4,i} > \frac{1}{4} \left( \frac{1}{\xi} - \epsilon_i \right), \quad i \in S, \]
where \( \epsilon_i \in \mathbb{R} \) denotes a positive constant for the \( i \)-th subsystem defined as
\[ \epsilon_i = \frac{1}{4k_{n_1,i}} + \frac{1}{4k_{n_2,i}} + \frac{1}{4k_{n_3,i}}, \quad i \in S. \]

B. Error system development

By taking the time derivative of (15) and using (16) and (17), the open-loop error system for the velocity filter term can be designed as [19]
\[ \dot{y}_j = -(a - y_j^2)^2 (y_j - \tanh(\epsilon_j)) - k_1 \eta_j, \]
where \( \eta \in \mathbb{R}^n \) denotes an auxiliary filtered tracking error defined as [19]
\[ \eta = \dot{e} + \tanh(e) + y. \]

By taking the time derivative of (22), pre-multiplying both sides of the resulting equation by \( M_i \), and using (1), the open-loop error system is
\[ M_i \dot{\eta} = M_i \ddot{q}_d + V_{m,i} \dot{q}_i + G_i + F_i \dot{q}_i + \tau_{d,i} - \tau_i + M_i \cosh^{-2}(\dot{e}) \dot{e} + M_i \dot{\eta}_c, \quad i \in S. \]

After adding and subtracting \( Y_d \dot{\theta}_i \) of (6) to (23) and utilizing (4), (10), (21), and (22), (23) can be rewritten as
\[ M_i \dot{\eta} = -V_{m,i} \dot{\eta} + Y_d \ddot{\theta}_i - \tau_i - k_i M_i \dot{\eta} + \chi_i, \]
where \( i \in S \), and the auxiliary function \( \chi_i(\cdot) \in \mathbb{R}^n \) denotes the disturbance and its upper bound is [19]
\[ \| \chi_i \| \leq \left( \sum_{j=1}^{n} \sum_{k=1}^{p} Y_{djk}^2 \right) \xi_{\dot{q}}, \xi_{\dot{q}}^2, \xi_{\dot{q}}^3, \xi_{\dot{q}}^4, \xi_{\dot{q}}^5, \]
where \( x \in \mathbb{R}^{3n} \) is defined as
\[ x = \left[ \tanh^T(e) \eta^T y^T \right]^T. \]

By substituting (13) into (24) and using (11), the closed-loop error system can be obtained as
\[ M_i \dot{\eta} = -V_{m,i} \eta + Y_d \ddot{\theta}_i + k_i \gamma_1 - \tanh(e) - k_i M_i \eta + \chi_i, \quad i \in S. \]

IV. STABILITY ANALYSIS OF SUBSYSTEMS

Since the trajectory for the switched system can diverge even when all the subsystems of the switched system are stable, the switching signal which determines the switching time instant must be properly developed. However, before designing the switching signal, the stability of each subsystem with its closed-loop error system in (25) is first analyzed based on the controller designed in (13).

Theorem 1. [19] Given a collection of the subsystem dynamics of the switched system in (1), the robust controllers designed for each subsystem in (13)-(18) with individual control gains satisfying the sufficient conditions described in (18) ensures that the position tracking of each subsystem is globally UUB in the sense that
\[ \| \epsilon \| \leq \| z \| \leq \tilde{d}_i, \quad \forall t \geq T_i(\tilde{d}_i, \| z(0) \|), \quad i \in S \]
in the region \( \mathcal{D}_1 \) defined as
\[ \mathcal{D}_1 \triangleq \left\{ (e, \eta, y) \in \mathbb{R}^n \times \mathbb{R}^n \times \left( -a \sqrt{n} \frac{\sqrt{\eta}}{\eta}, a \sqrt{n} \frac{\sqrt{\eta}}{\eta} \right) \right\}, \]
where the composite state vector \( z \in \mathbb{R}^{3n} \) is defined as
\[ z = \left[ e^T \eta^T y^T \right]^T. \]

In (26), \( \tilde{d}_i \in \mathbb{R} \) is a positive constant that defines the radius of a ball containing the position tracking error of \( i \)-th subsystem as
\[ \tilde{d}_i > (\gamma_1^{-1} \gamma_2) (\gamma_3^{-1} (\epsilon_i)), \quad i \in S, \]
and \( T_i(\tilde{d}_i, \| z(0) \|) \in \mathbb{R} \) denotes the following positive constant that represents the time duration from the initial time to the time to reach the ball for the \( i \)-th subsystem and is defined as
\[ T_i(\tilde{d}_i, \| z(0) \|) = \begin{cases} 0 & \text{if } \| z(0) \| \leq (\gamma_1^{-1} \gamma_2)(\gamma_3^{-1} (\epsilon_i)) \\ \gamma_2(\| z(0) \|) & \text{if } \| z(0) \| > (\gamma_1^{-1} \gamma_2)(\gamma_3^{-1} (\epsilon_i)) \end{cases} \]
where \( \epsilon_i \) was defined in (20), and the strictly increasing functions \( \gamma_1, \gamma_2, \gamma_3 : \mathbb{R} \rightarrow \mathbb{R} \) are defined as
\[ \gamma_1(\| z \|) \triangleq \lambda_1 \sinh \left( \frac{\| z \|}{\| z \|} \right) \]
\[ \gamma_2(\| z \|) \triangleq \lambda_2 \left( \| e^T \eta^T y^T \| \right)^{2} + \lambda_3 \sum_{j=1}^{n} \frac{y_j^2}{a - y_j} \]
\[ \gamma_3(\| z \|) \triangleq \frac{1}{2} - \frac{1}{4k_{n_4,i}} \tanh^2(\| z \|), \quad i \in S, \]
where $\lambda_1, \lambda_2 \in \mathbb{R}$ are positive constants defined as
\[
\lambda_1 \triangleq \min \left\{ \frac{1}{2}, \frac{m_1}{2} \right\}, \quad \lambda_2 \triangleq \max \left\{ 1, \frac{m_2}{2} \right\}.
\]

Proof: [19] Let $V_i : \mathbb{R}^{3n} \to \mathbb{R}$ be defined as
\[
V_i \triangleq \sum_{j=1}^{n} \ln(\cosh(e_{ij})) + \frac{1}{2} \eta_j^T M_i \eta_j + \frac{1}{2} \sum_{j=1}^{n} a_j y_j^2, \quad i \in \mathcal{S}.
\]

Each Lyapunov function in (32) is a positive-definite radially unbounded function in the set
\[
\mathcal{D} \triangleq \{(e, \eta, y) \in \mathbb{R}^n \times \mathbb{R}^n \times [-a, a]^n\}.
\]

Based on (9), the Lyapunov functions in (32) can be further lower and upper bounded by
\[
\gamma_1(\|z\|) \leq V_i \leq \gamma_2(\|z\|), \quad i \in \mathcal{S},
\]
where $z$ was defined in (28), and $\gamma_1$ and $\gamma_2$ were introduced in (29) and (30).

After taking the time derivative of (32), using (3), (19), (21), (22), and (25), the following inequality can be obtained from [19]:
\[
\dot{V}_i \leq -\frac{1}{2} - \frac{1}{4k_{n4,i}} \|z\|^2 + \varepsilon_i \text{ if } \|z\| < a, \quad i \in \mathcal{S}, \forall t \geq 0,
\]
where $\varepsilon_i$ was defined in (20). By applying the property in (9), (34) can be further upper bounded as
\[
\dot{V}_i \leq -\gamma_{3,i}(\|z\|) + \varepsilon_i \text{ if } \|z\| < a, \quad i \in \mathcal{S}, \forall t \geq 0,
\]
where $\gamma_{3,i}$ was defined in (31), and $z$ was defined in (28). The expression in (35) can be rewritten as [19]
\[
\dot{V}_i \leq -\gamma_{3,i}(\|z\|) + \varepsilon_i, \text{ if } z \in \mathcal{D}_1, \forall t \geq 0.
\]

From (19) and (29)-(31),
\[
\gamma_j(0) = 0, \quad j = 1, 2, \quad \gamma_{3,i}(0) = 0, \quad i \in \mathcal{S},
\]
\[
\lim_{\|z\| \to \infty} \gamma_j(z) = \infty, \quad j = 1, 2,
\]
\[
\lim_{\|z\| \to \infty} \gamma_{3,i}(z) = \left( 1 - \frac{1}{2} \right) - \frac{1}{4k_{n4,i}} \varepsilon_i < \left( 1 - \frac{1}{2} \right) - \frac{1}{4k_{n4,i}} \varepsilon_i, \quad i \in \mathcal{S}.
\]

Therefore, by selecting the initial condition of $p$ according to (17), then $p(t_0)$ with any given $e(t_0)$ and $\eta(t_0)$ will satisfy $z(t_0) \in \mathcal{D}_1$, so the global UUB tracking result in (26) can be ensured for each subsystem.

V. AVERAGE DWELL-TIME

The stability of each subsystem is globally UUB in $\mathcal{D}_1$ from the previous section, but it does not account for the stability when switching between subsystems. To ensure the position tracking error of the switched system is stable, the switching signal must be designed. By applying the scheme introduced by [13], the average dwell-time for the switched system based on the dynamics in (1) can be developed.

**Theorem 2.** The system consisting of the subsystems introduced in (1) with an appropriately designed average dwell-time $\tau^*$ and robust OFB controllers ensure that the position tracking error is semi-global UUB in the sense that
\[
\|e(t)\| \leq \|z(t)\| \leq r_{SG}, \quad \forall t \geq T_{SG}
\]
provided that the average dwell-time satisfies
\[
\tau^* = \frac{\ln \mu}{\beta_0 - \beta^*},
\]
where $r_{SG}, T_{SG}, \tau^*, \mu, \beta_0, \beta^* \in \mathbb{R}$ are known positive constants defined in the subsequent analysis.

Proof: To facilitate the application of the average dwell-time approach, two quadratic functions $\gamma'_2, \gamma'_3 : \mathbb{R} \to \mathbb{R}$ are defined as
\[
\gamma'_2(\|z\|) \triangleq \alpha_2 \|z\|^2, \quad \gamma'_3(\|z\|) \triangleq \alpha_3 \|z\|^2,
\]
where $\alpha_2, \alpha_3 \in \mathbb{R}$ are positive constants determined in the subsequent analysis. Assume that $\alpha_2$ and $\alpha_3$ are selected such that following two inequalities hold
\[
\gamma'_2(\|z\|) \geq \gamma_2(\|z\|), \quad z \in \mathcal{D}_{SG},
\]
\[
\gamma'_3(\|z\|) = \alpha_3 \|z\|^2 < \min_{i \in \mathcal{S}} \gamma_3,i(\|z\|), \quad z \in \mathcal{D}_{SG},
\]
where $\mathcal{D}_{SG}$ denotes a set defined as
\[
\mathcal{D}_{SG} \triangleq \{ z \in \mathbb{R}^{3n} \mid \|z\| \leq \left( \gamma_1 \circ \gamma_2 \right)(\|z(0)\|) \}
\]
where $(\gamma_1 \circ \gamma_2)(\cdot)$ accounts for the largest possible domain that can be reached by $z$ for any given $z(0)$ by using (33).

To ensure $\mathcal{D}_{SG} \subset \mathcal{D}_1$, the adjustable constant $a$ is selected sufficiently large to increase $\mathcal{D}_1$, thereby containing $\mathcal{D}_{SG}$, and will be determined in the subsequent analysis.

Let $\mathcal{D}_0 \subset \mathcal{D}_1$ denote a set defined as
\[
\mathcal{D}_0 \triangleq \{ z \in \mathcal{D}_1 \mid \|z\| \leq \bar{d}_{max} \}
\]
where $\bar{d}_{max} \in \mathbb{R}$ is a known positive constant defined as
\[
\bar{d}_{max} \triangleq \max_{i \in \mathcal{S}} \left\{ d_i \right\}.
\]
Based on the stability analysis in the previous section, given any initial condition $z(0), z \in \mathcal{D}_{SG} \cup \mathcal{D}_0, \forall t \geq 0$. Thus, the following inequality holds
\[
\|z\| \leq \max \left\{ (\gamma_1 \circ \gamma_2)(\|z(0)\|), \bar{d}_{max} \right\}, \quad \forall t \geq 0,
\]
where $(\gamma_1 \circ \gamma_2)(\|z(0)\|)$ dominates when the given initial condition satisfies $\|z(0)\| > \bar{d}_{max}$, otherwise $\bar{d}_{max}$ dominates. Regardless of the initial value, $\|z\|$ is at least bounded by $\bar{d}_{max}$ after some time period, and $\bar{d}_{max}$ can be decreased by increasing the control gain defined in (20).

To ensure $\mathcal{D}_{SG} \cup \mathcal{D}_0 \subset \mathcal{D}_1$ for any given $z(0)$, the adjustable constant $a$ must satisfy the following criterion based on (27) and (40)
\[
\max \left\{ (\gamma_1 \circ \gamma_2)(\|z(0)\|), \bar{d}_{max} \right\} \leq \frac{a}{\sqrt{n}}.
\]
By using (29) and (30), $(\gamma_1 \circ \gamma_2)(\cdot)$ in (41) can be further expressed as
\[
(\gamma_1 \circ \gamma_2)(\cdot) = \cosh^{-1}(\exp(\frac{2\lambda_1}{\lambda_3})).
\]
In addition, by selecting \(a\) sufficiently large, \(\gamma_2\) defined in (30) can be upper bounded as (by using the fact that \(\frac{1}{a-y_j^T} \leq \frac{1}{a||z(0)||^2}\), for \(1 \leq j \leq n\))

\[
\gamma_2(\|z\|) = \lambda_2 \left\| \begin{bmatrix} e^T \\ y^T \end{bmatrix} \right\|^2 + \frac{1}{2} \sum_{j=1}^{n} \frac{y_j^2}{a-y_j^2} \\
\leq \lambda_2 \left\| \begin{bmatrix} e^T \\ y^T \end{bmatrix} \right\|^2 + \frac{1}{2(a-\|z(0)\|^2)} \sum_{j=1}^{n} y_j^2 \\
\leq \lambda_2 \|e\|^2, \tag{43}
\]

where \(\lambda_2 \in \mathbb{R}\) is a positive constant defined as \(\lambda_2 \triangleq \max \left\{ \lambda_2, \frac{1}{2(a-\|z(0)\|^2)} \right\}\) and \(a\) satisfies

\[
a > \|z(0)\|^2. \tag{44}
\]

Based on (42) and (43), (41) can be further upper bounded as

\[
\max \left\{ \cosh^{-1}(\exp(\frac{\alpha_2(\|z(0)\|)}{\lambda_1})), \tilde{d}_{\max} \right\} \leq \frac{a}{\sqrt{n}}, \tag{45}
\]

where the inequality holds provided that \(a\) is selected sufficiently large. Therefore, \(\mathbb{D}_{SG} \cup \mathcal{D}_0 \subset \mathcal{D}_1\) holds if \(a\) satisfies (44) and (45).

According to (37) and (38), \(\alpha_2\), introduced in (37), is determined based on (43) as \(\alpha_2 = \lambda_2\), and \(\alpha_3\) can be determined by selecting it to be sufficiently small to satisfy (39) for \(z \in \mathbb{D}_{SG}, \forall t \geq 0\). By utilizing the two auxiliary functions defined in (33) and (37), (36) can be further upper bounded as

\[
V_i \leq - \left( \gamma_{3,i} \circ \gamma_{2,i}^{-1} \right) (V_i) + \varepsilon_i \leq - \left( \gamma_{3,i} \circ \gamma_{2,i}^{-1} \right) (V_i) + \varepsilon_i \leq - \frac{\alpha_2}{\alpha} V_i + \varepsilon_i, \quad i \in \mathbb{S}, \tag{46}
\]

for \(z \in \mathbb{D}_{SG}, \forall t \geq 0\). Based on (46), \(\mathbb{D}_{\alpha,i}\) denotes the UUB region of the \(i^{th}\) subsystem and is defined as

\[
\mathbb{D}_{\alpha,i} \triangleq \left\{ z \in \mathbb{D}_1 \mid V_i \leq \frac{\alpha_2}{\alpha_3} \varepsilon_i \right\}, \quad \forall i \in \mathbb{S},
\]

and the union of individual UUB region denoted as \(\mathbb{D}_{\bar{\alpha}}\) can be defined as \(\mathbb{D}_{\bar{\alpha}} \triangleq \bigcup_{i \in \mathbb{S}} \mathbb{D}_{\alpha,i}\), where the area of \(\mathbb{D}_{\bar{\alpha}}\) can be reduced by increasing the control gain \(k\), thereby reducing \(\varepsilon_i\) of the \(i^{th}\) subsystem.

Let the multiple Lyapunov function candidate \(V : \mathbb{R}^{3n} \rightarrow \mathbb{R}^n\) be defined as

\[
V(z) \triangleq V_\sigma(t)(z), \tag{47}
\]

where \(\sigma : [0, \infty) \rightarrow \mathbb{S}\) denotes a piecewise constant switching signal that determines the sequence of switching between subsystems. Applying the same concept as [21], it is assumed from (46) that each subsystem is UUB with a decay rate along Lyapunov function less than \(\frac{\alpha_2}{\alpha_3}\) if \(z \in \mathbb{D}_{SG} \setminus \mathbb{D}_{\bar{\alpha}}\).

Let \(t_1, t_2, \ldots\) be the time instants at which a switching event occurs, and \(m \in \mathbb{N}\) denotes the number of switchings. Then, any switching time sequence with \(m\) switchings can be expressed as \(0 < t_1 < \cdots < t_m < t < t_{m+1}\). Let \(q_m \in \mathbb{S}\) denote the value of \(\sigma\) between switches that is defined as

\[
q_m \triangleq \sigma(t), \quad t \in [t_{j-1}, t_j), \quad \forall j \in \mathbb{N}.
\]

Then, using (46) and (47), it is straightforward to show

\[
V(z) \leq \max \left\{ e^{-\beta_{q_m}(t-t_m)} V_{q_m}(z(t_m)), \frac{\alpha_2 \varepsilon}{\alpha_3} \right\}, \quad q_m \in \mathbb{S}, \tag{48}
\]

for \(z \in \mathbb{D}_{SG} \cup \mathcal{D}_\alpha, \forall t \geq 0\). In (48), \(\beta_{q_m} \in \mathbb{R}\) is a positive constant that denotes a decay rate of the Lyapunov function \(V_i\) along the closed-loop dynamics of the \(i^{th}\) subsystem and satisfies \(\beta_{q_m} < \frac{\alpha_2}{\alpha_3}\), and \(\varepsilon \in \mathbb{R}\) is defined as

\[
\varepsilon \triangleq \max \left\{ \varepsilon_i \right\}, \quad i \in \mathbb{S}.
\]

Based on (33), the following inequalities hold

\[
V_{q_j}(z) \leq \mu V_{q_k}(z), \quad V_{q_k}(z) \leq \mu V_{q_j}(z), \quad j \neq k, \tag{49}
\]

where \(q_j, q_k \in \mathbb{S}\) index any two subsystems, and \(\mu \in \mathbb{R}\) is a positive constant that denotes the maximum ratio of any two Lyapunov functions of the subsystems and is defined as

\[
\mu \triangleq \sup_{t} \left\{ \frac{\gamma(z)}{\gamma_1(z)} \right\}, \quad z \in \mathbb{D}_{SG} \cup \mathcal{D}_\alpha.
\]

By using (49) and defining \(\beta_0 \in \mathbb{R}\) to be positive constant as \(\beta_0 \triangleq \min_{i \in \mathbb{S}} \beta_i\), the expression in (48) can be upper bounded as [21]

\[
V(z) \leq \max \left\{ e^{-\beta_{0}(t-t_m)} V_{q_m}(z(t_m)), \frac{\alpha_2 \varepsilon}{\alpha_3} \right\}
\]

\[
\leq \max \left\{ e^{-\beta_{0}(t-t_m)} \mu V_{q_{m-1}}(z(t_{m-1})), \frac{\alpha_2 \varepsilon}{\alpha_3} \right\}
\]

\[
\leq \max \left\{ e^{-\beta_{0}(t-t_m)} \mu e^{-\beta_{0}(t_{m-1}-t_{m-1})} V_{q_{m-1}}(z(t_{m-1})), \frac{\alpha_2 \varepsilon}{\alpha_3} \right\}
\]

\[
\leq \cdots \leq \max \left\{ e^{-\beta_{0}t} \mu V_{q_{0}}(z(0)), \frac{\alpha_2 \varepsilon}{\alpha_3} \right\}
\]

\[
= \max \left\{ e^{-\beta_{0}t} \mu N_{\sigma}(t) V(z(0)), \frac{\alpha_2 \varepsilon}{\alpha_3} \right\}, \tag{50}
\]

for \(z \in \mathbb{D}_{SG} \cup \mathcal{D}_\alpha, \forall t \geq 0\). In (50), \(N_{\sigma} \in \mathbb{R}\) denotes the number of switchings during time interval \([0, t], t \in [t_m, t_{m+1})\). Moreover, in (50) the inequality holds even if the switching sequence is arbitrarily specified.

By selecting a desired decay rate \(\beta^* \in \mathbb{R}\) for the switched system, where

\[
0 < \beta^* < \beta_0,
\]

(50) can be used to show

\[
e^{-\beta^*t} \leq e^{-\beta^*t}, \quad t \in [t_m, t_{m+1}). \tag{51}
\]
Based on (51), the average dwell-time $\tau^* \in \mathbb{R}$ can be determined as
\[
\tau^* = \frac{\ln \mu}{\beta_0 - \beta^*},
\tag{52}
\]
and the number of switchings is finite and can be upper bounded by
\[
N_\sigma \leq \frac{t}{\tau^*}, \quad t \in [t_m, t_{m+1}).
\]
Then, (50) can be expressed as
\[
V(z) \leq \max \left\{ e^{-\beta^* t} V(z(0)), \frac{\alpha_2 \varepsilon}{\alpha_3} \right\}.
\tag{53}
\]

Based on (53), $r_{SG} \in \mathbb{R}$ denotes the radius of the UUB ball of the switched system and can be determined by using (33) as
\[
r_{SG} \triangleq \left( \frac{\gamma_2 \circ \gamma_1^{-1}}{\alpha_3} \right) \frac{\alpha_2 \varepsilon}{\alpha_3}.
\]
By using (53), $T_{SG} \in \mathbb{R}$ denotes the time to reach the UUB ball for any given $z(0)$ and can be determined as
\[
T_{SG} \geq \begin{cases} 
- \ln \left( \frac{\alpha_2 \varepsilon}{\alpha_3} \right) & \text{if } V(z(0)) > \frac{\alpha_2 \varepsilon}{\alpha_3}, \varepsilon > 0, \\
0 & \text{otherwise}.
\end{cases}
\tag{54}
\]

In (53), the decay rate $\beta^*$ can be arbitrary selected on $(0, \beta_0)$ which characterizes the decay rate of the entire switched system. The time to reach the UUB ball for any given $V(z(0))$ can be decreased by increasing $\beta^*$, as indicated in (52). However, increases in $\beta^*$ require a larger average dwell-time, as indicated in (52). Thus, there is an interplay between average dwell-time and the time to reach the UUB ball. Based on this average dwell-time scheme, the position tracking error of the switched system is semi-global UUB with arbitrary switching sequences.

VI. CONCLUSION

A robust OFB controller with a time-dependent switching signal is developed for a switched Euler-Lagrange system, which consists of subsystems with parametric uncertainties and additive bounded disturbances. A switching signal is designed based on the average dwell-time. A Lyapunov function analysis consisting of multiple Lyapunov functions is developed that yields semi-global UUB OFB tracking with arbitrary switching sequences.

REFERENCES